Magnetized Schwarzschild solution in five-dimensional gravity

Tonatiuh Matos
Departamento de Física, Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional, Apartado Postal 14-740, 07000 México, Distrito Federal, México
(Received 23 May 1988)

Two new exact solutions for the five-dimensional Kaluza-Klein field equations are generated. The first one is a solution without a magnetic field but with a scalar potential. The second one is an exact solution with a gravitational potential as the Schwarzschild solution and a magnetic field containing a flux along the azimuthal direction and a scalar potential equal to one far away from a black hole.

I. INTRODUCTION

In this work a new static solution with a magnetic field for the Kaluza-Klein theories[1] without sources is determined. For these theories, electromagnetism is a consequence of the projection of a single five-dimensional field to space-time. There are different attempts to formulate the field equations. We will discuss a variant in which the vacuum fields are characterized by the vanishing of the five-dimensional Ricci tensor, i.e.,

$$R_{\mu\nu}=0, \quad \mu,\nu=1,\ldots,5 , \tag{1}$$

and the five-dimensional metric $\gamma_{\mu\nu}$ is the projection tensor

$$\gamma_{\mu\nu}=g_{\mu\nu}+X_\mu X_\nu I^{-2} , \tag{2}$$

where X^μ is a Killing vector, $I^2=X^\mu X_\mu$, and $g_{\mu\nu}$ is the space-time metric. In the adapted coordinates system where $X^\mu=\delta^\mu_5$, the space-time metric g_{ij}, and the electromagnetic four potential A_k are related with the five-dimensional metric according to

$$\begin{gathered} I^2=\gamma_{55}, \quad A_k=I^{-2}\gamma_{5k} , \\ g_{ik}=\gamma_{ik}-I^2A_iA_k, \quad i,k=1,\ldots,4 \end{gathered} \tag{3}$$

(see Ref. 2). By introducing a second Killing vector field Y^μ, one may define in a covariant manner, a set of five potentials:

$$\begin{gathered} I^2=X^\mu X_\mu, \quad f=-IY^\mu Y_\mu+I^{-1}(X^\mu Y_\mu)^2, \quad \psi=I^{-2}X_\mu Y^\mu , \\ \chi_{,\mu}=2\epsilon_{\alpha\beta\gamma\delta\mu}X^\alpha Y^\beta X^{\gamma;\delta}, \quad \epsilon_{,\mu}=2\epsilon_{\alpha\beta\gamma\delta\mu}X^\alpha Y^\beta Y^{\gamma;\delta} , \end{gathered} \tag{4}$$

where I^2 corresponds to the scalar potential, f to the gravitational potential, ψ to the electrostatic potential, χ to the magnetostatic potential, and ϵ to the rotational potential. These potentials define a Riemannian space V_5^P with metric

$$dS^2=\frac{1}{2f^2}[df^2+(d\epsilon+\psi d\chi)^2]+\frac{1}{2f}\left[\kappa^2 d\psi^2+\frac{1}{\kappa^2}d\chi^2\right]+\frac{2}{3}\frac{d\kappa^2}{\kappa^2} , \tag{5}$$

where $k^2=I^3$.

This metric is endowed with an eight-parameter group of motions (see Ref. 3). The Jordan theory[4] and the Brans-Dicke theory[5] are contained in (5) after an appropriate conformal transformation. The potentials (4) are equivalent to the Ernst potentials of the Einstein-Maxwell theory (see Ref. 6).

We assume that the five potentials $\Psi^A=(\kappa,f,\psi,\chi,\epsilon)$, depend only on two coordinates. Let these two coordinates be ρ and z (the Weyl canonical coordinates) and let $\zeta=\rho+iz$. Then $\Psi^A=\Psi^A(\zeta,\bar\zeta)$ where $\bar\zeta$ is the complex conjugate of ζ. Substituting the potentials (4) into (1), one finds that the field equations can be written as

$$(\rho\Psi^A_{,\zeta})_{,\bar\zeta}+(\rho\Psi^A_{,\bar\zeta})_{,\zeta}+2\rho\left\{ {A \atop BC} \right\}\Psi^B_{,\zeta}\Psi^C_{,\bar\zeta}=0 , \tag{6}$$

where the $\left\{ {A \atop BC} \right\}$ are the Christoffel symbols of the metric (5) (see Ref. 7). In this report, our main goal is to find a set of solutions to Eq. (6).

II. TRANSFORMATION OF THE FIELD EQUATIONS

Let us suppose that the potentials Ψ^A depend only on one parameter λ which depend on ζ and $\bar\zeta$; i.e.,

$$\Psi^A=\Psi^A(\lambda(\zeta,\bar\zeta)), \quad A=1,\ldots,5 . \tag{7}$$

Under such an assumption, the field equation (6) becomes

$$2\rho\left[\Psi^A_{,\lambda\lambda}+\left\{ {A \atop BC} \right\}\Psi^B_{,\lambda}\psi^C_{,\lambda}\right]\lambda_{,\zeta}\lambda_{,\bar\zeta}+\Psi^A_{,\xi}[(\rho\lambda_{,\zeta})_{,\bar\zeta}+(\rho\lambda_{,\bar\zeta})_{,\zeta}]=0 . \tag{8}$$

Now let the parameter λ be a solution of the Laplace equation

$$(\rho\lambda_{,\zeta})_{,\bar\zeta}+(\rho\lambda_{,\bar\zeta})_{,\zeta}=0 , \tag{9}$$

then the equations for the five potentials Ψ^A reduce to

$$\Psi^A_{,\lambda\lambda}+\left\{ {A \atop BC} \right\}\Psi^B_{,\lambda}\Psi^C_{,\lambda}=0 . \tag{10}$$

All solutions of (10) have been found in Ref. 7.

In this paper we investigate only two solutions of (10): namely,

$$(\mathrm{A})\ \kappa^2 = {}^3\sqrt{4c}\,e^{3r_2\lambda/2},\quad f=-2\,dc\,e^{r_1\lambda+r_2\lambda/2},$$
$$\psi=\epsilon=\chi=0\ , \tag{11a}$$

$$(\mathrm{B})\ \kappa^2=\frac{c^2}{\sqrt{d}}\left[\frac{4e^{r_1\lambda}}{c\lambda+b}\right]^{3/2},\quad f=\frac{\sqrt{d}\,e^{-3r_1\lambda/2}}{c\lambda+b}\ ,$$
$$\chi=\frac{2\sqrt{2}c}{c\lambda+b},\quad \psi=\epsilon=0\ , \tag{11b}$$

where b, c, d, r_1, and r_2 are constants.

As a solution of (9), we consider the function

$$e^{r_1\lambda}=e^{2u}=1-\frac{2m}{r} \tag{12}$$

written in Boyer-Lindquist coordinates

$$\rho=\sqrt{r^2-2mr}\,\sin\theta,\quad z=(r-m)\cos\theta\ .$$

The expressions (11a), (11b), and λ given in (12) fulfill (6) in spherical coordinates. In what follows we shall investigate both solutions.

III. SOLUTION (A)

The solution (11a) contains only scalar and gravitational fields. For $\delta_0=\frac{1}{2}r_2$, $(4c)^{1/6}=I_0$ and $d(c)^{5/6}=-4$ from (4), one obtains

$$I=I_0e^{\delta_0 u},\quad g_{44}=-e^{2u}\ ,$$
$$g_{33}=\frac{\rho^2}{I_0^2}e^{-2(1+\delta_0)u},\quad g_{34}=0 \tag{13}$$

with vanishing electromagnetic potential. [We have taken as usual $(x^1,x^2,x^3,x^4)=(r,\theta,\phi,t)$.] To determine the components g_{11} and g_{22}, one integrates the differential equation

$$(\ln F)_{,\zeta}=\frac{(\ln\rho)_{,\zeta\zeta}}{(\ln\rho)_{,\zeta}}+\frac{\mathrm{Tr}(\rho A)^2}{4\rho\rho_{,\zeta}}\ , \tag{14}$$

where the 3×3 matrix $A=g_{,\zeta}g^{-1}$ with the 3×3 matrix g defined in terms of the components of the metric tensor $(g)_{ij}=g_{ij}$ for $i,j=3,\ldots,5$ (see Ref. 8 or 9). The differential equations (14) and its complex conjugate yield a system of differential equations for the function $F=g_{11}=g_{22}$. Integrating the quoted equations one arrives at

$$F\,d\zeta\,d\bar{\zeta}=\frac{\left[1-\frac{2m}{r}\right]^{\delta_0^2}}{\left[1-\frac{2m}{r}+\frac{m^2}{r^2}\sin^2\theta\right]^{(\delta_0+\delta_0^2)}}\times\left[\frac{dr^2}{1-\frac{2m}{r}}+r^2d\theta^2\right]. \tag{15}$$

Thus, the full metric is

$$dS^2=\frac{\left[1-\frac{2m}{r}\right]^{\delta_0^2}}{\left[1-\frac{2m}{r}-\frac{m^2}{r^2}\sin^2\theta\right]^{(\delta_0+\delta_0^2)}}\left[\frac{dr^2}{1-\frac{2m}{r}}+r^2d\theta^2\right]+\frac{1}{I_0^2\left[1-\frac{2m}{r}\right]^{\delta_0}}r^2\sin^2\theta\,d\phi^2-\left[1-\frac{2m}{r}\right]c^2dt^2\ ,$$
$$I^2=I_0^2\left[1-\frac{2m}{r}\right]^{\delta_0}, \tag{16}$$

where δ_0 and I_0 are arbitrary constants and m is the mass parameter. It is easily seen that when the δ_0 parameter vanishes, one gets the Schwarzschild solution in (16) (with $I_0=1$). For $m=0$ the metric (16) becomes flat.

IV. SOLUTION (B)

The solution (11b) possesses scalar, gravitational, and magnetic potentials. We take again λ as in (12) and $d=b=2\sqrt{2}$, $r_1=1/\eta$. If we do so and take into account the definitions (4) we find

$$g_{44}=-e^{2u},\quad g_{33}=\rho^2e^{-u}(1-\eta u),\quad g_{34}=0,\quad I^2=\frac{e^{-u}}{1-\eta u},\quad A_\phi=-\eta m\cos\theta\ , \tag{17}$$

while A_r, A_θ, and A_t vanish. The function F for this metric can be now found. Substituting the quantities (17) into (14), one arrives at

$$dS^2=\left[1-\frac{m^2\sin^2\theta}{r^2\left[1-\frac{2m}{r}\right]}\right]^{1/4}\left[1-\frac{2m}{r}\right]^{1/2}\left[1-\eta\ln\left[1-\frac{2m}{r}\right]^{1/2}\right]\left[\frac{dr^2}{1-\frac{2m}{r}}+r^2d\theta^2\right]$$
$$+\left[1-\frac{2m}{r}\right]^{1/2}\left[1-\eta\ln\left[1-\frac{2m}{r}\right]^{1/2}\right]r^2\sin^2\theta\,d\phi^2-\left[1-\frac{2m}{r}\right]c^2dt^2\,,$$
$$A_r=A_\theta=A_t=0,\quad A_\phi=-\eta m\cos\theta,\quad I^2=\frac{1}{\left[1-\frac{2m}{r}\right]^{1/2}\left[1-\eta\ln\left[1-\frac{2m}{r}\right]^{1/2}\right]}\,. \tag{18}$$

The metric (18) is an exact solution of the field equations (1) with scalar field I^2 and magnetic field A_ϕ. It is asymptotically flat and the scalar potential reduces to one as r approaches infinity. Nevertheless, close to a black hole, I^2 must be taken into account, the coefficients of $d\theta^2$ and $d\phi^2$ go very slowly to zero and the coefficient of dt^2 goes even faster to zero. Near to the black hole the scalar potential very slowly gets bigger, but it is singular for $r=2m$ as is the factor of dr^2 too.

V. THE LIMIT $2m/r \ll 1$

For $r \gg 1$, the metric (18) becomes flat, and I^2 goes to one. Nevertheless, the third component of the electromagnetic four potential A_ϕ remains finite. We can establish the behavior of the magnetic field at first order, if we take the metric flat. We suppose that the metric (18) describes a magnet. Then the magnetic induction $\mathbf{B}$ is given by

$$\mathbf{B}=\text{curl}\,\mathbf{A}=\frac{\eta m}{r}\left[-\frac{\cos2\theta}{\sin\theta}\hat{\mathbf{e}}_r+\cos\theta\hat{\mathbf{e}}_\theta\right].$$

The hypothetical flux $\mathbf{J}'$ which cause this magnetic induction can be now obtained:

$$\text{curl}\mathbf{B}=-\frac{\eta m}{r^2}\cos\theta(2+\csc^2\theta)\hat{\mathbf{e}}_\phi=\frac{4\pi}{c}\mathbf{J}'\,,$$

which is a flux along the azimuthal direction. The magnetization $\mathbf{M}$ is given by

$$\mathbf{M}=\frac{1}{2c}\mathbf{r}\times\mathbf{J}'=\frac{1}{8\pi}\cos\theta(2+\csc^2\theta)\hat{\mathbf{e}}_\theta\,.$$

Thus the magnetic field vector $\mathbf{H}$ is

$$\mathbf{H}=\mathbf{B}-4\pi\mathbf{M}=-\frac{\eta m}{r}\left[\frac{\cos2\theta}{\sin\theta}\hat{\mathbf{e}}_r+\frac{1}{2}\frac{\cos\theta}{\sin^2\theta}\hat{\mathbf{e}}_\theta\right].$$

It can be seen that $\text{curl}\mathbf{M}=0$. That means that $\text{curl}\mathbf{H}=\text{curl}\mathbf{B}=(4\pi/c)\mathbf{J}=(4\pi/c)\mathbf{J}'$. $\mathbf{J}$ is a real flux along the e_ϕ axis. It reminds us of the Van Allen rings around the planets when they have a magnetic field. $\mathbf{J}$ is in this theory a consequence of the existence of the magnetic field, it is not a source. The magnetic field $\mathbf{H}$ is a consequence of the projection in four dimensions of a single five-dimensional field. It is easily seen that the field lines of $\mathbf{H}$ are in the surface

$$r=a^2e^{2\sin^2\theta}\cos^2\theta\,,$$

where a^2 is a constant. They are not the field lines of a magnetic dipole, but they seem like wings of a butterfly. They might be seen as a first approximation of the field lines of the magnetic field, because we have supposed that the metric is flat.

VI. CONCLUSIONS

We have generated two classes of solutions, one of them without magnetic field, which for a choice of one parameter, reduces to the Schwarzschild solution and the scalar potential becomes equal to 1. The other one has a magnetic field, as a consequence of the projection of the five-dimensional field. In this case, one real flux along the azimuthal direction is necessarily produced as a consequence of the existence of a magnetic field. Both solutions are asymptotically flat. In the second case, the scalar potential is rather big near a black hole, and equal to one far away from it.

ACKNOWLEDGMENT

I wish to thank Dr. E. Piña Garza for some helpful discussions.

[1]Th. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Phys. Math. Kl. 966 (1921); O. Klein, Z. Phys. **37**, 875 (1926).

[2]E. Schmutzer, in *Proceedings of the Ninth International Conference on General Relativity and Gravitation*, edited by E. Schmutzer (DVW, Berlin, 1982)

[3]G. Neugebauer, Habilitationsschrift, Jena, 1969.

[4]P. Jordan, *Schwerkraft and Weltall* (Vieweg, Braunschweig, 1955).

[5]C. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961).

[6]D. Kramer, H. Stephani, M. MacCallum, and E. Herlt, in *Exact Solutions of Einstein Field Equations*, edited by E. Schmutzer (DVW, Berlin, 1980).

[7]T. Matos, Ann. Phys. (Leipzig) (to be published).

[8]T. Matos, Rev. Mex. Fis. (to be published).

[9]V. A. Belinsky and R. Ruffini, Phys. Lett. **89B**, 195 (1980).