SL(3, ℝ) REPRESENTATION FOR INVARIANCE TRANSFORMATIONS IN FIVE-DIMENSIONAL GRAVITY ☆

Tonatiuh MATOS

Departamento de Física, Centro de Investigación y de Estudios Avanzados del I.P.N., Apartado Postal 14-740, Mexico 07000, DF, Mexico

Received 28 April 1988; accepted for publication 1 July 1988
Communicated by J.P. Vigier

The invariance transformations of the axisymmetric five-dimensional vacuum Einstein equations are written in a representation of the group SL(3, ℝ) in the potential space (analogous to the Ernst potential space). Using this formulation, an exact class of stationary axisymmetric solutions is generated, which contains among others Belinsky–Ruffini, Kramer and the Kerr–NUT solutions, as particular cases.

1. Introduction

In this paper we give a matrix representation for invariance transformations in Kaluza–Klein theories [1]. As an example how to use this, we generate a new axisymmetric solution, which contains, for instance, the solutions due to Belinsky and Ruffini [2], Kramer [3], Neugebauer [4] and the Kerr–NUT solution [5] as particular cases. It is well-known that the axisymmetric Ernst equations [6] contain an isometry group SU(1,1) in vacuum and an isometry group SU(2,1) for the Einstein–Maxwell theory [7,8]. A study of these transformations and their consequences is given in refs. [5,8].

Since the group SU(1,1) is isomorphic to the group SL(2, ℝ), it is possible to write the Ernst equations also in this last representation. This will be done in this paper. Furthermore, we give a representation of the isometry group SL(3, ℝ) of the five-dimensional Einstein equations when the five-dimensional metric contains three Killing vectors.

The five-dimensional Einstein equations with a compact fifth dimension are known as the Kaluza–Klein theory [1]. It is a unified field theory of gravitation and electromagnetism. There are only a few known exact solutions of this theory. Kühnel and Schmutzer [4] published a static solution for a charged particle. Neugebauer [7] defined in a covariant form five potentials κ, f, ψ, χ, and ϵ when the five-dimensional metric contains two Killing vectors X^μ and Y^μ in the following form:

$$\kappa^{4/3}=I^2=X^\mu X_\mu,\quad f=-IY^\mu Y_\mu+I^{-1}(X^\mu Y_\mu)^2,\quad \psi=I^{-2}X^\mu Y_\mu\,,$$

$$\chi_{,\mu}=2\epsilon_{\alpha\beta\gamma\delta\mu}X^\alpha Y^\beta X^{\gamma,\delta},\quad \epsilon_{,\mu}=2\epsilon_{\alpha\beta\gamma\delta\mu}X^\alpha Y^\beta Y^{\gamma,\delta}$$

($\epsilon_{\alpha\beta\gamma\delta\mu}$ is the five-dimensional Levi-Civita pseudotensor).

These five potentials are the local coordinates of a potential space V_5^P with the metric

$$dS^2=\frac{1}{2f}[df^2+(d\epsilon+\psi\, d\chi)^2]+\frac{1}{2f}\left(\kappa^2\,d\psi^2+\frac{1}{\kappa^2}\,d\chi^2\right)-(\xi-\tfrac{3}{2})\frac{d\kappa^2}{\kappa^2}\,. \tag{1}$$

☆ This work was partially carried out in a Ph. D. stay at the FSU, Jena, GDR.

0375-9601/88/$ 03.50 © Elsevier Science Publishers B.V.
(North-Holland Physics Publishing Division)

The potentials $\psi^A = (\kappa, f, \psi, \chi, \epsilon)$, $A = 1, ..., 5$ respectively, are the scalar, gravitational, electrostatic, magnetostatic, and rotational potential [7]. Kramer [3] gave a method to find stationary axisymmetric solutions without electromagnetism (but in the presence of the scalar potential) and Neugebauer [9] found a method, consisting of a coordinate transformation in the five-dimensional Riemann space to generate charged solutions from vacuum solutions. A stationary axisymmetric soliton solution was reported Belinsky and Ruffini [2]. A systematic construction of spherically symmetric solutions has been developed by Dobiasch and Maison [10]. Lessner [11] used the Neugebauer [4] and Kramer [3] methods together to generate new stationary axisymmetric solutions. Clement [12] showed that a static solution of the $(M+1)$-dimensional Einstein equations corresponds to each stationary solution of the M-dimensional Einstein equations and gave a relation between the components of the M- and $(M+1)$-dimensional metrics for the axisymmetric case. A linear problem equivalent to the five-dimensional field equations in the potential space has been constructed [13] and the corresponding N-soliton solution has been derived [14]. In a forth-coming paper we will analyse all the subspaces of the potential space of dimension one and two [15], and generate a set of new solutions.

2. The chiral form of the field equations

The explicit form of the five-dimensional Einstein field equations in the potential space V_5^P are given in ref. [7], and in ref. [13] for the axisymmetric case. The equations for this last case can be rewritten in the form

$$(\rho g_{,z} g^{-1})_{,\bar{z}} + (\rho g_{,\bar{z}} g^{-1})_{,z} = 0\,, \tag{2}$$

where Weyl's canonical coordinates ρ and ζ are related with $z = \rho + i\zeta$ and its complex conjugate $\bar{z}$. The matrix g in (2) is a symmetric matrix which is an element of the group SL(3, $\mathbb{R}$), i.e.

$$g = g^{\mathrm{T}}, \quad g = \bar{g}, \quad \det g = 1 \tag{3}$$

(T denotes matrix transposition). A suitable parametrization of g in terms of the potentials ψ^A is given by

$$g = \frac{1}{f\kappa^{2/3}} \begin{pmatrix} -2(f^2+\epsilon^2-f\kappa^2\psi^2) & 2\epsilon & 2^{-1/2}(\epsilon\chi+f\kappa^2\psi) \\ 2\epsilon & -2 & -2^{-1/2}\chi \\ 2^{-1/2}(\epsilon\chi+f\kappa^2\psi) & -2^{-1/2}\chi & -\frac{1}{4}(\chi^2-\kappa^2 f) \end{pmatrix}. \tag{4}$$

The SL(3, $\mathbb{R}$) symmetry transformations can be written in the form

$$g = C g_0 C^{\mathrm{T}}\,, \tag{5}$$

where the constant matrix C is also an element of SL(3, $\mathbb{R}$). The field equations (2) and the conditions (3) are preserved under (5). A straightforward calculation shows that $\mathrm{Tr}(\mathrm{d}g\,\mathrm{d}g^{-1}) = 4\mathrm{d}S^2$. If we substitute $\psi = \chi = 0$, $\kappa = 1$ in (4), we obtain the axisymmetric Ernst equations in (2), $E = f + i\epsilon$ being the Ernst potential. An equivalent approach was developed by Neugebauer and Kramer [16] for the Einstein–Maxwell theory.

3. The new solution

It is not difficult to calculate the inverse matrix of g. We get

$$g^{-1} = -\frac{1}{2}\frac{\kappa^{2/3}}{f} \begin{pmatrix} 1 & \epsilon-\chi\psi & 2\sqrt{2}\psi \\ \epsilon-\chi\psi & f^2+(\epsilon-\chi\psi)^2-f\chi^2\kappa^{-2} & 2\sqrt{2}[f\chi\kappa^{-2}+\psi(\epsilon-\chi\psi)] \\ 2\sqrt{2}\psi & 2\sqrt{2}[f\chi\kappa^{-2}+\psi(\epsilon-\chi\psi)] & -8(f\kappa^{-2}-\psi^2) \end{pmatrix}. \tag{6}$$

We can now write down the potentials ψ^A in terms of the components of the matrices g and g^{-1},

$$\kappa^{4/3}=\frac{4g_{11}^{-1}}{g_{22}},\quad f^2=\frac{1}{g_{11}^{-1}g_{22}},\quad \chi=2\sqrt{2}\frac{g_{23}}{g_{22}},\quad \psi=\frac{1}{2\sqrt{2}}\frac{g_{13}^{-1}}{g_{11}^{-1}},\quad \epsilon=-\frac{g_{12}}{g_{22}}, \tag{7}$$

where g_{ij} are the components of the matrix g and g_{ij}^{-1} the components of the matrix g^{-1}. We assume that $\psi=\chi=0$ for the matrix g_0 in (5), then the matrix g in (5) can be evaluated. We use the relations (7) to get

$$\kappa^{4/3}=\frac{D}{4W}\kappa_0^{4/3},\quad f^2=\frac{f_0^2}{DW},\quad \psi=\frac{1}{2\sqrt{2D}}\left[(u\epsilon_0+q)(w\epsilon_0+t)+f_0(wuf_0-sz\kappa_0^{-2})\right],$$

$$\chi=\frac{2\sqrt{2}}{W}\left[(d\epsilon_0-e)(i\epsilon_0-h)+f_0(dif_0-kj\kappa_0^2)\right],\quad \epsilon=-\frac{1}{W}\left[(a\epsilon_0-b)(d\epsilon_0-e)+f_0(adf_0-cj\kappa_0^2)\right],$$

$$D=(u\epsilon_0+q)^2+f_0(u^2f_0-s^2\kappa_0^{-2}),\quad W=(d\epsilon_0-e)^2+f_0(d^2f_0-j^2\kappa_0^2), \tag{8}$$

for the matrix

$$C=\begin{pmatrix} a & b & c\\ d & e & j\\ i & h & k\end{pmatrix}=\begin{pmatrix} q & p & t\\ u & v & w\\ s & y & z\end{pmatrix}^{-1}. \tag{9}$$

(In order to obtain a better notation, we substituted κ_0 by $2^{-3/2}\kappa_0$.)

After the transformation (8), the new solution is endowed eight free parameters. Nevertheless if we start from a seed solution, in which ϵ_0 vanishes and f_0 and κ_0 reduce to one for a certain limit, we must choose the matrix C such that

$$qt+wu-sz=0,\quad be+ad-cj=0,\quad eh+di-kj=0,\quad q^2+u^2-s^2=1,\quad e^2+d^2-j^2=1, \tag{10}$$

in order to obtain a new solution with the same properties, and in which ψ and χ vanish for this limit. So, the new solution will be endowed with only three new free parameters. Using the transformations (8), we can obtain a solution for the potentials ψ^A from an arbitrary seed solution without electromagnetism. A particular case are the transformations [11] for

$$C=\begin{pmatrix} c & 0 & -(c^2-1)^{1/2}\\ 0 & 1 & 0\\ -(c^2-1)^{1/2} & 0 & c\end{pmatrix}.$$

Let us consider an example. We take the Kerr–NUT solution together with a κ_0 potential as a seed solution. In this case we have

$$f_0=\frac{\Sigma-2mr-2l\theta}{\Sigma},\quad \epsilon_0=\frac{2(m\theta-lr)}{\Sigma},\quad \kappa_0=\left(\frac{r-m+\sigma}{r-m-\sigma}\right)^{\delta},\quad \theta=\cos v+l,\quad \Sigma=r^2+\theta^2,$$

where r and v are the Boyer–Lindquist coordinates; the constants a, m and l are respectively the rotation, mass and NUT parameters. We obtain the following solution,

$$\kappa^{4/3}=\frac{1}{4}\left(\frac{r-m+\sigma}{r-m-\sigma}\right)^{-2/3\delta}\frac{D}{W},\quad f^2=\frac{(\Sigma-2mr-2l\theta)^2}{DW}\Sigma^2\left(\frac{r-m+\sigma}{r-m-\sigma}\right)^{2\delta},$$

$$\psi=\frac{1}{2\sqrt{2D}}\left[\left(\frac{r-m+\sigma}{r-m-\sigma}\right)^{2\delta}[2u(m\theta-lr)+q\Sigma][2w(m\theta-lr)+t\Sigma]\right.$$

$$\left.+wu(\Sigma-2mr-2l\theta)^2\Sigma^2\left(\frac{r-m+\sigma}{r-m-\sigma}\right)^{2\delta}-sz(\Sigma-2mr-2l\theta)\Sigma\right],$$

$$\chi=\frac{2\sqrt{2}}{W}\left\{[2d(m\theta-lr)-e\Sigma][2i(m\theta-lr)-h\Sigma]+(\Sigma-2mr-2l\theta)\left[dl(\Sigma-2mr-2l\theta)-kj\Sigma\left(\frac{r-m+\sigma}{r-m-\sigma}\right)^{2\delta}\right]\right\},$$

$$\epsilon=-\frac{1}{W}\left\{[2a(m\theta-lr)-b\Sigma][2d(m\theta-lr)-e\Sigma]+(\Sigma-2mr-2l\theta)\left[ad(\Sigma-2mr-2l\theta)-cj\Sigma\left(\frac{r-m+\sigma}{r-m-\sigma}\right)^{2\delta}\right]\right\},$$

$$D=\left[\left(\frac{r-m+\sigma}{r-m-\sigma}\right)^{\delta}[2u(m\theta-lr)+q\Sigma]\right]^2+\left[u(\Sigma-2mr-2l\theta)\left(\frac{r-m+\sigma}{r-m-\sigma}\right)\right]^2-s^2(\Sigma-2mr-2l\theta)\Sigma,$$

$$W=[2d(m\theta-lr)-e\Sigma]^2+d^2(\Sigma-2mr-2l\theta)^2-j^2\left(\frac{r-m+\sigma}{r-m-\sigma}\right)^{2\delta}\Sigma(\Sigma-2mr-2l\theta)\,. \qquad (11)$$

This solution contains important special cases. For instance, the Belinsky-Ruffini solution [2] can be obtained by setting

$$C=\begin{pmatrix} q & 0 & -s \\ 0 & 1 & 0 \\ -s & 0 & q \end{pmatrix}, \qquad \delta=\tfrac{1}{2}\,.$$

The Kerr–NUT solution is derived from (11) by the substitution $C=\mathrm{diag}(1,1,1)$, $\delta=0$, or the Kerr solution with $l=0$. The Kramer solutions [3] can be get by $C=\mathrm{diag}(1,1,1)$ and $l=0$.

The solution (11) is a stationary axisymmetric solution for the five potentials ψ^A in five-dimensional gravity. The parameters (9) must be chosen to get a physical solution of (2). The invariance transformations (5) can be used as a method to obtain new solutions in five-dimensional gravity.

Acknowledgement

I wish to thank Drs. G. Neugebauer and D. Kramer for stimulating discussions.

References

[1] Th. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Phys. Math. Kl. (1921) 966;
O. Klein, Z. Phys. 37 (1926) 895.
[2] V. Belinsky and R. Ruffini, Phys. Lett. B 89 (1980) 195.
[3] D. Kramer, Acta Phys. Pol. B 2 (1971) 807.
[4] A. Kühnel and E. Schmutzer, Ann. Phys. 7 (1961) 243.
[5] D. Kramer, H. Stephani, M. MacCallum and E. Herlt, Exact solutions of Einstein's field equations (Berlin, 1980).
[6] F.J. Ernst, Phys. Rev. 167 (1968) 1175.
[7] G. Neugebauer, Habilitationsschrift, Jena (1969).
[8] J. Kinnersley, J. Math. Phys. 14 (1973) 651.
[9] G. Neugebauer, Wiss. Z. Friedrich-Schiller-Univ. Jena Math. Naturwiss. Reihe 21 (1972).
[10] P. Dobiach and D. Maison, Gen. Rel. Grav. 14 (1982) 231.
[11] G. Lessner, Preprint.
[12] G. Clement, Gen. Rel. Grav. 18 (1986) 137.
[13] T. Matos, Astron. Nachr. 307 (1986) 5, 317.
[14] T. Matos, Gen. Rel. Grav. 19 (1987) 5.
[15] T. Matos, Ann. Phys. (Leipzig), to be published.
[16] G. Neugebauer and D. Kramer, in: Galaxies, axisymmetric systems and relativity, ed. M.A.H. MacCallum (Cambridge Univ. Press, Cambridge, 1985).