Class. Quantum Grav. **17** (2000) 3917–3933. Printed in the UK PII: S0264-9381(00)50116-6

# Class of Einstein–Maxwell dilatons, an ansatz for new families of rotating solutions

Tonatiuh Matos†, Darío Núñez‡ and Maribel Rios§

† Departamento de Física, Centro de Investigación y de Estudios Avanzados del IPN, AP 14-700, 07000 México, DF, Mexico

‡ Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México AP 70-543, 04510 México, DF, Mexico

§ Instituto de Física y Matemáticas Universidad Michoacana de San Nicolás de Hidalgo Apdo Postal 2-82, 58040 Morelia, Michoacán, Mexico

Received 13 January 2000

**Abstract.** The functional potential formalism is used to analyse stationary axisymmetric spaces in Einstein–Maxwell–dilaton theory. Performing a Legendre transformation, a Hamiltonian is obtained, which allows us to rewrite the dynamical equations in terms of three complex functions only. Using an ansatz resembling the one used by the harmonic maps ansatz, we express these three functions in terms of the harmonic parameters, studying the cases where these parameters are real, and when they are complex. For each case, the set of equations in terms of these harmonic parameters is derived, and several classes of solutions to the Einstein–Maxwell equations with arbitrary coupling constant to a dilaton field are presented. Most of the known solutions of charged and dilatonic black holes are contained as special cases and can be non-trivially generalized in different ways.

PACS numbers: 0420, 0420F

## 1. Introduction

Scalar fields as a fundamental interaction in physics are one of the main predictions of Kaluza–Klein and superstring theories. Scalar fields are also a fundamental component of the Brans–Dicke theory and of inflationary models. Furthermore, in the standard model of Weinberg, Glashow and Salam, scalar fields are needed as a primordial component for giving mass to the particles. More recently, using an exact solution of the scalar–tensor field equations of gravity, it has been shown that scalar fields are a very good candidate to be the dark matter in spiral galaxies [1, 2]. However, it is fair to say that there are still some open questions in this issue which should be solved in order to give more solid evidence of their presence, and thus establish proof for the existence of scalar fields in nature.

The possibility of the existence of scalar fields together with the spontaneous scalarization in compact stars [3] implies that astrophysical objects could contain inherently scalar fields. In other words, if scalar fields exist the way that was established in [1, 2] a compact star will prefer to have one in order to save energy. Even when these fundamental scalar fields have not been observed, they are one of the main ingredients of modern physics. Of course the question arises, why if they are so important in physics have we never seen one? The answer to this question could be that they interact very weakly with matter. It can be shown that many of the theories containing scalar fields are in concordance with measurements in weak gravitational fields [4, 5]. We expect that scalar fields are important in strong gravitational fields such as at the origin of the universe or in pulsars or black holes. A great effort has been made in order to detect scalar fields in strong gravitational fields [6]. In some sense, inflation at the origin of the universe could be proof of such an interaction. Nevertheless, this effort has been made using perturbative methods [4] or using static exact solutions [5, 7, 8]. The problem with these two approaches is that for the first, perturbative methods are very efficient only in weak fields and for the second, astrophysical objects are in general non-static, thus the approach is not realistic. Even for binary pulsars, perturbative methods have shown great success because the distance between the pulsars is such that the gravitational field is too strong to be understood with Newtonian mechanics but weak enough to be described by pertubative methods in general relativity [6]. Nevertheless, the gravitational interaction is too weak for deciding which theory containing scalar fields could be the right one. It has been possible to discard a series of theories which did not agree with measurements or to bound some parameters of some other theories using perturbative methods or static exact solutions [4, 5]. But the most interesting effects of scalar fields are expected to be very near to a black hole or a pulsar and are expected to be non-perturbative. If we want to understand scalar fields in a strong regime one method is to find rotating exact solutions of the theory containing scalar fields and compare them to observations. The problem then is that the field equations are very complicated for them to be solved in an exact manner. In past work [7] we gave a very powerful method for finding exact static solutions of the Einstein–Maxwell–dilaton field equations; using harmonic maps, we found classes of solutions with arbitrary electromagnetic fields and gravitational arbitrary multipole momentums. In the present work we want to: (i) give the details of the calculations made in [7]; (ii) complete the schema of that work, and (iii) present a way of deriving exact rotating dilatons with arbitrary coupling between the scalar field and the electromagnetic one. In order to do so, let us start from the Lagrangian

$$\mathcal{L} = \sqrt{-g}\,\left(R - 2\,(\nabla\phi)^2 - \mathrm{e}^{-2\alpha\phi}F^2\right), \tag{1}$$

where $g$ is the determinant of the metric tensor, $R$ is the scalar curvature, $\phi$ the dilaton field and $F$ the Maxwell one. The constant $\alpha$ is a free parameter which governs the strength of the coupling of the dilaton to the Maxwell field. When $\alpha = 0$, the action reduces to the Einstein–Maxwell scalar theory. When $\alpha = 1$, the action is part of the low-energy action of string theory. For $\alpha = \sqrt{3}$, the Lagrangian (1) leads to the Kaluza–Klein field equations obtained from the dimensional reduction of the five-dimensional Einstein vacuum equations. However, we will consider this theory for all values of $\alpha$.

On the other hand, the harmonic maps ansatz has proved to be an excellent tool for finding exact solutions of systems of nonlinear partial differential equations [9]; in particular, this method has been very useful in solving the chiral equations derived from a nonlinear $\sigma$ model [10]. Einstein equations in vacuum can be reduced to a nonlinear $\sigma$ model with structural group $SL(2, R)$ in the spacetime and to a structural group $SU(1, 1)$ in the potential spaces, i.e., in terms of the Ernst potentials. The electro-vacuum case can also be reduced to a nonlinear $\sigma$ model with structural group $SU(2, 1)$ in terms of the extended Ernst potentials [11, 12]. The Kaluza–Klein field equations can be cast into a $SL(3, R)$ nonlinear $\sigma$ model in the spacetime as well as in the potential space [9, 13]. This is possible because the corresponding potential space, defined below, is a symmetric Riemannian space only for $\alpha = 0$ and $\alpha = \sqrt{3}$, but this is not the case for the low-energy limit in superstring theory, where $\alpha = 1$. In this work we will extend the techniques of the harmonic maps ansatz [9, 14, 15], even for non-symmetric Riemannian spaces, maintaining $\alpha$ as an arbitrary constant. In the present work, the reduction of the field equations to a nonlinear $\sigma$ model is not needed. Throughout this work, we will be using the MTW [16] sign conventions.

We will analyse spacetimes characterized by two Killing vector fields $X$ and $Y$ and introduce coordinates $t$ and $\varphi$ which are chosen such that $X = \frac{\partial}{\partial t}$ and $Y = \frac{\partial}{\partial \varphi}$. The corresponding line element can then be expressed as

$$\mathrm{d}s^2 = -f\,(\mathrm{d}t - \omega \mathrm{d}\varphi)^2 + f^{-1}\left[\mathrm{e}^{2k}\left(\mathrm{d}\rho^2 + \mathrm{d}z^2\right) + \rho^2\mathrm{d}\varphi^2\right], \tag{2}$$

where $f,\ \omega$, and $k$ are functions of $\rho$ and $z$ only. The electromagnetic potential has the form $A_\mu = (A_0, 0, 0, A_3)$, and again $A_0, A_3$, and the dilatonic field, $\phi$, are functions of $\rho$ and $z$ only.

The work is composed as follows. In section 2 we introduce the abstract potential space, obtaining a Lagrangian whose variation with respect to the potentials reproduces the field equations. Performing a Legendre transformation, we obtain a Hamiltonian and finally introduce new functions which greatly simplify the system of equations. In section 3, by means of an ansatz resembling one of the harmonic maps, we express that system of equations in terms of harmonic parameters, analysing two cases: where there are two real harmonic parameters and where there is one complex parameter. In section 4 we present several new classes of solutions to the Einstein–Maxwell–dilaton theory. Finally, in section 5 we present our conclusions and mention some possible future developments.

## 2. Functional space formulation

Trying to solve directly the field equations for the Einstein–Maxwell–dilaton theory for the line element (2) can be a very difficult task. Instead, we will apply the functional geodesic formulation by defining an abstract space whose coordinates are defined by the metric functions and the fields entering the system. In order to introduce an ansatz resembling the harmonic map ansatz into the functional geodesic formulation (see [12] and [17] for an explanation), we will explain the general idea of the harmonic map ansatz method. The field equations of the theory can be written as

$$\left(\rho\Psi^A{}_{,\varsigma}\right)_{,\bar\varsigma} + \left(\rho\Psi^A{}_{,\bar\varsigma}\right)_{,\varsigma} + 2\rho\left\{{}^{\,A}_{B\,C}\right\}\Psi^B_{,\varsigma}\Psi^C_{,\bar\varsigma} = 0 \tag{3}$$

where $\varsigma = \rho + \mathrm{i}z$ and $\bar\varsigma$ is its complex conjugate. $\Psi^A$ are the potentials of the geodesic formulation and $\{{}^{\,A}_{B\,C}\}$ are the Christoffel symbols of the Riemannian space $V^A$ defining the potential space of the theory. Now we look for invariant transformations of equation (3), i.e., transformations of the form $\Psi^A = \Psi^A(\lambda^i)$ that leave the field equations (3) invariant, where $\lambda^i$ are potentials fulfilling the same field equations (3). The potentials $\lambda^i$ define the Riemannian space $V_p$. In terms of the potentials $\lambda^i$, the field equations (3) read

$$2\rho\left[\Psi^A{}_{,i\,j} + \left\{{}^{\,A}_{B\,C}\right\}\Psi^B_{,i}\Psi^C_{,j}\right]\lambda^i_{,\varsigma}\lambda^j_{,\bar\varsigma} + \Psi^A_{,i}\left[\left(\rho\lambda^i{}_{,\varsigma}\right)_{,\bar\varsigma} + \left(\rho\lambda^i{}_{,\bar\varsigma}\right)_{,\varsigma}\right] = 0, \tag{4}$$

where $,\,i = \partial/\partial\lambda^i$. In terms of the Christoffel symbols of the abstract Riemannian space $V_p$, (4) reads

$$\Psi^A{}_{,i;j} + \left\{{}^{\,A}_{B\,C}\right\}\Psi^B_{,i}\Psi^C_{,j} = 0 \tag{5}$$

where we have used the field equations for the $\lambda^i$ and the fact that the $\lambda^i$ are linearly independent.

In what follows we will introduce the functional geodesic formulation for Lagrangian (1). The method is fully explained in [17]. Essentially, the formulation takes advantage of the fact that the introduction of a line element in the Lagrangian for the Einstein–Hilbert action, equation (1), and performing the variation, are operations which commute for some cases, in particular for the axisymmetric stationary one. Thus, introducing the operator $D = (\partial_\rho, \partial_z)$, taking out a total divergence term and eliminating the terms with $Dk$ by means of a Legendre transformation, we find that the original Lagrangian, given by equation (1), can be rewritten as:

$$\mathcal{L} = \frac{\rho}{2f^2}\, Df^2 - \frac{f^2}{2\rho}\, D\omega^2 + \frac{2\rho}{\alpha^2\kappa^2}\, D\kappa^2 + \frac{2f\kappa^2}{\rho}\left[(\omega DA_0 + DA_3)^2 - \frac{\rho^2}{f^2}\, DA_0^2\right], \tag{6}$$

where $\kappa^2 = \mathrm{e}^{-2\alpha\phi}$.

The Euler–Lagrange equations, obtained directly from extremizing the action for a Lagrangian such as: $D(\frac{\partial\mathcal{L}}{\partial DZ^a}) - (\frac{\partial\mathcal{L}}{\partial Z^a}) = 0$, with $Z^a = (f, \omega, A_0, A_3, \kappa)$, are:
the Klein–Gordon equation:

$$D^2\kappa + \left(\frac{D\rho}{\rho} - \frac{D\kappa}{\kappa}\right) D\kappa - \frac{\alpha^2\kappa^3 f}{\rho^2}\left[(\omega DA_0 + DA_3)^2 - \frac{\rho^2}{f^2}\, DA_0^2\right] = 0, \tag{7}$$

the Maxwell equations:

$$\begin{aligned} &D\left(\frac{f\kappa^2}{\rho}\,(\omega DA_0 + DA_3)\right) = 0,\\ &D\left(\kappa^2\left[\frac{f\omega}{\rho}(\omega DA_0 + DA_3) - \frac{\rho}{f}\, DA_0\right]\right) = 0, \end{aligned} \tag{8}$$

and the main Einstein equations:

$$\begin{aligned} &D^2 f + \left(\frac{D\rho}{\rho} - \frac{Df}{f}\right) Df + \frac{f^3}{\rho^2}\, D\omega^2\\ &\qquad - \frac{2\kappa^2 f^2}{\rho^2}\left[(\omega DA_0 + DA_3)^2 + \frac{\rho^2}{f^2}\, DA_0^2\right] = 0,\\ &D^2\omega - \left(\frac{D\rho}{\rho} - \frac{2Df}{f}\right) D\omega + \frac{4\kappa^2}{f}\,(\omega DA_0 + DA_3)\, DA_0 = 0, \end{aligned} \tag{9}$$

which are equivalent to the usual field equations. Notice that the metric function $k$ does not appear, reflecting the fact that it is determined by quadratures in terms of the rest of the functions. The Lagrangian (6) is sometimes viewed as describing the line element in the potential space, and thus the motion equations can be seen as geodesics in that space.

From the fact that $D\tilde{D} = 0$ for any analytic function, with $\tilde{D} = (-\partial_z, \partial_\rho)$, we conclude from the second Maxwell equation, equation (8), that there exists a potential $\chi$, such that

$$\tilde{D}\chi = \frac{2f\kappa^2}{\rho}\,(\omega DA_0 + DA_3). \tag{10}$$

And with this potential, the second Einstein equation, (9), can be rewritten as $D\left(\frac{f^2}{\rho}\, D\omega + \psi\tilde{D}\chi\right) = 0$, with $\psi = 2A_0$, so that there exists another potential, $\epsilon$, defined by

$$\tilde{D}\epsilon = \frac{f^2}{\rho}\, D\omega + \psi\tilde{D}\chi. \tag{11}$$

The use of the potentials $\chi$ and $\epsilon$, will be helpful in the procedure of defining harmonic functions, so we rewrite the field equations in terms of these functions:

$$
\begin{aligned}
&D^2\kappa + \left(\frac{D\rho}{\rho} - \frac{D\kappa}{\kappa}\right) D\kappa + \frac{\alpha^2\kappa^3}{4f}\left(D\psi^2 - \frac{1}{\kappa^4}\,D\chi^2\right) = 0,\\
&D^2\psi + \left(\frac{D\rho}{\rho} + \frac{2D\kappa}{\kappa} - \frac{Df}{f}\right) D\psi - \frac{1}{\kappa^2 f}\,(D\epsilon - \psi D\chi)\,D\chi = 0,\\
&D^2\chi + \left(\frac{D\rho}{\rho} - \frac{2D\kappa}{\kappa} - \frac{Df}{f}\right) D\chi + \frac{\kappa^2}{f}\,(D\epsilon - \psi D\chi)\,D\psi = 0, \qquad (12)\\
&D^2 f + \left(\frac{D\rho}{\rho} - \frac{Df}{f}\right) Df + \frac{1}{f}(D\epsilon - \psi D\chi)^2 - \frac{\kappa^2}{2}\left(D\psi^2 + \frac{1}{\kappa^4}\,D\chi^2\right) = 0,\\
&D^2\epsilon - D\psi D\chi - \psi D^2\chi + \left(\frac{D\rho}{\rho} - \frac{2Df}{f}\right)(D\epsilon - \psi D\chi) = 0,
\end{aligned}
$$

where we have used the fact that $\tilde{D}A\tilde{D}B = DADB$, for any functions $A$, $B$. The equation for $\chi$ is obtained from $D\tilde{D}A_3 = 0$, and the one for $\epsilon$ is obtained from $D\,\tilde{D}\omega = 0$. Equations (12) are equivalent to equations (3). This new set of field equations can be obtained from the Lagrangian [11, 14]

$$
\mathcal{L} = \frac{\rho}{2f^2}\left[Df^2 + (D\epsilon - \psi D\chi)^2\right] + \frac{2\rho}{\alpha^2\kappa^2}\,D\kappa^2 - \frac{\rho\kappa^2}{2f}\left(D\psi^2 + \frac{1}{\kappa^4}\,D\chi^2\right). \qquad (13)
$$

Notice, however, that this new Lagrangian is not obtained if the transformation is made directly on the original Lagrangian, given by equations (1) or (6). This fact implies that the transformations defined by equations (10) and (11) must have a degeneracy. A detailed analysis of this issue is currently under way and will be published elsewhere.

As mentioned above, this Lagrangian, equation (13), can be given by the line element, $DS^2$, of a potential space: $\mathcal{L} = DS^2 = G_{AB}D\Psi^A D\Psi^B$, with $\Psi^A = (f, \epsilon, \chi, \psi, \kappa)$, so that the equations of motion, equations (12), obtained from variations of this Lagrangian with respect to the 'coordinates', $\Psi^A$, can be thought as 'geodesics in such potential space'.

It is important to recall some of the geometric properties of this potential space, $DS^2$. It defines a Riemannian space with constant scalar curvature, $R = -(12 + \alpha^2)$. All the covariant derivatives of the Riemann tensor are proportional to $\alpha^2 - 3$, thus the corresponding Riemannian space is symmetric only for the case $\alpha^2 = 3$ (and for the case $\alpha = 0$, not treated here). Therefore, for an arbitrary $\alpha$, the space is not symmetric and thus the harmonic map ansatz formulation cannot be applied.

In this way, we see that in order to reformulate the Einstein–Maxwell–dilaton field equations for arbitrary $\alpha$, including the low-energy limit of string theory, we cannot use the harmonic map formulation, but we can try instead to mimic that formulation by means of an appropriately chosen ansatz. In this quest, it is necessary to obtain the algebraic structure associated with the potential space. An analysis of the Killing vectors, $\xi^a$, implies that there are eight of them for the space described by equation (13):

$$\xi_1^a = a_1\,(0, 1, 0, 0, 0), \qquad (14a)$$

$$\xi_2^a = a_2\,(0, 0, 1, 0, 0), \qquad (14b)$$

$$\xi_3^a = a_3\left(f, \epsilon, \frac{\chi}{2}, \frac{\psi}{2}, 0\right), \qquad (14c)$$

$$\xi_4^a = a_4\,(0, \chi, 0, 1, 0), \qquad (14d)$$

$$\xi_5^a = a_5\,(0, 0, \chi, -\psi, \kappa), \qquad (14e)$$

$$\xi_6^a = a_6\left[(2\epsilon - \chi\psi)f, \epsilon^2 - f^2 + f\psi^2\kappa^2,\right.$$
$$\left. f\psi\kappa^2 + \epsilon\chi, \psi\epsilon - \psi^2\chi - \frac{f\chi}{\kappa^2}, \frac{\alpha^2\psi\chi\kappa}{2}\right], \qquad (14f)$$

$$\xi_7^a = a_7\left[f\chi, f\psi\kappa^2 + \epsilon\chi,\right.$$
$$\left. f\kappa^2 + \left(1+\alpha^2\right)\frac{\chi^2}{4}, \epsilon + \left(1-\alpha^2\right)\frac{\psi\chi}{2}, \alpha^2\frac{\chi\kappa}{2}\right], \qquad (14g)$$

$$\xi_8^a = a_8\left[f\psi, \left(3-\alpha^2\right)\frac{\chi\psi^2}{4}, \left(3-\alpha^2\right)\frac{\chi\psi}{2} - \epsilon,\right.$$
$$\left. \frac{f}{\kappa^2} + \left(1+\alpha^2\right)\frac{\psi^2}{4}, -\alpha^2\frac{\psi\kappa}{2}\right], \qquad (14h)$$

where $a_1, \ldots, a_8$, are arbitrary constants. For $\alpha^2 \neq 3$, only the first five Killing vectors remain independent. Thus, the algebra associated with the potential space for arbitrary values of $\alpha$ is $SL(3, R)$ for $\alpha^2 = 3$, and a subalgebra of $SL(3, R)$, actually the upper triangle in the matrix representation, for $\alpha^2 \neq 3$.

The commutation relations, $[\xi_i^a, \xi_j^b] = \xi^a_{i\ ;b}\xi^b{}_j - \xi^a_{j\ ;b}\xi^b{}_j$ for the first five Killing vectors are:

$$[\xi_1, \xi_2] = 0, \quad [\xi_1, \xi_3] = -\xi_1, \quad [\xi_1, \xi_4] = 0, \quad [\xi_1, \xi_5] = 0,$$
$$[\xi_2, \xi_3] = -\tfrac{1}{2}\xi_2, \quad [\xi_2, \xi_4] = -\xi_1, \quad [\xi_2, \xi_5] = -\xi_2, \qquad (15)$$
$$[\xi_3, \xi_4] = -\tfrac{1}{2}\xi_4, \quad [\xi_3, \xi_5] = 0, \quad [\xi_4, \xi_5] = \xi_4.$$

Notice that we have several subalgebras with three vectors, but all of them have at least one of the commutators equal to zero. These results will be used in studying the type of target space that can be chosen in doing a map to a bi-dimensional space.

Finally, as we have mentioned, the remaining metric function $k$ is determined by quadratures in terms of the other field functions [7], explicitly:

$$k_\rho = \frac{\rho}{4f^2}\left[f_\rho{}^2 - f_z{}^2 + \epsilon_\rho{}^2 - \epsilon_z{}^2 + \left(\psi^2 + \frac{f}{\kappa^2}\right)\left(\chi_\rho{}^2 - \chi_z{}^2\right) - 2\psi\left(\epsilon_\rho\chi_\rho - \epsilon_z\chi_z\right)\right.$$
$$\left. + f\kappa^2\left(\psi_\rho{}^2 - \psi_z{}^2\right) + \left(\frac{2f}{\alpha\kappa}\right)^2\left(\kappa_\rho{}^2 - \kappa_z{}^2\right)\right], \qquad (16)$$

$$k_z = \frac{\rho}{2f^2}\left[f_\rho f_z + \epsilon_\rho\epsilon_z + \kappa^2 f\psi_\rho\psi_z - \psi(\epsilon_\rho\chi_z + \epsilon_z\chi_\rho)\right.$$
$$\left. + \left(\psi^2 + \frac{f}{\kappa^2}\right)\chi_\rho\chi_z + \left(\frac{2f}{\alpha\kappa}\right)^2\kappa_\rho\kappa_z\right]. \qquad (17)$$

Continuing with the quest for reformulating the Lagrangian, equation (13), notice that we can perform a Legendre transformation and define 'momenta' as $P_a = \frac{\partial\mathcal{L}}{\partial D\Psi^a}$, we can construct a new function along the lines of the standard Hamiltonian, although in our case it does not have the usual properties of evolution associated with the Hamiltonians. This Hamiltonian has the explicit form:

$$\mathcal{H} = \frac{f^2}{2\rho}\left(P_f{}^2 + P_\epsilon{}^2\right) + \frac{\alpha^2\kappa^2}{8\rho}P_\kappa{}^2 - \frac{f}{2\rho}\left[\frac{P_\psi{}^2}{\kappa^2} + \kappa^2\left(P_\chi + \psi P_\epsilon\right)^2\right]. \qquad (18)$$

The equations of motion are now $D\Psi^a = \frac{\partial\mathcal{H}}{\partial P_a}$ and $DP_a = -\frac{\partial\mathcal{H}}{\partial\Psi^a}$, that is:

$$Df = \frac{f^2}{\rho}\,P_f, \tag{19a}$$

$$D\epsilon = \frac{f^2}{\rho}\left[P_\epsilon - \frac{\kappa^2\psi}{f}\,(P_\chi + \psi P_\epsilon)\right], \tag{19b}$$

$$D\psi = -\frac{f}{\rho\kappa^2}\,P_\psi, \tag{19c}$$

$$D\chi = -\frac{f\kappa^2}{\rho}\,(P_\chi + \psi P_\epsilon), \tag{19d}$$

$$D\kappa = \frac{\alpha^2\kappa^2}{4\rho}\,P_\kappa, \tag{19e}$$

and

$$DP_f = -\frac{f}{\rho}\left({P_f}^2 + {P_\epsilon}^2\right) + \frac{1}{2\rho}\left[\frac{{P_\psi}^2}{\kappa^2} + \kappa^2\,(P_\chi + \psi\,P_\epsilon)^2\right], \tag{19f}$$

$$DP_\epsilon = 0, \tag{19g}$$

$$DP_\psi = \frac{f\kappa^2}{\rho}\,(P_\chi + \psi P_\epsilon)\,P_\epsilon, \tag{19h}$$

$$DP_\chi = 0, \tag{19i}$$

$$DP_\kappa = -\frac{f}{\rho\kappa}\left[\frac{{P_\psi}^2}{\kappa^2} - \kappa^2\,(P_\chi + \psi P_\epsilon)^2\right] - \frac{\alpha^2\kappa}{4\rho}\,{P_\kappa}^2. \tag{19j}$$

Now we can define new functions in order to simplify the form of this last set of equations, and to be able to introduce the harmonic maps:

$$A = \frac{f}{2\rho}\,(P_f - \mathrm{i}P_\epsilon),$$

$$B = \frac{\sqrt{f}}{2\rho}\left[\frac{P_\psi}{\kappa} - \mathrm{i}\kappa\,(P_\chi + \psi P_\epsilon)\right],$$

$$C = \frac{\alpha^2\kappa}{4\rho}\,P_\kappa,$$

or in terms of the velocities:

$$A = \frac{1}{2f}\,[Df - \mathrm{i}\,(D\epsilon - \psi D\chi)],$$

$$B = -\frac{1}{2\sqrt{f}}\left(\kappa D\psi - \frac{\mathrm{i}}{\kappa}\,D\chi\right),$$

$$C = \frac{D\kappa}{\kappa},$$

so we can rewrite the equations for the ‘momenta’ equations (19), as the following system:

$$\begin{aligned}
&\frac{1}{\rho}D(\rho A) = A\,(A - A^*) + BB^*,\\
&D(\rho B) = -\tfrac{1}{2}\,B(A - 3A^*) - CB^*,\\
&D(\rho C) = -\frac{\alpha^2}{2}\,(B^2 + B^{*2}),
\end{aligned} \tag{20}$$

where $^*$ denotes complex conjugate. Notice that in this way we have reduced the system of field equations to a set of three first-order differential equations for the three functions $A$, $B$, and $C$.

## 3. The ansatz

Now we introduce the harmonic maps ansatz. Let $V_p$ be a $p$-dimensional Riemannian space, and $\lambda^i$ be an harmonic parameter in $V_p$, i.e.

$$\left(\rho\lambda^i_{,\zeta}\right)_{,\bar{\zeta}} + \left(\rho\lambda^i_{,\bar{\zeta}}\right)_{,\zeta} + 2\rho\Gamma^i{}_{jk}\lambda^j_{,\zeta}\lambda^k_{,\bar{\zeta}} = 0, \tag{21}$$

where $\Gamma^i{}_{jk}$ are the Christoffel symbols on the Riemannian space $V_p$. We suppose that all the potentials $f$, $\epsilon$, ... depend on the $\lambda^i$, $f = f(\lambda^i)$, $\epsilon = \epsilon(\lambda^i)$, ..., etc (for details see [9,10,12] ). We recall that this work belongs to a programme aimed at obtaining exact solutions of the field equations derived from the Lagrangian (6) by means of the harmonic maps ansatz (see [9]) for the case of arbitrary $\alpha$. However, as we have already mentioned, some difficulties appear for arbitrary $\alpha$ and we cannot follow the procedures carried out before. Firstly, as was shown above, for $\alpha = 0, \sqrt{3}$, the isometry group has eight parameters, whereas for arbitrary $\alpha(\neq 0, \sqrt{3})$, there exist only five parameters [11]. Secondly, the isometry group of this Lagrangian is trivial except for $\alpha = 0$ and $\alpha = \sqrt{3}$, in the sense that the invariant transformations of this Lagrangian for arbitrary $\alpha$ lead only to gauge transformations. Thirdly, since for arbitrary $\alpha(\neq 0, \sqrt{3})$ the potential space is not symmetric, it is not possible to obtain a nonlinear $\sigma$ model for this system. All these problems prevent us from taking the same way as previous studies. Therefore, we propose another method for solving the differential equations. Here we will study the two-dimensional subspace of the $V_p$-space. We start with a two-dimensional Riemannian space $V_2$ with constant curvature, parameterizing this Riemannian space with two harmonic parameters, $\lambda$ and $\tau$, such that $\lambda, \tau \in \mathbb{R}$. The line element is

$$\mathrm{d}s^2 = \frac{2\left(\mathrm{d}\lambda^2 + \mathrm{d}\tau^2\right)}{\left(1 - \sigma\left(\lambda^2 + \tau^2\right)\right)^2} = \frac{\mathrm{d}\xi\mathrm{d}\xi^*}{(1 - \sigma\xi\xi^*)^2}, \tag{22}$$

where $\sigma$ is a real constant proportional to the potential space curvature, and $\xi = \lambda + \mathrm{i}\tau$, for the case of complex parameters. We know that this is a maximally symmetric space, so it has three Killing vectors. If the electromagnetic field vanishes then any value for $\alpha$ is similar because there is no interaction between scalar and electromagnetic fields. Then $\sigma$ can be set to one as for $\alpha^2 = 0, 3$. But if there is electromagnetic interaction then the situation is different. As we have shown, the subalgebras for the potential space with arbitrary $\alpha$, with three Killing vectors, are such that one of them has to be set to zero, so we conclude that if the electromagnetic field does not vanish the only case of maximally symmetric $V_2$ that can be taken is the one with $\sigma = 0$. In this case, the parameters satisfy the usual Laplace equation: $D(\rho D\lambda) = 0$, $D(\rho D\tau) = 0$. As in the harmonic maps ansatz case, let us express the functions $A, B, C$ in terms of these parameters as follows:

$$\begin{aligned} A &= a_1(\lambda, \tau)\, D\lambda + a_2(\lambda, \tau)\, D\tau, \\ B &= b_1(\lambda, \tau)\, D\lambda + b_2(\lambda, \tau)\, D\tau, \\ C &= c_1(\lambda, \tau)\, D\lambda + c_2(\lambda, \tau)\, D\tau. \end{aligned} \tag{23}$$

Using the harmonic equations, i.e. the Laplace equation, for these parameters in the field equations (20), and recalling the fact that $(D\lambda)^2$, $(D\tau)^2$, and $D\lambda D\tau$ are independent functions, from the system of equations for $A, B, C$, equation (20), we obtain the following set of equations:

$$a_{1,\lambda} - a_1\,(a_1 - a_1{}^*) - b_1 b_1{}^* = 0, \tag{24a}$$

$$b_{2,\lambda} + \frac{b_1}{2}\,(a_1 - 3\,a_1{}^*) + c_1 b_1{}^* = 0, \tag{24b}$$

$$c_{1,\lambda} + \frac{\alpha^2}{2}\left(b_1{}^2 + b_1{}^{*2}\right) = 0, \tag{24c}$$

$$a_{2,\tau} - a_2\,(a_2 - a_2{}^*) - b_2 b_2{}^* = 0, \tag{25a}$$

$$b_{2,\tau} + \frac{b_2}{2}\,(a_2 - 3\,a_2{}^*) + c_2 b_2{}^* = 0, \tag{25b}$$

$$c_{2,\tau} + \frac{\alpha^2}{2}\,(b_2{}^2 + b_2{}^{*2}) = 0, \tag{25c}$$

$$a_{1,\tau} + a_{2,\lambda} - 2a_1 a_2 + a_1{}^* a_2 + a_1 a_2{}^* - b_1 b_2{}^* - b_1{}^* b_2 = 0, \tag{26a}$$

$$2b_{1,\tau} + 2b_{2,\lambda} + b_2\,(a_1 - 3a_1{}^*) + b_1\,(a_2 - 3a_2{}^*) + 2c_1 b_2{}^* + 2c_2 b_1{}^* = 0, \tag{26b}$$

$$c_{1,\tau} + c_{2,\lambda} + \alpha^2\,(b_1 b_2 + b_1{}^*\,b_2{}^*) = 0. \tag{26c}$$

Equations (24)–(26) are equivalent to the field equations (4). Taking the original potential as functions of the harmonic parameters, we can express the functions $A$, $B$, and $C$ from equation (23) as:

$$\begin{aligned} A &= \frac{1}{2f}\,[f_{,\lambda} - \mathrm{i}\,(\epsilon_{,\lambda} - \psi\,\chi_{\lambda})]\,D\lambda + \frac{1}{2f}\,[f_{,\tau} - \mathrm{i}\,(\epsilon_{,\tau} - \psi\,\chi_{,\tau})]\,D\tau, \\ B &= -\frac{1}{2\sqrt{f}}\left(\kappa\psi_{,\lambda} - \frac{\mathrm{i}}{\kappa}\chi_{,\lambda}\right)D\lambda - \frac{1}{2\sqrt{f}}\left(\kappa\psi_{,\tau} - \frac{\mathrm{i}}{\kappa}\,\chi_{,\tau}\right)D\tau, \\ C &= \frac{\kappa_{,\lambda}}{\kappa}\,D\lambda + \frac{\kappa_{,\tau}}{\kappa}\,D\tau, \end{aligned} \tag{27}$$

from which, and from equation (23), we can make the following identification:

$$\begin{aligned} &a_1 = \frac{1}{2f}\,[f_{,\lambda} - \mathrm{i}\,(\epsilon_{,\lambda} - \psi\,\chi_{,\lambda})]; \qquad a_2 = \frac{1}{2\,f}\,[f_{,\tau} - \mathrm{i}\,(\epsilon_{,\tau} - \psi\,\chi_{,\tau})], \\ &b_1 = -\frac{1}{2\sqrt{f}}\left(\kappa\psi_{,\lambda} - \frac{\mathrm{i}}{\kappa}\,\chi_{,\lambda}\right); \qquad b_2 = -\frac{1}{2\sqrt{f}}\left(\kappa\psi_{,\tau} - \frac{\mathrm{i}}{\kappa}\,\chi_{,\tau}\right), \\ &c_1 = \frac{\kappa_{,\lambda}}{\kappa}; \qquad c_2 = \frac{\kappa_{,\tau}}{\kappa}. \end{aligned} \tag{28}$$

## 4. Classes of solutions

Now we proceed to present some solutions to the Einstein–Maxwell–dilaton system in terms of the harmonic functions $\lambda$, $\tau$. Taking $a_1, b_1, c_1$ as functions of $\lambda$ only, and $a_2, b_2, c_2$ as functions of $\tau$ only, we obtain two sets of equations, one in terms of $\lambda$, the other in terms of $\tau$, with the equations (26) being only constraint equations. We will see three such cases:

### *4.1. First class*

Letting all the functions be complex, we obtain that $a_2 = b_2 = c_2 = 0$, and

$$\begin{aligned} a_1 &= -\frac{1 + \mathrm{i}\sqrt{\gamma^2 - \frac{1}{4}}}{2\,(\lambda + \beta_0)}, \\ b_1 &= -\frac{\sqrt{\gamma}\,\left(\sqrt{\gamma - \frac{1}{2}} + \mathrm{i}\,\sqrt{\gamma + \frac{1}{2}}\right)}{\alpha\,(\lambda + \beta_0)}, \\ c_1 &= -\frac{\gamma}{\lambda + \beta_0}, \end{aligned} \tag{29}$$

where $\beta_0$ is an arbitrary constant and $\gamma = \frac{\alpha}{2}\sqrt{\frac{3}{4-\alpha^2}}$. This solution is valid for $1 < \alpha < 2$. Using equations (28), we find that the potentials are given by:

$$
\begin{aligned}
f &= \frac{f_0}{\lambda + \beta_0},\\
\epsilon &= \frac{\epsilon_0}{\lambda + \beta_0} + \epsilon_{10}\,(\lambda + \beta_0)^{-(\gamma + \frac{1}{2})},\\
\chi &= \frac{2\kappa_0}{\alpha}\sqrt{\frac{f_0\gamma}{\gamma + \frac{1}{2}}}\left[(\lambda + \beta_0)^{-(\gamma+\frac{1}{2})} - \beta_0^{-(\gamma+\frac{1}{2})}\right],\\
\psi &= \frac{2}{\kappa_0\alpha}\sqrt{\frac{f_0\gamma}{\gamma - \frac{1}{2}}}\left[(\lambda + \beta_0)^{\gamma-\frac{1}{2}} - \beta_0^{\gamma-\frac{1}{2}}\right],\\
\kappa &= \kappa_0\,(\lambda + \beta_0)^{-\gamma},
\end{aligned}
\tag{30}
$$

where $\epsilon_0 = f_0\sqrt{\frac{\gamma+\frac{1}{2}}{\gamma-\frac{1}{2}}}\left(\frac{4-\alpha^2}{\alpha^2}\gamma + \frac{1}{2}\right)$, and $\epsilon_0 = -\frac{4\,f_0\,\gamma\,\beta_0^{\gamma-\frac{1}{2}}}{\alpha^2\sqrt{\gamma^2-\frac{1}{4}}}$. From these potentials, using equations (10, 11), we find that

$$
\begin{aligned}
\omega_{,\rho} &= -\frac{\sqrt{\gamma^2 - \frac{1}{4}}}{f_0}\,\rho\lambda_{,z}, \qquad \omega_{,z} = \frac{\sqrt{\gamma^2 - \frac{1}{4}}}{f_0}\,\rho\lambda_{,\rho},\\
A_{3,\rho} &= -\frac{\sqrt{\gamma f_0}}{\kappa_0\alpha}\,(\lambda + \beta_0)^{\gamma-\frac{3}{2}}\left(\omega\sqrt{\gamma - \frac{1}{2}}\,\lambda_{,\rho} - \frac{\rho}{f_0}\sqrt{\gamma + \frac{1}{2}}\,(\lambda + \beta_0)\,\lambda_{,z}\right),\\
A_{3,z} &= -\frac{\sqrt{\gamma f_0}}{\kappa_0\alpha}\,(\lambda + \beta_0)^{\gamma-\frac{3}{2}}\left(\omega\sqrt{\gamma - \frac{1}{2}}\,\lambda_{,z} + \frac{\rho}{f_0}\sqrt{\gamma + \frac{1}{2}}\,(\lambda + \beta_0)\,\lambda_{,\rho}\right).
\end{aligned}
\tag{31}
$$

Finally, from equation (17), we find that the last metric coefficient, $k$, in this case is constant, $k = k_0$, and we take $k_0 = 0$. This solution represents a rotating object with scalar and electromagnetic fields which depend on the form of $\lambda$. In this way, we have obtained a family of solutions for the Einstein–Maxwell–dilaton theory in which all the fields are non-trivially involved. Let us give an example. Writing the metric (2) in spherical-like coordinates $\rho = r\sin\theta$, $z = r\cos\theta$ , we can choose $\lambda = M/r$, which gives us the line element of the spacetime as ($f_0 = \beta_0 = \frac{1}{2}$):

$$
\begin{aligned}
\mathrm{d}s^2 = &-\frac{1}{1 + \frac{2M}{r}}\left(\mathrm{d}t - 2\sqrt{\gamma^2 - \tfrac{1}{4}}M\cos\theta\mathrm{d}\phi\right)^2\\
&+\left(1 + \frac{2M}{r}\right)\left(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin\theta^2\mathrm{d}\phi^2\right),
\end{aligned}
\tag{32}
$$

the scalar field is

$$
\varphi = \ln\left(\beta_0 + \frac{M}{r}\right)^{\frac{\gamma}{\alpha}} - \varphi_0,
\tag{33}
$$

where $\varphi_0 = \ln\kappa_0^{\alpha}$, and the electric and magnetic potentials are

$$
A_0 = \sqrt{\frac{\gamma}{\gamma - \frac{1}{2}}}\frac{\left[\left(1 + \frac{2M}{r}\right)^{\gamma-\frac{1}{2}} - 1\right]}{2^{\gamma}\kappa_0\alpha},
\tag{34}
$$

$$
A_3 = \sqrt{\gamma\left(\gamma + \tfrac{1}{2}\right)}\frac{M}{2^{\gamma-1}\kappa_0\alpha}\left[\left(\beta_0 + \frac{2M}{r}\right)^{\gamma-\frac{1}{2}} - 1\right]\cos\theta.
\tag{35}
$$

Metric (32) contains a mass-like parameter, $M$, and is not asymptotically flat, except for $\gamma^2 = \frac{1}{4}$. This spacetime represents a multipole electromagnetic field coupled to a multiple gravitational one. The electric charge is $q = \sqrt{\gamma\left(\gamma - \frac{1}{2}\right)}M/(2^{\gamma-1}\kappa_0\alpha)$, and the magnetic monopole charge $Q = \sqrt{\gamma\left(\gamma - \frac{1}{2}\right)}M\left(\beta_0^{\gamma-\frac{1}{2}} - 1\right)/\left(2^{\gamma-1}\kappa_0\alpha\right)$. Notice, however, that the scalar field is tied to the gravitational and electromagnetic charges, in the sense that when $M = 0$ or $\gamma = 0$, the scalar field becomes trivial. In order to overcome this problem, we think that more potentials have to be included. Also, this solution has no horizons, and is singular for $r = 0$. Thus, we consider that this solution is a good one for describing the spacetime with gravitational, electromagnetic and dilatonic fields in a region between $r > 0$ and $r = r_0$ where it has to be matched to an exterior solution. A detailed analysis of the physical properties of this particular exact solution, as well as other solutions belonging to the family of solutions presented in this section, is under way and will be published soon.

### 4.2. Second class

We take the case where the ansatz is such that $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$, and that $a_1, b_1, c_1$ depend only on $\lambda$, and $a_2, b_2, c_2$ depend only on $\tau$. With these conditions the system of equations, equations (24), implies that $b_1 b_2 = 0$. Taking $b_1 = 0$, in turn implies that $a_1 = c_1 = \text{const}$, and the final set of equations reduces to the following:

$$a_{2,\tau} - b_2^2 = 0, \qquad b_{2,\tau} - b_2\,(a_2 - c_2) = 0, \qquad c_{2,\tau} + \alpha^2\, {b_2}^2 = 0.$$

Solving for $a_2$, we get a Riccati type equation:

$$a_{2,\tau\tau} - \left(1+\alpha^2\right)\left({a_2}^2\right)_{,\tau} + 2\,k_0\,a_{2,\tau} = 0, \tag{36}$$

with $k_0$ an integration constant. We obtain two sets of solutions for $a_2$:

$$a_2 = \frac{k_1 \mathrm{e}^{q_1\,\tau} - \mathrm{k}_2 \mathrm{e}^{(\mathrm{q}_1+2(1+\alpha^2))\,\tau}}{k_1 \mathrm{e}^{q_1\,\tau} + k_2 \mathrm{e}^{(q_1+2(1+\alpha^2))\,\tau}} + \frac{k_0}{(1+\alpha^2)}, \tag{37}$$

and

$$a_2 = \frac{1}{(1+\alpha^2)}\left(-\frac{k_3}{k_3\,\tau + k_4} + k_0\right), \tag{38}$$

which imply

$$\begin{aligned} b_2 &= \frac{2\sqrt{-k_1 k_2\,(1+\alpha^2)}\,\mathrm{e}^{2(q_1+1+\alpha^2)\,\tau}}{k_1 \mathrm{e}^{q_1\,\tau} + k_2 \mathrm{e}^{(q_1+2(1+\alpha^2))\,\tau}}, \\ c_2 &= \frac{k_0}{(1+\alpha^2)} - \alpha^2\,\frac{k_1 \mathrm{e}^{q_1\tau} - k_2 \mathrm{e}^{(q_1+2(1+\alpha^2))\,\tau}}{k_1 \mathrm{e}^{q_1\tau} + k_2 \mathrm{e}^{(q_1+2(1+\alpha^2))\,\tau}}, \end{aligned} \tag{39}$$

and

$$\begin{aligned} b_2 &= \frac{k_3}{(1+\alpha^2)^{\frac{1}{2}}\,(k_3\tau + k_4)}, \\ c_2 &= \frac{1}{(1+\alpha^2)}\left(k_0 + \frac{\alpha^2 k_3}{k_3\tau + k_4}\right), \end{aligned} \tag{40}$$

where $k_i, q_i$ are constants. Performing the integration in equations (28), the following expressions for the potentials are obtained:

$$\begin{aligned} f &= f_0 \frac{e^{\lambda_0\lambda+\tau_0\tau}}{(m_1\Sigma_1+m_2\Sigma_2)^\gamma}, \\ \kappa^2 &= {\kappa_0}^2\,(m_1\Sigma_1+m_2\Sigma_2)^\beta\, e^{\lambda_0\lambda+(\tau_0-t_1-t_2)\,\tau}, \\ \psi &= \frac{m_3\Sigma_1+m_4\Sigma_2}{(m_1\Sigma_1+m_2\Sigma_2)}, \\ \chi &= 0, \\ \epsilon &= 0, \end{aligned} \tag{41}$$

and

$$\begin{aligned} f &= \frac{f_0 e^{\gamma k_1\tau+\lambda_0\lambda}}{(k_3\tau+k_4)^\gamma}, \\ \kappa^2 &= {\kappa_0}^2\,(k_3\tau+k_4)^\beta e^{\gamma k_1\tau+\lambda_0\lambda} \\ \psi &= -2 f_0^{1/2}\kappa_0\,\left(1+\alpha^2\right)^{1/2}\,(k_3\tau+k_4), \\ \chi &= 0, \\ \epsilon &= 0, \end{aligned} \tag{42}$$

where $\gamma = \frac{2}{1+\alpha^2}$, $\beta = \alpha^2\gamma$, $f_0, \kappa_0, t_i, \lambda_0, \tau_0, m_i$ are constants. For the first set of solutions $\Sigma_i = e^{t_i\,\tau}$ and the constants are related by

$$4m_1m_2f_0 + \kappa_0^2\,\left(1+\alpha^2\right)\,(m_1m_4-m_2m_3)^2 = 0. \tag{43}$$

And for the second set of solutions $t_1 = -t_2$, $\Sigma_1 = \tau$, $\Sigma_2 = 1$, and the constants satisfy the relationship

$$4\,m_1^2\,f_0 - \kappa_0^2\,\left(1+\alpha^2\right)\,(m_1\,m_4-m_2m_3)^2 = 0. \tag{44}$$

In this way, with the ansatz chosen, we obtained families of solutions for the Einstein–Maxwell–dilaton theory without magnetic field and without rotation. The generic solutions are in terms of two arbitrary harmonic functions and a large number of constants. The respective magnetic solutions can be obtained using the invariant trasformations.

$$\phi \to -\phi, \qquad F_{\mu\nu} = 1/2 e^{-2\alpha\phi}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} \tag{45}$$

Thus, we can generate particular solutions with very different physical properties. These sets of solutions were presented as a rapid communication in [7], which included several interesting known solutions, such as the static dilatonic version of the Kastor–Traschen solution [18], and several generalizations of it; the spherically symmetric dilatonic black hole [19]; and solutions with arbitrary magnetic field coupled to the dilaton [20].

In order to present explicitly some families of solutions, let us now rewrite solutions (30) in a more convenient form. If we perform the following transformation, $f \to f/f_0, \kappa^2 \to \kappa^2/\kappa_0^2$, $\lambda \to \lambda_0\lambda + \tau\tau_0$, $\tau_0 \to -t_1 - t_2$, and $g = m_1\,e^{t_1\,\tau} + m_2\,e^{t_2\,\tau}$ in (41), then solution (41) transforms into

$$\begin{aligned} f &= \frac{e^\lambda}{g^\gamma}\,, \\ \kappa^2 &= \frac{e^{-\lambda+\tau_0\tau}}{g^\beta}\,, \\ \psi &= \frac{(m_3\Sigma_1+m_4\Sigma_2)}{g}, \\ \chi &= 0\,, \\ \epsilon &= 0, \end{aligned} \tag{46}$$

or, analogously for solution (42), we perform the transformation $f \to f/f_0$, $\kappa^2 \to \kappa^2/\kappa_0^2$, $\lambda \to \lambda_0\lambda + \gamma k_1\tau$, $\psi \to -\psi\, \kappa_0 \left(1+\alpha^2\right)^{1/2}/f_0^{1/2}$, and $g = k_3\tau + k_4$; then solution (42) now reads

$$f = \frac{e^{\lambda}}{g^{\gamma}}, \qquad \kappa^2 = e^{\lambda} g^{\beta}, \qquad \psi = \frac{1}{g}, \qquad \chi = 0, \qquad \epsilon = 0. \tag{47}$$

In these two cases the differential equation for the metric function $k$ in (17) can be separated into a gravitational, a scalar and an electromagnetic part. In order to do so, we substitute (46) and (47) into (17) to obtain the differential equation for $k$, thus arriving at

$$k_{,\zeta} = \frac{\rho}{2}\left[(\lambda_{,\zeta})^2 + \frac{1}{\alpha^2}\left((\lambda_{,\zeta} - \tau_0\tau_{,\zeta})^2 - 2q_1q_2\beta(\tau_{,\zeta})^2\right)\right], \tag{48}$$

where $\zeta = \rho + i z$. Let us now perform the following separation

$$k = k_g + k_e + k_s, \tag{49}$$

where we have defined the gravitational part of $k$ as

$$k_{g,\zeta} = \frac{\rho}{2}(\lambda_{,\zeta})^2, \tag{50}$$

the scalar part as

$$k_{s,\zeta} = \frac{\rho}{2\alpha^2}(\lambda_{,\zeta} - \tau_0\tau_{,\zeta})^2 \tag{51}$$

and the electromagnetic part as

$$k_{e,\zeta} = \frac{\rho}{\alpha^2}q_1q_2\beta(\tau_{,\zeta})^2, \tag{52}$$

where now the constants $a_1, \ldots, q_1, q_2$, and $\kappa_0$, satisfy the relationship

$$4a_1^2 - \kappa_0^2\left(1+\alpha^2\right)(a_1a_4 - a_2a_3)^2 = 0. \tag{53}$$

It is important to note that the electrostatic potential $\psi$ is completely determined by the harmonic function $\tau$. This means that the electrostatic (magnetostatic) potential is determined only by $\tau$, so we can obtain solutions with arbitrary electromagnetic fields writing the corresponding solution of the Laplace equation for $\tau$. The corresponding magnetic solution can be obtained using the invariant transformations (45). The most important well-known solutions can be derived from this method. Some examples are given in [21]. New solutions have been derived in [22–24].

On the other hand, a star is basically a gravitational monopole together with a magnetic dipole field. Using the invariant transformations (45), we can construct a class of solution with these characteristics. For the gravitational potential we now choose a gravitational monopole, in order to reproduce the most important gravitational features of the star and of the Schwarzschild solution. In order to do so we write the line element (2) in Boyer–Lindquist coordinates $\rho = \sqrt{r^2 - 2mr}\sin\theta$, $z = (r-m)\cos\theta$ and take $\lambda = \ln(1 - \frac{2m}{r})$, the differential equation for $k_g$ can be integrated and the spacetime metric reads

$$ds^2 = e^{2(k_s+k_e)}g^{\gamma}\frac{dr^2}{1-2m/r} + g^{\gamma}r^2\left(e^{2(k_s+k_e)}d\theta^2 + \sin^2\theta d\varphi^2\right) - \frac{(1-2m/r)}{g^{\gamma}}dt^2. \tag{54}$$

The scalar and the electromagnetic fields for this solution read

$$\kappa^2 = e^{-2\alpha\phi} = \frac{e^{-2\alpha\phi_0}}{(1-2m/r)g^{\beta}} \qquad A_{3,\rho} = -Q\rho\tau_{,z}., \quad A_{3,z} = Q\rho\tau_{,\rho}. \tag{55}$$

Metric (54), (55) is a static asymptotically flat metric with mass parameter $m$, and magnetostatic charge $Q$. The scalar parameter depends on the form of $\tau$. Observe that the electromagnetic field is always integrable because $\tau$ fulfills the Laplace equation $D(\rho D\tau) = (\rho\tau_{,z})_{,z} + (\rho\tau_{,\rho})_{,\rho} = 0$. So we can construct solutions representing magnetic monopoles, dipoles, etc. With this form of the metric we can interpret $g$ and $e^{2k_e}$ as the contribution of the electromagnetic field and $e^{2k_s}$ as the contribution of the scalar field to the metric. Asymptotically, the scalar field behaves like $e^{-2\alpha\phi_0}/g^{\beta}(1 + 2m/r + \ldots)$, where $g \sim 1 + O(1/r^n)$. This means that the scalar field deviates from a constant in orders of $2m/r$. For a star like the Sun $2m/r \sim 10^{-6}$, for a white dwarf $2m/r \sim 10^{-4}$ and only for a compact star like a neutron star $2m/r \sim 10^{-1}$, i.e., this metric represents the spacetime of an object very similar to a magnetized Schwarzschild one. Only for a very compact object are both metrics different. This fact is in agreement with the concept of spontaneous scalarization: if scalar fields exist, compact stars will prefer to possess one in order to save energy, but even when a star possesses one, it will be very difficult to detect it (see [8]). We can carry out the following classification of solution (54), (55).

(a) Suppose that $\tau = 0$, this implies that $g = 1$ and $A_3 = 0$, so that the metric (54) is

$$ds^2 = e^{2k_s}\frac{dr^2}{1 - 2m/r} + r^2\left(e^{2k_s}d\theta^2 + \sin^2\theta d\varphi^2\right) - (1 - 2m/r)dt^2. \quad (56)$$

This metric is almost spherically symmetric and represents a gravitational body (gravitational monopole) with a scalar field. The scalar field deforms the spherical symmetry in the factor $e^{2k_s}d\theta^2$. If $k_s = 0$ we recover the spherical symmetry. This metric has been used as a model for a star [5], finding that the physical differences between (56) and the Schwarzschild solution are too small to be measured, even for a compact star like a pulsar. This implies that a star like the Sun could possess a scalar field and we will not be able to measure it.

(b) We now take $\tau \neq 0$, $g = 1$ , but with arbitrary magnetic field $A_3$. The metric (54) then reads

$$ds^2 = e^{2(k_s+k_e)}\frac{dr^2}{1 - 2m/r} + r^2\left(e^{2(k_s+k_e)}d\theta^2 + \sin^2\theta d\varphi^2\right) - (1 - 2m/r)dt^2. \quad (57)$$

In this case the scalar and the magnetostatic potentials are the ones that deform the spherical symmetry of the metric. Furthermore, if we make $k_s + k_e = 0$ and $\tau = \lambda$, $(p + q - 1)^2 - 2pq\beta = 0$, we recover the spherical symmetry and the metric transforms into the Schwarzschild line element. This metric represents a gravitational body with an arbitrary magnetostatic field coupled to a scalar field. The scalar and mass parameters here are proportional.

(c) If we choose $\tau = \lambda$, metric (54) is spherically symmetric, the constants fulfil the constraint $(p + q - 1)^2 - pq\beta = 0$, and the metric (54) reads

$$ds^2 = g^{\gamma}\frac{dr^2}{1 - 2m/r} + g^{\gamma}r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) - \frac{(1 - 2m/r)}{g^{\gamma}}dt^2. \quad (58)$$

In here is contained the Gibbons–Maeda black hole [19] by choosing $\tau$ to be the harmonic function corresponding to a monopole. The generalized version of solution [19] for any electromagnetic multipole field is given by this metric.

(d) Finally, we choose $\tau = \lambda$ and $p + q = 1$, which implies $k_s = 0$. The metric (54) is now given by

$$ds^2 = e^{2k_e}g^{\gamma}\frac{dr^2}{1 - 2m/r} + g^{\gamma}r^2\left(e^{2k_e}d\theta^2 + \sin^2\theta d\varphi^2\right) - \frac{(1 - 2m/r)}{g^{\gamma}}dt^2. \quad (59)$$

This metric is again almost spherically symmetric, but now it is deformed by the electromagnetic field, the factor $e^{2k_e}d\theta^2$ is the deformation of the spherical symmetry due to the electromagnetic field.

### 4.3. Third class

Now let us study the case when the electromagnetic field vanishes. Here it is convenient to take the harmonic parameter complex. Choosing the simple ansatz $a_2 = b_1 = b_2 = c_1 = c_2 = 0$, $\sigma = 1$, using equation (22) with the complex parameter $\xi$, and repeating the procedure, we find that the system of equations (24–26) takes the form

$$a_{1,\xi} - a_1^2 - \frac{2\xi^* a_1}{1 - \xi\xi^*} = 0,$$
$$a_{1,\xi^*} + a_1 a_1^* = 0,$$

with solution

$$a_1 = \frac{1 + \xi^*}{(1 + \xi)(\xi\xi^* - 1)}. \tag{60}$$

Performing the integration to obtain the potential functions, one finally gets

$$\begin{aligned} f &= \frac{\xi\xi^* - 1}{(1 + \xi)(1 + \xi^*)}, \\ \epsilon &= \mathrm{i}\,\frac{\xi - \xi^*}{(1 + \xi)(1 + \xi^*)}, \end{aligned} \tag{61}$$

which we identify with the Ernst solution; that is, we have obtained the Tomimatsu–Sato family of solutions, including of course Kerr as a particular case [12].

### 4.4. Fourth class

Finally, we obtain some stationary solutions using this method. The most simple ansatz to solve equations (24–26) is supposing that all the functions $a_1, a_2, \ldots$ etc are constants; the differential equations then become algebraic equations which can be easily solved. For this case we take the ansatz

$$\begin{aligned} a_1 &= \mathrm{i}p, \qquad a_2 = 0, \\ b_1 &= p(1 - \mathrm{i}), \qquad b_2 = 0, \\ d_1 &= -2p, \qquad c_2 = 0, \end{aligned} \tag{62}$$

where $p$ is an arbitrary constant. Integrating the potentials for this case we obtain

$$\begin{aligned} f &= 1, \\ \kappa^2 &= {\kappa_0}^2\, e^{-4p\lambda}, \\ \psi &= -\frac{e^{2p\lambda}}{\kappa_0} + \psi_0, \\ \chi &= \kappa_0\, e^{-2p\lambda} + \chi_0, \\ \epsilon &= \kappa_0\, \psi_0\, e^{-2p\lambda}. \end{aligned} \tag{63}$$

Substituting in the metric (2) and in the equation for the function $k$, (17), the metric can be integrated to obtain

$$ds^2 = -(dt - \omega d\varphi)^2 + \left[e^{2k}\left(d\rho^2 + dz^2\right) + \rho^2 d\varphi^2\right], \tag{64}$$

where $\omega$ and the function $k$ can be integrated from the differential equations:

$$\omega_{,\rho} = -2p\rho\lambda_{,z} \qquad \omega_{,z} = 2p\rho\lambda_{,\rho}$$
$$k_{,\zeta} = \frac{p^2\left(3\alpha^2+4\right)}{\alpha^2}\rho(\lambda_{,\zeta})^2. \tag{65}$$

The integrability conditions for $\omega$ and $k$ are guaranteed because $\lambda$ fulfils the Laplace equation. Metric (65) represents a rotating degenerated object (the gravitational potential $f = 1$), with electromagnetic and scalar potentials, where the coupling constant $\alpha$ between scalar and electromagnetic fields remains arbitrary. The particular multipole development depends on the harmonic function $\lambda$. As an example, if $\lambda = \cos\theta/r$, the solution represents a pure magnetic monopole with a multipole electrostatic field, without a gravitational one. In this case $\omega = -2p\sin^2\theta/r$, and the magnetic field is proportional to it. This, therefore, represents a magnetic dipole, with an exponentially decaying scalar field.

## 5. Conclusions

We have presented a detailed description of the functional space formulation, joined with the harmonic maps one, in such a way that starting with the Einstein–Hilbert action for the stationary axisymmetric spacetime, we obtained an effective Lagrangian for the field variables, then by means of a Legendre transformation, obtained a Hamiltonian and, with a canonical transformation, reduced the system of dynamical equations to three first-order coupled differential equations for three unknowns, $A$, $B$, $C$, two complex and one real. The harmonic maps ansatz enabled us to reduce the system of five second-order partial differential equations (12) for the Einstein–Maxwell–dilaton system with two Killing vectors to a system of five (two complex and one real) first-order ordinary differential equations (20). Using the harmonic maps ansatz [9], we rewrote that system of equations in such a way that, for different ansatz, it allowed us to generate large classes of solutions to the Einstein–Maxwell theory non-minimally coupled to a dilatonic field. In this way, we consider that we have described a robust formulation which allows us to generate large classes of solutions, placing us on the right track, we think, to obtain exact solutions for astrophysically important cases, like the one describing a compact charged rotating object surrounded by a scalar field which could be useful for solving the dark matter problem at a galactic level. Actually, in [1, 26], we have worked out this idea and the scalar field is a good candidate for the dark matter in the galactic halo of spiral galaxies. We have presented several particular cases which include most of the well known solutions, as well as new ones, where we have explicitly presented the form of the fields and of the charges, for particular choices of the potentials. Also, the formulation described in the present work is not only a technique to generate exact solutions, but it also has several other possible directions worth studying; for instance, from the effective Lagrangian (6), an off-shell Lagrangian for the nonlinear $\sigma$ model for some families of solutions can be obtained [25]. It has not been possible to make this analogy for cases with different values of $\alpha$, due to the fact that for them the potential space is not symmetric. Finally we think that the Hamiltonian obtained in the present formulation is worth further study within the gravity quantization endeavour.

## 6. Acknowledgments

This work has been supported by DGAPA-UNAM, project IN121298, and by CONACYT, Mexico, project 3697-E. TM thanks the relativity group in Jena, Germany for their hospitality, and the DAAD for their support in the course of this work.

## References


[1] Matos T and Guzmán F S 2000 *Class. Quantum Grav.* **17** L9–16
(Matos T and Guzmán F S 1998 *Preprint* gr-qc/9810028)
Matos T and Guzmán F S 2000 *Ann. Phys., Lpz.* **9** S133–6
(Matos T and Guzmán F S 2000 *Preprint* astro-ph/0002126)
[2] Guzmán F S, Matos T and Villegas H 1998 *Preprint* astro-ph/9811143
[3] Damour T and Esposito-Farese G 1993 *Phys. Rev. Lett.* **70** 2220
[4] Damour T and Esposito-Farese G 1992 *Class. Quantum Grav.* **9** 2093
[5] Matos T, Obregon O and Villegas H 1998 *Mod. Phys. Lett.* A **13** 3161
Matos T and Villegas H 1999 to be published in *Astrophys. J.*
[6] Esposito-Farese G 1996 *Preprint* gr-qc/9612039
[7] Matos T, Nuñez D and Quevedo H 1995 *Phys. Rev.* D **51** R310
[8] Matos T 1998 *Gen. Rel. Grav.* **30** 5
[9] Matos T 1994 *J. Math. Phys.* **35** 1302
[10] Matos T 1995 Exact solutions of G-invariant chiral equations *Math. Notes* **58** 1178–82
[11] Neugebauer G 1969 *Habilitationsschrift* (Jena: Universität Jena Press)
[12] Kramer D, Stephani H, MacCallum M and Held E 1980 *Exact Solutions of Einstein Field Equations* (Berlin: DVW)
[13] Matos T 1986 *Astron. Nachr.* **307** 317
[14] Neugebauer G and Kramer D 1969 *Ann. Phys., Lpz.* **24** 62
[15] Matos T, Rodríguez G and Becerril R 1992 *J. Math. Phys.* **33** 3521
[16] Misner Ch W, Thorne K S and Wheeler J A 1973 *Gravitation* (San Francisco: Freeman) ch 21
[17] Núñez D, Quevedo H and Sánchez A 1998 *Rev. Mex. Fis.* **44** 1820
[18] Kastor D and Traschen J 1993 *Phys. Rev.* D **47** 5370
Horne J H and Horowitz G T 1993 *Phys. Rev.* D **48** R5457
[19] Gibbons G and Maeda K 1988 *Nucl. Phys.* B **298** 741
Garfinkle D, Horowitz G T and Strominger A 1991 *Phys. Rev.* D **43** 3140
[20] Matos T 1994 *Phys. Rev.* D **49** 4296
[21] Rios M 1995 *Thesis* CINVESTAV
[22] Matos T and Macias A 1994 *Mod. Phys. Lett.* A **9** 3707
[23] Macias A and Matos T 1996 *Class. Quantum Grav.* **13** 345
[24] Matos T and Becerril R 1992 *Rev. Mex. Fis.* **38** 69
[25] Cortez J, Nuñez D and Quevedo H *Int. J. Mod. Phys.* in press
[26] Matos T, Guzmán F S and Nuñez D 2000 *Phys Rev.* D **57** **62** 061301
(Matos T, Guzmán F S and Nuñez D 2000 *Preprint* astro-ph/0003398)
Matos T, Nuñez D, Guzman F and Ramirez E *Astrophys. J.* submitted