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Subspaces and Subgroups in Five-Dimensional Gravity

By T. Matos

Friedrich-Schiller-Universität Jena, Jena, GDR

Abstract. A systematic investigation of the one- and two-dimensional subspaces of the potential space in the projective field theory is made. From the one-dimensional subspaces some stationary charged solutions can be generated. The two-dimensional subspaces have an isometry group SU(1,1) which corresponds to the Einstein equations in vacuum, an O(2,1) group which corresponds to stationary charged solutions with constant scalar field, and an Abelian group which has no counterpart in the Einstein-Maxwell theory.

Unterräume und Untergruppen in der fünfdimensionalen Gravitationstheorie

Inhaltsübersicht. Es wird eine systematische Untersuchung der ein- und zweidimensionalen Unterräume des Potentials der projektiven Feldtheorie unternommen. Für den eindimensionalen Unterraum können einige stationäre geladene Lösungen erzeugt werden. Die zweidimensionalen Unterräume besitzen eine Isometriegruppe SU(1,1), die der Einstein-Gleichung im Vakuum entspricht, eine O(2,1)-Gruppe, die stationär geladenen Lösungen mit konstantem Skalarfeld entsprechen, und eine Abelsche Gruppe, die keine Entsprechung in der Einstein-Maxwell-Theorie findet.

1. Introduction

The five-dimensional projective field theory is a unified theory of gravitation and electromagnetism [1]. In this theory it is assumed that gravitation and electromagnetism define a five-dimensional Riemannian space admitting a non-zero Killing vector field X^μ with $I^2 = X^\mu X_\mu \neq 0$. We will discuss a variant in which the vacuum fields are characterized by zero five-dimensional Ricci-tensor, i.e.

$$\overset{5}{R}_{\mu\nu} = 0; \; \mu, \; \nu = 1, \ldots, 5, \tag{1}$$

and the spacetime metric is the projection tensor $g_{\mu\nu} = \gamma_{\mu\nu} - I^{-2}X_\mu X_\nu$. In the stationary case the space admits a second Killing vector field Y^μ with $Y^\mu Y_\mu < 0$. Following [2, 10] we can define, in a covariant manner, the five real potentials I, f, ψ, χ and ε according to $I^2 = X^\mu X_\mu$; $f = -IY^\mu Y_\mu - I^{-1}(X^\mu X_\mu)^2$; $\psi = I^{-2}X^\mu Y_\mu$;

$$\chi_{,\mu} = 2\varepsilon_{\alpha\beta\gamma\delta\mu}X^\alpha Y^\beta X^{\gamma;\delta}; \; \varepsilon_{,\mu} = 2\varepsilon_{\alpha\beta\gamma\delta\mu}X^\alpha Y^\beta Y^{\gamma;\delta} \tag{2}$$

($\varepsilon_{\alpha\beta\gamma\delta\mu}$ is the five-dimensional Levi-Civita pseudotensor). In the coordinate system with $X^\mu = \delta_5^\mu$, $Y^\mu = \delta_4^\mu$ one finds that $\psi^A = (\varkappa, f, \psi, \chi, \varepsilon)$; $A = 1, \ldots, 5$, respectively, are the scalar potential (with $I^2 = \varkappa^2$), the gravitational, electro- and magnetostatic, and the rotational potential. These five potentials ψ^A are the coordinates in a Riemannian space V_p^5 with the metric [3]

$$ds^2 = G_{AB}\, d\psi^A\, d\psi^B,$$
$$= \frac{1}{2f^2}[df^2 + (d\varepsilon + \psi\, d\chi)^2] + \frac{1}{2f}\left(\varkappa^2\, d\psi^2 + \frac{1}{\varkappa^2} d\chi^2\right) + \left(\zeta - \frac{3}{2}\right)\frac{d\varkappa^2}{\varkappa^2}, \tag{3}$$

which admits an isometry group $SL(3, R)$.

In the present paper we investigate the one- and two-dimensional subgroups of (3). An analogous investigation for the Einstein-Maxwell theory [3] has proved to be a very helpful tool to find new solutions in general relativity. In reference [4] some classes of the subspaces of the five-dimensional gravity (in our notation for $b = 0$, see eq. (17) below) were investigated; solutions without spatial symmetry were generated.

2. The Field Equations

In the stationary case the field equations (1) can be derived in terms of the potentials (2) from the Lagrangian [2, 3]

$$\mathscr{L} = \frac{\alpha}{2f^2}[f_{,i}f^{,i} + (\varepsilon_{,i} + \psi\chi_{,i})(\varepsilon^{,i} + \psi\chi^{,i})] + \frac{\alpha}{2f}(\varkappa^2\psi_{,i}\psi^{,i} + \varkappa^{-2}\chi_{,i}\chi^{,i}) + \frac{2}{3}\frac{\alpha}{\varkappa^2}\varkappa_{,i}\varkappa^{,i}; \quad i = x^1, x^2. \tag{4}$$

The field equations $\dfrac{\delta\mathscr{L}}{\delta\psi^A} = 0$ take then the form

$$\psi^{A;a}_{,a} + \begin{Bmatrix} A \\ BC \end{Bmatrix} \psi^B_{,a}\psi^{C,a} = 0, \tag{5}$$

$$\begin{Bmatrix} A \\ BC \end{Bmatrix} = \frac{1}{2} G^{AD}\left(\frac{\partial G_{BD}}{\partial\psi^C} + \frac{\partial G_{CD}}{\partial\psi^B} - \frac{\partial G_{BC}}{\partial\psi^D}\right).$$

For the axisymmetric case these last equations can be rewritten as chiral-field equation [7]

$$(\varrho g_{,z}g^{-1})_{,\bar z} + (\varrho g_{,\bar z}g^{-1})_{,z} = 0, \tag{6}$$

where $z = \varrho + i\zeta$ and its complex conjugate $\bar z$ are related to Weyl's canonical coordinates ϱ and ζ. The matrix g in (6) is a symmetric matrix which is an element of the group $SL(3, R)$, i.e.

$$\begin{aligned} &\text{a)}\ g = g^T, \\ &\text{b)}\ g = \bar g, \\ &\text{c)}\ \det g = 1, \end{aligned} \tag{7}$$

(the operation T denotes matrix transposition). A suitable parametrization of g in terms of the potentials ψ^A is given by

$$g = -\frac{2}{f\varkappa^{2/3}}\begin{pmatrix} f^2 + \varepsilon^2 - f\varkappa^2\psi^2 & -\varepsilon & -\frac{1}{2\sqrt{2}}(\varepsilon\chi + f\varkappa^2\psi) \\ -\varepsilon & 1 & \frac{1}{2\sqrt{2}}\chi \\ -\frac{1}{2\sqrt{2}}(\varepsilon\chi + f\varkappa^2\psi) & \frac{1}{2\sqrt{2}}\chi & \frac{1}{8}(\chi^2 - \varkappa^2 f) \end{pmatrix}. \tag{8}$$

The $SL(3, R)$ symmetry transformations can be written in the form (see also ref [2])

$$g = Cg_0C^T, \tag{9}$$

where the constant matrix C is also an element of $SL(3, R)$. The field equations (6) and the condition (7) are preserved under (9). A straightforward calculation shows that

$\mathrm{Tr}(dg\, dg^{-1}) = 4ds^2$. If we substitute $\chi = \psi = 0$, $\varkappa = 1$ in (8), we obtain from (6) the axisymmetric Ernst equations, $\epsilon = f + i\varepsilon$ being the Ernst potential. An equivalent approach was made by Neugebauer and Kramer [7] for the Einstein-Maxwell theory. We follow the line outlined in [6].

3. The Matrix A

Let us define a new 3×3 matrix A_i by

$$A_i = g_{,i} g^{-1}, \tag{10a}$$

where the comma means derivative with respect to a set of parameters $\lambda^i = \lambda^i(z, \bar{z})$ which are also solution of the equations (5) in the axisymmetric case

$$(\varrho \lambda^i_{,z})_{,\bar{z}} + (\varrho \lambda^i_{,\bar{z}})_{,z} + \varrho \Gamma^i_{jk} \lambda^j_{,z} \lambda^k_{,\bar{z}} = 0. \tag{10b}$$

The potentials ψ^A depend only on these parameters $\psi^A = \psi^A(\lambda^i)$; $i = 1, 2, \ldots, n$; $n \leq 4$.

The equation (6) and the definition (10a) imply the following properties for the matrix A_i

$$\text{a)}\ A_{i;j} + A_{j;i} = 0, \tag{11a}$$

$$\text{b)}\ A_{i,j} - A_{j,i} = [A_i, A_j] = A_i A_j - A_j A_i. \tag{11b}$$

The matrix equation (11a) is a Killing equation for each component of the matrix A_i. From the equations (11) it follows

$$A_{i_1;i_2;\ldots;i_{j+1}} = -\frac{1}{2} [A_{i_1;i_2;\ldots;i_j}, A_{i_{j+1}}]. \tag{12}$$

With the aid of the covariant derivative of the well-known relation for Killing-vectors and Riemannian-tensors $\zeta_{n;b;a} = R^m{}_{abn} \xi_m$ we conclude that $A_i R^i{}_{jkl;m} = 0$ for each component of the matrix A_i. Then the relation

$$R^i_{jkl;m} = 0 \tag{13}$$

holds. The relation (13) means that the Riemann space V_p^5 is symmetric. Another important property of the matrix A_i is that one can derive the field equations (6) from the Lagrangian

$$\mathscr{L} = \varrho\, \mathrm{Tr}(A_z A_{\bar{z}}) \tag{14}$$

(variation with respect to $\delta \mathscr{L} / \delta g_{ij} = 0$).

Now we are interested to see how the matrix $A_i = g_{0,i} g_0^{-1}$ changes if we transform the matrix g_0 as in (9). In this case we find that the corresponding matrix $A_i' = g_{,i} g^{-1}$ is equivalent to the matrix A_i,

$$A_i' = C A_i C^{-1}; \tag{15}$$

therefore it is sufficient to investigate the normal forms of the matrix A_i. From the relations (7) for the matrix g the properties

$$\text{a)}\ A_i g = g A_i^T,$$

$$\text{b)}\ A_i = \bar{A}_i, \tag{16}$$

$$\text{c)}\ \mathrm{Tr}\, A_i = 0$$

for the matrix A_i follow. Each matrix with the properties (16b) and (16c) is equivalent to one of the matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & 0 \end{pmatrix} \text{ or } \begin{pmatrix} q & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2q^2 & -q \end{pmatrix}, \tag{17}$$

where a is the determinant of A_i and $b = \frac{1}{2} \operatorname{Tr} A_i^2$. Then it is sufficient to investigate these two cases.

4. The One-dimensional Subspaces of V_p^5

Let us assume that the potential ψ^A depend only on a simple potential λ. The ansatz

$$\psi^A = \psi^A(\lambda) \tag{18}$$

reduces the potential space V_p^5 to an one-dimensional subspace. The equation (11a) is transformed into

$$g_{,\lambda} = Ag, \tag{19}$$

where A can be cast into the normal forms (17) and g has the form (8). We substitute the first of the normal forms (17) into (19). From the property (16a) we conclude that the relations for the potentials ψ^A

$$\begin{aligned} &1)\ \varepsilon\chi + f\varkappa^2\psi = -2\sqrt{2}, \\ &2)\ \chi^2 - \varkappa^2 f + 8a\varepsilon = 8b, \\ &3)\ a(f^2 + \varepsilon^2 - f\varkappa^2\psi^2) - b\varepsilon - \frac{1}{2\sqrt{2}}\chi = 0 \end{aligned} \tag{20}$$

hold. Now we shall solve the differential equation (19). First we obtain a linear differential equation for the component g_{12} of g

$$g_{12,\lambda\lambda\lambda} - bg_{12,\lambda\lambda} - ag_{12} = 0. \tag{21}$$

Then we have to solve the characteristic polynomial of the matrix A

$$\lambda^3 - b\lambda - a = 0. \tag{22}$$

The polynomial (22) has different solutions which depend on the values of a and b. Therefore we have to differ between the following cases:

(i) The eigenvalues r_j of the matrix A are real, with $r_1 + r_2 + r_3 = 0$. We can use in this case a diagonal form for the matrix A'.

(ii) Two eigenvalues are equal, the invariant factors of dimension one and two are equal to one. We can use in this case a Jordan normal form for the matrix A'.

(iii) Two eigenvalues are imaginar, one is equal to zero.

(iv) One eigenvalue is real, two are complex $r_j = (-2m, m + in, m - in)$.

In the case (iv) the solution of the equation (21) reads

$$g_{12} = e^{m\lambda}[c \cos n\lambda + c_{12}e^{-2m\lambda}], \tag{23}$$

Table 1. One-dimensional subspaces

Case	Eigenvalue	A'	A	$\varkappa^2$
(i)	$r_1, r_2, r_3 \in \mathbb{R}$ $r_1 + r_2 + r_3 = 0$		$\begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix}$	$\sqrt[3]{4c_{33}}\, e^{\frac{2}{3} r_3 \lambda}$
(ii)	$r_1, r_1, -2r_1$ $a_1 = a_2 = 1$		$\begin{pmatrix} -2r_1 & 0 & 0 \\ 0 & r_1 & 1 \\ 0 & 0 & r_2 \end{pmatrix}$	$c_{23}^2 \left[\frac{4e^{r_1\lambda}}{c_{11}^{1/3}(c_{23}\lambda + c_{22})}\right]^{3/2}$
(iii)	$r_1, \bar{r}_1, 0$ $r_1 = -\bar{r}_1$ $b < 0$ $\Lambda = \sqrt{\lvert b \rvert}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & b & 0 \end{pmatrix}$	$\frac{8b(b\varepsilon^2 - 1)}{f}$
(iv)	$-2m, m + in,$ $m - in$		$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & 0 \end{pmatrix}$ $a \neq 0$	$\frac{\chi^2 + 8(a\varepsilon - b)}{f}$
$q \neq 0$	$r_1, r_1, -2r_1$	$\begin{pmatrix} q & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -2q^2 & -q \end{pmatrix}$	$\begin{pmatrix} -2r_1 & 0 & 0 \\ 0 & r_1 & 0 \\ 0 & 0 & r_1 \end{pmatrix}$	$\sqrt[3]{4c_{33}}\, e^{\frac{3}{2} r_1 \lambda}$
$q = 0$	$0, 0, 0$		$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\frac{8c_{23}^2}{\sqrt{-c_{11}}\,(-c_{23}\lambda - c_{22})^{3/2}}$

(where $c_{11}, c_{12}, c_{22}, c_{23}, c_{33}$ are constants, $(\varrho\lambda_{,z})_{,\bar{z}} + (\varrho\lambda_{,\bar{z}})_{,z} = 0$.)

where m, n, c and c_{12} are constants. Now we can calculate the components g_{22} and g_{23}

$$g_{22} = e^{m\lambda}[c(m \cos n\lambda + n \sin n\lambda) - 2mc_{12}e^{-3m\lambda}],$$

$$g_{23} = e^{m\lambda}\left[c\,\frac{a(m \cos n\lambda + n \sin n\lambda) + b(m^2 + n^2) \cos n\lambda}{m^2 + n^2} + \frac{c_{12}(2mb - a)}{2m}\, e^{-3m\lambda}\right]. \tag{24}$$

We compare these expressions with the matrix g in (8) and write down the potentials ε and χ

$$\varepsilon = -\frac{g_{12}}{g_{22}}, \quad \chi = \frac{2\sqrt{2}\, g_{23}}{g_{22}}. \tag{25}$$

f	ψ	χ	ε
$-2c_{11}c_{33}e^{r_1\lambda+\frac{1}{2}r_3\lambda}$	0	0	0
$\dfrac{\sqrt{c_{11}}\,e^{-\frac{3}{2}r_1\lambda}}{(c_{23}\lambda+c_{22})^{1/2}}$	0	$\dfrac{2\sqrt{2}\,c_{23}}{c_{23}\lambda+c_{22}}$	0
$\left[\dfrac{c\sin\Lambda\lambda+\Lambda c_{11}}{b\,c_{12}\sin\Lambda\lambda+\Lambda c_{13}} + \dfrac{2\sqrt{2}\,(b\varepsilon^2-1)^2-\varepsilon^2(b\varepsilon^2+1)}{b\varepsilon^2+1}\right]^{1/2}$	$\dfrac{b\varepsilon^2-1}{2\sqrt{2}\,b(b\,\varepsilon^2+1)}$	$-2\sqrt{2}\,b\varepsilon$	$-\dfrac{c_{12}\cos\Lambda\lambda}{\frac{bc_{12}}{\Lambda}\sin\Lambda\lambda+c_{13}}$
$\left\{\dfrac{1}{a}\left[\dfrac{1}{2\sqrt{2}}\chi+b\varepsilon-a\varepsilon^2 + \dfrac{a(2\sqrt{2}+\varepsilon\chi)^2}{\chi^2+8(a\varepsilon-b)}\right]\right\}^{1/2}$	$\dfrac{\varepsilon\chi+2\sqrt{2}}{\chi^2+8(a\varepsilon-b)}$	Equation (25)	Equation (25)
$-2c_{11}c_{33}e^{-\frac{3}{2}r_1\lambda}$; $c_{11}, c_{33} \neq 0$	0	0	0
$\dfrac{\sqrt{-c_{11}}}{(-c_{23}\lambda-c_{22})^{1/2}}$; $c_{11}=-\dfrac{1}{c_{23}^2}$, $c_{23}\neq 0$	$\dfrac{c_{12}}{2\sqrt{2}\,c_{23}}$	$\dfrac{2\sqrt{2}\,c_{23}}{c_{23}\lambda+c_{22}}$	$-\dfrac{c_{12}}{c_{23}\lambda+c_{22}}$

The other potentials can be calculated by means of the relations (20). The results of the other cases are given in Table 1.

By substituting the second normal form in (17) we obtain two cases: $q = 0$ and $q \neq 0$. If $q =$ constant, we can tranform the matrix A in a diagonal matrix A'. Both results are also given in Table 1.

It is easy to see, that we get the case $q \neq 0$ from case (i) in Tab. 1 by putting $r_1 = r_3$.

5. Some Solutions Corresponding to One-dimensional Subspaces

We give some examples for the classes listed in Tab. 1.

Example 1. We take the case (i) and substitute $r_1 = 1/H$, $r_2 = -1$, $r_3 = 1 - 1/H$, $I_0 = 2c_{11}c_{33}$ and $\lambda = \ln V$. We get the solution

$$I = I_0 V^{\frac{1}{2}\left(1-\frac{1}{H}\right)}, \quad f = -I_0 V^{\frac{1}{2}\left(1+\frac{1}{H}\right)},$$
$$\varkappa^2 = I^3, \quad \varepsilon = \chi = \psi = 0,$$

which is exactly the Heckmann-Jordan-Fricke solution (HJF) [8]. The line element of this solution is given by

$$ds^2 = e^\sigma\, dr^2 + r^2 d\,\Omega^2 - e^\nu\, dx^{1^2} + I\, dx^{5^2},$$

$$r = r_0 \left[e^{\frac{1}{2}\lambda} (e^{-h\lambda} + e^{h\nu})\right]^{-1}, \quad h^2H^2 = \frac{1}{4}(H^2 - H + 1),$$

$$e^\sigma = 4h^2 \left[\left(h + \frac{1}{2}\right) e^{h\lambda} + \left(h - \frac{1}{2}\right) e^{-h\lambda}\right]^{-2}, \quad h\nu = \lambda.$$

Example 2. We take the case (i) again and perform a transformation (15) to get the new matrix

$$A_i = \begin{pmatrix} h_1 & 0 & h_2 \\ 0 & -(h_1 + h_4) & 0 \\ h_3 & 0 & h_4 \end{pmatrix}, \tag{26}$$

where h_1, h_2, h_3 and h_4 are constants with the following restrictions

$$h_2 = \sqrt{2}\,(h_1 - h_4)\left[hd\, P_1 + \frac{ec}{I_0^2} P_2\right],$$

$$h_3 = \frac{4(h_1 - h_4)^2\, cdeh}{h_2(I_0^2 + 16\, cdeh)},$$

$$P_{1,2} = 1 \pm \sqrt{\frac{I_0^2}{I_0^2 + 16\, cdeh}}; \quad I_0, c, d, e, h = cte.$$

The eigenvalues of the matrix A are

$$r_{1,2} = \frac{h_1 + h_4}{2} \pm \frac{1}{2}\sqrt{(h_1 - h_4)^2 + 4h_2h_3},$$

$$r_3 = -(h_1 + h_4).$$

We can choose the constants in such a form that the discriminant becomes greater than zero. In this way the matrix (26) is of type (i) in Table 1. We transform also the matrix (8) as in (9) and get the following solution

$$f^2 = -\frac{c_1c_2(r_1 - r_2)\, e^{r_2\lambda} e^{(h_1+h_4)\lambda}}{h_3c_{22}[c_1(r_1 - h_1) + c_2(r_2 - h_1)\, e^{(r_2-r_1)\lambda}]},$$

$$\psi = \frac{1}{2\sqrt{2}} \frac{c_1h_2 + c_2h_2\, e^{(r_2-r_1)\lambda}}{c_1(r_1 - h_1) + c_2(r_2 - h_1)\, e^{(r_2-r_1)\lambda}},$$

$$f\varkappa^{\frac{2}{3}} = -\frac{2}{c_{22}} e^{(h_1+h_4)\lambda}, \quad \varepsilon = \chi = 0,$$

where c_1, c_2 and c_{22} are constants. We substitute now $I_0 = 2/c_{22}$, $e^{r_1\lambda} = V^{1/H}$, $e^{(h_1+h_4)\lambda} = V$, $c_1 = 2\sqrt{2}\, hd$ and $c_2 = 2\sqrt{2}\, ec/I_0^2$ to get

$$f^2 = \frac{I_0^2 V^{1/H-1}}{1 + G(1 - V^{2/H-1})}, \quad \psi = \frac{\frac{ec}{I_0^2} V^{2/H-1} + hd}{1 + G(1 - V^{2/H-1})},$$

$$G = -\frac{c_2}{h_2}(r_2 - h_1), \quad f\varkappa^{2/3} = -I_0^2 V,$$

which is the Kühnel-Schmutzer-solution (KS) [9]. In reference [10] it was shown that the KS-solution can be generated from the HJF-solution by means of a coordinate transformation in the five-dimensional space. We have shown that these two solutions are included in the same subspace.

6. The Two-dimensional Subspaces of V_p^5

Now let us assume that the potentials ψ^A and therefore the matrices g depend on two parameters λ and τ. Because of eq. (13) the curvature K of the corresponding two-dimensional subspaces is constant. Hence the Killing equation (11a) refers now to the two-dimensional space with line element

$$4\,ds^2 = \frac{4d\lambda\,d\tau}{(1+K\lambda\tau)^2} = \mathrm{Tr}(dg\,dg^{-1}), \tag{27}$$

where K is the constant scalar curvature.

The Riemannian space (27) has three Killing vectors. Thus we can write the matrix A_i as a combination of three independent Killing vectors $\overset{a}{\xi} = (\overset{a}{\xi}_1, \overset{a}{\xi}_2)$ $a = 1, 2, 3$

$$A_i = \overset{1}{\xi}_i\sigma_1 + \overset{2}{\xi}_i\sigma_2 + \overset{3}{\xi}_i\sigma_3. \tag{28}$$

The independent Killing vectors of (27) are

$$\begin{aligned}
\overset{1}{\xi} &= \frac{1}{2} V^{-2}(K\tau^2 + 1, K\lambda^2 + 1),\\
\overset{2}{\xi} &= V^{-2}(-\tau, \lambda),\\
\overset{3}{\xi} &= \frac{1}{2} V^{-2}(K\tau^2 - 1, 1 - K\lambda^2),\\
V &= 1 + K\lambda\tau.
\end{aligned} \tag{29}$$

We substitute the Killing vectors (29) into (28) and this equation in turn into (12) (with $j = 0$) to get the following relations for the matrices σ_a:

$$\begin{aligned}
[\sigma_1, \sigma_2] &= -4K\sigma_3,\\
[\sigma_2, \sigma_3] &= 4K\sigma_1,\\
[\sigma_3, \sigma_1] &= -4\sigma_2.
\end{aligned} \tag{30}$$

Now we investigate further properties of the matrices σ_a. In the direction of the geodesic parameter $\lambda = \tau = \tau(s)$ the identity

$$\frac{dg}{ds} g^{-1} = \sigma_1 \tag{31}$$

holds. From the substitution of (31) in (27) one obtains

$$\mathrm{Tr}\,\sigma_1^2 = 4. \tag{32}$$

A representation of the matrices σ_a with the commutation relations (30) and the condition (32) does not exist for any K; we conclude that only the cases $K = 0$, $K = -1$ and $K = -\frac{1}{4}$ are possible. We investigate these three cases.

Let K be equal to zero. Then a representation for the matrices σ_a is given by

$$\sigma_1 - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & 2 & 0 \end{pmatrix}, \quad \sigma_2 = 0, \quad \sigma_3 = \begin{pmatrix} -\frac{4}{3} c_3 & b_3 & c_3 \\ a c_3 & \frac{2}{3} c_3 & b_3 \\ a b_3 & 2 b_3 + a c_3 & \frac{2}{3} c_3 \end{pmatrix}, \tag{33}$$

where a_3 and c_3 are arbitrary constants. From the condition (16a) we get again the three relations (20) with which we can express three potentials as functions of the other two. The two independent potentials can be written as exponential functions of λ and τ, whose forms depend on the value of a, a_3 and c_3. (We give an example in the next section).

Let K be equal to -1. A representation of the matrices σ_a is

$$\sigma_1 = 2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \sigma_2 = 2 \begin{pmatrix} 0 & b & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \sigma_3 = 2 \begin{pmatrix} 0 & -b & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad ab = 1. \tag{34}$$

The general solution of the equation (10) for the matrix g with (28) and (29) is given by

$$f = -\frac{(1 - \lambda\tau)}{(1 + \lambda)(1 + \tau)}, \quad \varepsilon = -\frac{i(\tau - \lambda)}{(1 + \lambda)(1 + \tau)}, \quad \varkappa = 1, \tag{35}$$
$$\psi = \chi = 0,$$

where we have used the matrix (8). This case covers all solutions of the Einstein vacuum equations without electromagnetism and $\varkappa$ equal to one.

For $K = -1/4$ we get a new subspace. A representation of the matrices σ_a is

$$\sigma_1 = 2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 0 & d \\ 0 & 0 & c \\ b & a & 0 \end{pmatrix}, \quad \sigma_3 = 2 \begin{pmatrix} 0 & 0 & -d \\ 0 & 0 & c \\ b & -a & 0 \end{pmatrix}, \tag{36}$$
$$ac = bd = \frac{1}{2}.$$

We solve again the equation (10a) for the matrix g and obtain

$$f = -\frac{(4 - \lambda\tau)^2}{(\lambda + 2)^2(\tau + 2)^2}, \quad \varepsilon = \frac{4(\lambda - \tau)^2}{(\lambda + 2)^2 (\tau + 2)^2},$$
$$\psi = \frac{i(\lambda - \tau)}{(\lambda + 2)(\tau + 2)}, \quad \chi = \frac{8i(\lambda - \tau)}{(\lambda + 2)(\tau + 2)}, \quad \varkappa = 2\sqrt{2}, \tag{37}$$

which is a solution of the field equations (5). The line element is given by

$$ds^2 = \frac{d\lambda \, d\tau}{\left(1 - \frac{1}{4} \lambda\tau\right)^2}, \tag{38}$$

and the field equations for λ and τ are given in (10b) with $\lambda^i = (\lambda, \tau)$. The Christoffel symbols can be calculated from the line element (38).

Let us perform the transformation $2\zeta \to \lambda$, $2\eta \to \tau$ in (37) and (38). It is easy to

see that

$\chi = 8\psi$, $\varepsilon = -$

We define $\Phi = 2\psi$ anc

$\Phi = \frac{i(\zeta -}{(\zeta + 1)(}$

with

$ds^2 = \frac{4 ds\zeta\, d\eta}{(1 - \zeta\eta}$

Now we can formulate

Theorem. From each ele theory ($\mathfrak{E}$, Φ) ($\mathfrak{E} = \bar{\mathfrak{E}}$, $\Phi =$

the substitution $\varkappa = 2\sqrt{2}$,

7. Some Solutions Corresp

Example 1. Let us giv values of the matrix σ_1, all σ_1 into a diagonal form. Th

$f^2 = \frac{c_{11}}{c_{22}} \exp \{($

$\varkappa^2 = \frac{8}{f} \frac{c_{33}}{c_{22}} \exp$

$r_1 + r_2 + r_3 =$

$a_3 + e_3 + i_3 =$

$\varepsilon = \chi = \psi = 0.$

For instance, if $r_1 = \frac{1}{2}\sqrt{}$

$a_3 = r_2$, $e_3 = r_1$, $i_3 = r_3$, the

$f = e^\lambda$, $\varkappa^2 = e^{\sqrt{3}}$

The matrices σ_a form an Al

$ds^2 = d\lambda\, d\tau.$

Example 2. In the case K The two-dimensional metric

$ds^2 = \frac{d\lambda d\tau}{(1 - \lambda\tau)^2)}$

A solution of (10b) is $\bar{\tau} = \lambda$

coordinates. The substitutic Kerr-NUT solution for the

Example 3. In the case K

The two-dimensional metric

see that

$$\chi = 8\psi, \ \varepsilon = -4\psi^2, \ \varkappa = 2\sqrt{2}. \tag{39}$$

We define $\Phi = 2\psi$ and $\mathfrak{E} = -f + \Phi^2$ to obtain

$$\Phi = \frac{i(\zeta - \eta)}{(\zeta + 1)(\eta + 1)}, \ \mathfrak{E} = \frac{(1 - \zeta)(1 - \eta)}{(1 + \zeta)(1 + \eta)}, \tag{40}$$

with

$$ds^2 = \frac{4ds\zeta \, d\eta}{(1 - \zeta\eta)^2}.$$

Now we can formulate the following theorem:

Theorem. From each electrostatic (magnetostatic) solution of the Maxwell-Einstein theory $(\mathfrak{E}, \Phi)$ $(\mathfrak{E} = \bar{\mathfrak{E}}, \Phi = \bar{\Phi})$ we can generate a solution of the projective theory by the substitution $\varkappa = 2\sqrt{2}$, $f = -\mathfrak{E} + \Phi^2$, $\psi = \frac{1}{2}\Phi$, $\chi = 4\Phi$, $\varepsilon = -\Phi^2$.

7. Some Solutions Corresponding to Two-dimensional Subspaces

Example 1. Let us give an example for the case $K = 0$. Let r_1, r_2 and r_3 be the eigenvalues of the matrix σ_1, all of them being real. In this case we can transform the matrix σ_1 into a diagonal form. Then the solution can be written as

$$\begin{aligned} f^2 &= \frac{c_{11}}{c_{22}} \exp\{((r_1 - a_3) - (r_2 - e_3))\lambda + ((r_1 + a_3) - (r_2 + e_3))\tau\}, \\ \varkappa^2 &= \frac{8}{f} \frac{c_{33}}{c_{22}} \exp\{((r_3 - i_3) - (r_2 - e_3))\lambda + ((r_3 + i_3) - (r_2 + e_3))\tau\}, \\ & r_1 + r_2 + r_3 = 0, \ r_1^2 + r_2^2 + r_3^2 = 4, \ a = r_1 r_2 r_3, \\ & a_3 + e_3 + i_3 = 0, \ c_{11}c_{22}c_{33} = 1; \ a_3, e_3, i_3 \in \mathbb{R}, \\ & \varepsilon = \chi = \psi = 0. \end{aligned} \tag{41}$$

For instance, if $r_1 = \frac{1}{2}\sqrt{3}\,(-1 + \sqrt{3})$, $r_2 = \frac{1}{2\sqrt{3}}(-1 - \sqrt{3})$, $r_3 = \frac{1}{\sqrt{3}}$, and $a_3 = r_2$, $e_3 = r_1$, $i_3 = r_3$, the potentials read

$$f = e^{\lambda}, \ \varkappa^2 = e^{\sqrt{3}\,\tau}, \quad c_{11} = c_{22} = 2, \ c_{33} = \frac{1}{4}.$$

The matrices σ_a form an Abelian group and the two-dimensional metric has the form

$$ds^2 = d\lambda \, d\tau.$$

Example 2. In the case $K = -1$ the matrices σ_a have a $SU(1,1)$ group of isometries. The two-dimensional metric has the from

$$ds^2 = \frac{d\lambda d\tau}{(1 - \lambda\tau)^2)}.$$

A solution of (10b) is $\bar{\tau} = \lambda = \dfrac{m + il}{r - m + ia\cos\nu}$ where r and ν are Boyer-Lindquist coordinates. The substitution of these parameters into (35) gives an analogue of the Kerr-NUT solution for the projective field theory.

Example 3. In the case $K = -\frac{1}{4}$ the matrices σ_a form an O(2,1) group of isometries. The two-dimensional metric can be written as in (40). Starting with the Reissner-Nordström solution we can construct a corresponding solution in the projective theory. For the Reissner-Nordström solution the Ernst potential and the electromagnetic potential read

$$\mathfrak{E} = 1 - \frac{2m}{r}, \Phi = \frac{e}{r},$$

Using Theorem 1 we obtain the corresponding solution for the projective theory

$$\varkappa = 2\sqrt{2}, f = -\frac{r^2 - 2mr - e^2}{r^2},$$

$$\psi = \frac{1}{2}\frac{e}{r}, \ \chi = 4\frac{e}{r}, \ \varepsilon = -\frac{e^2}{r^2}.$$

The components of the electromagnetic 4-potential are

$$A_1 = A_2 = 0,$$

$$A_3 = \frac{c}{2}\cos\vartheta,$$

$$A_4 = \frac{1}{2}\frac{e}{r},$$

where c, e and m are constants.

Further new solutions can be generated by transforming the matrix g with the aid of the invariance transformations (15).

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Dentro de Investigacion y de Estudios
Avanzados del IPN
Apto. Post. 14–740
Departamento de Fisica
07000 Mexico
D.F. Mexico