

**Class of Einstein-Maxwell-dilaton-axion space-times**Tonatiuh Matos<sup>\*,†</sup>*Departamento de Física, Centro de Investigación y de Estudios Avanzados del IPN, Apartado Postal 14-740, 07000 Distrito Federal, México*Galaxia Miranda<sup>\*,‡</sup>*Departamento de Física, Escuela Superior de Física y Matemáticas del IPN, Edificio 9, 07738 Distrito Federal, México*Rubén Sánchez-Sánchez<sup>§</sup>*Centro de Investigación en Ciencia Aplicada y Tecnología Avanzada del IPN, Legaria 694, 11500 Distrito Federal, México*Petra Wiederhold<sup>||</sup>*Departamento de Control Automático, Centro de Investigación y de Estudios Avanzados del IPN, Apartado Postal 14-740, 07000 Distrito Federal, México*

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We use the harmonic maps ansatz to find exact solutions of the Einstein-Maxwell-dilaton-axion (EMDA) equations. The solutions are harmonic maps invariant to the symplectic real group in four dimensions  $Sp(4, \mathbb{R}) \sim O(5)$ . We find solutions of the EMDA field equations for the one- and two-dimensional subspaces of the symplectic group. Specially, for illustration of the method, we find space-times that generalize the Schwarzschild solution with dilaton, axion, and electromagnetic fields.

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**I. INTRODUCTION**

The new discoveries of the last years have changed our perspective and understanding of the Universe. Specially, the discovery of the dark matter and the dark energy have opened new big questions about the nature of the matter in cosmos. Doubtless, it is time to propose new paradigms in order to give some light to these questions. One of the most accepted candidates to be the nature of the dark energy is a scalar field [1], and maybe it is less known that scalar fields are also very good candidates to be the nature of the dark matter [2].

At the same time, theories like superstrings propose the existence of several scalar fields. In particular, at low energy the superstrings theory contains at least two scalar fields called the dilaton and the axion. There are some attempts to compare these two scalar fields with the dark matter and dark energy [3,4], but the main problem for this is to go from the higher dimension theory to the four-dimensional one [5]. In some cases, it seems that this theory could explain the Universe, but this question is still open.

In this work we study the Einstein-Maxwell-dilaton-axion (EMDA) system, from the effective point of view, i.e., we start from the corresponding Lagrangian and derive

the field equations. Later we use the harmonic maps ansatz to solve the system of six coupled, nonlinear differential equations for the axial symmetric stationary case.

The method of harmonic maps to find exact solutions of the Einstein, Einstein-Maxwell, and Einstein-Maxwell-dilaton fields has been used with great success. This ansatz was first used by Neugebauer and Kramer to find exact solutions to Einstein-Maxwell equations [6], and in [7] this ansatz was generalized to the Einstein-Maxwell-dilaton system with a coupling constant  $\alpha$  between the dilaton and the Maxwell fields given by  $\alpha = \sqrt{3}$ . Later on this ansatz was generalized in [8] for an arbitrary  $\alpha$ . The ansatz has been used also for solving the Einstein-Maxwell-phantom system with arbitrary  $\alpha$  [9]. Here we apply the harmonic maps ansatz to solve the equations of motion for the Einstein-Maxwell-dilaton-axion theory in the target space.

This work is organized as follows. In Sec. II, we introduce the fields of the potential space we are working with. In Sec. III, we write the field equations as a nonlinear  $\sigma$  model to be used in Sec. IV, where we use the harmonic maps ansatz to solve the system. In Sec. V, we solve the field equations for the one-dimensional subgroups of  $Sp(4, \mathbb{R}) \sim O(5)$ , and, in Sec. VI, for the subgroup  $SO(2, 1)$ . In Sec. VII, some conclusions and perspectives are given. In the Appendix we review the use of the harmonic maps ansatz for the chiral equations, the nonlinear  $\sigma$  models.

**II. THE EFFECTIVE ACTION FOR EMDA**

Gravity with two scalar fields, the dilaton and the axion and a  $U(1)$  vector field can be described with the action

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$$S = \frac{1}{16\pi} \int \left[ -R + \frac{1}{3} e^{-4\Phi} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + 2\partial_\mu \Phi \partial^\mu \Phi - e^{-2\Phi} F_{\mu\nu} F^{\mu\nu} \right] \sqrt{-g} d^4x, \quad (1)$$

where we start with a space-time metric in four dimensions with the dilaton  $\Phi$  coupled to the  $U(1)$  vector field, the Maxwell field, with coupling constant  $\alpha = 1$  as in superstrings theory, such that  $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$  is the corresponding Maxwell tensor plus the Pecci-Quinn pseudoscalar  $\mathbf{a}$ . The Maxwell tensor can be written as  $\mathbf{F} = d\mathbf{A}$ . The antisymmetric tensor of three indices  $H^{\mu\nu\lambda}$  is the Kalb-Ramon tensor defined as

$$H_{\mu\nu\lambda} = (\partial_\mu B_{\nu\lambda} + \partial_\lambda B_{\mu\nu} + \partial_\nu B_{\lambda\mu}) - (A_\mu F_{\nu\lambda} + A_\lambda F_{\mu\nu} + A_\nu F_{\lambda\mu}). \quad (2)$$

In this description, the electromagnetic 4-potential  $A_\mu$  has two nonzero components

$$A_\mu = \frac{1}{\sqrt{2}} (\psi, 0, 0, \sqrt{2}A_\varphi).$$

On the other hand, the Kalb-Ramon tensor has only one component  $B_{03} = b$ .

The symmetry group  $Sp(4, \mathbb{R})$  for the EMDA model acts on the set of the six potentials:  $f$ , the gravitational;  $\epsilon$ , the rotational;  $\psi$ , the electrostatic;  $\chi$ , the magnetostatic;  $\Phi$ , the dilatonic; and  $\mathbf{a}$  the axionic potential. The group  $Sp(4, \mathbb{R})$  is homomorphic to the group  $O(5)$ , but in this work we will use the representations of  $Sp(4, \mathbb{R})$ . The three potentials  $f$ ,  $\psi$ , and  $\chi$  are dual to the three potentials  $\epsilon$ ,  $\chi$ , and  $\mathbf{a}$ . Here  $\mathbf{a}$  is a Pecci-Quinn pseudoscalar field dual to the Kalb-Ramon tensor  $H^{\mu\nu\sigma}$

$$H^{\mu\nu\sigma} = \frac{1}{2} e^{4\Phi} E^{\mu\nu\sigma\tau} \frac{\partial \mathbf{a}}{\partial x^\tau}.$$

The effective action for the bosonic sector of a heterotic string of ten dimensions compactified into four and with one vector field  $U(1)$  can be rewritten as

$$S = \frac{1}{16\pi} \int \left\{ -R + 2\partial_\mu \Phi \partial^\mu \Phi + \frac{1}{2} e^{4\Phi} \partial_\mu \mathbf{a} \partial^\mu \mathbf{a} - e^{-2\Phi} F_{\mu\nu} F^{\mu\nu} - \mathbf{a} F_{\mu\nu} {}^* F^{\mu\nu} \right\} \sqrt{-g} d^4x. \quad (3)$$

Here  ${}^*F = \frac{1}{2} E^{\mu\nu\lambda\tau} F_{\lambda\tau}$  is the dual of the Maxwell tensor. Also, we have that  $E^{\mu\nu\lambda\tau} = \epsilon^{\mu\nu\lambda\tau} \text{sign}(g)/\sqrt{-g}$  is the Levi-Civita pseudotensor. To reduce the system to three dimensions we need a nonzero, timelike Killing vector. With this ansatz it is possible to write the four-dimensional metric  $g_{\mu\nu}$  in terms of the three-dimensional  $h_{ij}$  one as

$$ds^2 = ds_{(4)}^2 = g_{\mu\nu} dx^\mu dx^\nu = f(dt - \omega_i dx^i)^2 - \frac{1}{f} h_{ij} dx^i dx^j. \quad (4)$$

(We use the following convention: Latin indices run in three dimensions, for example  $i, j = 1, 2, 3$  and Greek indices run in four dimensions, for example  $\alpha, \beta = 0, 1, 2, 3$ .) Here, the three-dimensional metric is given by

$$ds_{(3)}^2 = h_{ij} dx^i dx^j = 2e^{2\Gamma} dz d\bar{z} + \rho^2 d\varphi^2, \quad (5)$$

or, in Weyl coordinates we use the complex variable  $z = \frac{1}{\sqrt{2}}(\rho + i\zeta)$ , thus metric (5) transforms into the Lewis-Papapetrou form

$$ds_{(3)}^2 = e^{2\Gamma} (d\rho^2 + d\zeta^2) + \rho^2 d\varphi^2.$$

We will use also the Boyer-Lindquist coordinates  $\rho = \sqrt{r^2 - 2mr + \sigma^2} \sin(\theta)$  and  $\zeta = (r - m) \cos(\theta)$ . In these coordinates, the 3-metric (5) reads

$$ds_{(3)}^2 = e^{2\Gamma} \left[ ((r - m)^2 + K^2 \cos^2(\theta)) \times \left( \frac{dr^2}{r^2 - 2mr + \sigma^2} + d\theta^2 \right) + (r^2 - 2mr + \sigma^2) \sin^2(\theta) d\varphi^2 \right]. \quad (6)$$

The variation of the action (3) gives the Euler-Lagrange equations for the fields, to obtain the following: the coupled Maxwell equation with two scalar fields

$$\nabla_\mu (e^{-2\Phi} F^{\mu\nu} + \mathbf{a} {}^* F^{\mu\nu}) = 0, \quad (7)$$

the dilaton and axion equations

$$\nabla^\mu \nabla_\mu \Phi = \frac{1}{2} e^{-2\Phi} F^2 + \frac{1}{2} e^{4\Phi} (\partial \mathbf{a})^2, \quad (8)$$

$$\nabla_\mu (e^{4\Phi} g^{\mu\nu} \partial_\nu \mathbf{a}) + F_{\mu\nu} {}^* F^{\mu\nu} = 0, \quad (9)$$

and the main Einstein equations

$$R_{\mu\nu} = 2\Phi_{,\mu} \Phi_{,\nu} + \frac{1}{2} e^{4\Phi} a_{,\mu} a_{,\nu} + e^{-2\Phi} \left( 2F_{\mu\tau} F^\tau{}_\nu + \frac{1}{2} F^2 g_{\mu\nu} \right). \quad (10)$$

If there exists a timelike Killing vector, it is possible to decompose the Maxwell tensor into two fields, the electrostatic  $\psi$  and the magnetostatic  $\chi$  potentials. With the help of these two quantities we can obtain the electric  $E_i = F_{0i}$  and magnetic  $F_{ij} = \epsilon^{ijk} B_k$  components of the Maxwell tensor as

$$F_{i0} = \frac{1}{\sqrt{2}} \partial_i \psi, \quad (11)$$

$$e^{-2\Phi} F^{ij} + \mathbf{a} {}^* F^{ij} = \frac{f}{\sqrt{2h}} \epsilon^{ijk} \partial_k \chi. \quad (12)$$

The first relationship (11) can be deduced from the Bianchi identity

$$\nabla_\mu {}^*F^{\mu\nu} = 0.$$

Another important quantity for this work is the twist 3-tensor  $\tau_i$ , this is derived from the rotational  $\varepsilon$ , the magneto-static  $\chi$  and electrostatic  $\psi$  potentials as

$$\tau_i = \partial_i \varepsilon + \psi \partial_i \chi - \chi \partial_i \psi.$$

The metric function  $\omega_i = \omega_i(r, \theta)$  in the 4-metric in the Lewis-Papapetrou form (4) is computed from the relation

$$\tau^i = -\frac{f}{\sqrt{h}} \varepsilon^{ijk} \partial_j \omega_k.$$

Thus, if we know the potentials, we can integrate the elements of the four-dimensional metric.

### III. THE NONLINEAR $\sigma$ MODEL OF THE EMDA THEORY

The most important feature we use here to find exact solutions for the EMDA field equations is the fact that the Euler-Lagrange equations (7)–(10), can be obtained from the following action of the three-dimensional nonlinear  $\sigma$  model [10]

$$\begin{aligned} S^{(3)} = & \int \left[ R^{(3)} - \frac{1}{2f^2} [(\nabla f)^2 + (\nabla \varepsilon + \psi \nabla \chi - \chi \nabla \psi)^2] \right. \\ & - 2(\nabla \Phi)^2 - \frac{1}{2} \exp(4\Phi) (\nabla \kappa)^2 \\ & \left. + \frac{1}{f} [e^{2\Phi} (\nabla \psi - a \nabla \chi)^2 + e^{-2\Phi} (\nabla \psi)^2] \right] \sqrt{h} d^3 x. \end{aligned} \quad (13)$$

Alternatively this can be written as

$$S_{(\sigma)} = \int (R^{(3)} - \mathcal{G}_{AB} \partial_i \varphi^A \partial_j \varphi^B h^{ij}) \sqrt{h} d^3 x, \quad (14)$$

with the line element of the target space given by

$$\begin{aligned} dl^2 = & \mathcal{G}_{AB} d\varphi^A d\varphi^B \\ = & \frac{1}{2f^2} [df^2 + (d\varepsilon + \psi d\chi - \chi d\psi)^2] \\ & - \frac{1}{f} [e^{2\Phi} (d\chi - a d\psi)^2 + e^{-2\Phi} d\psi^2] + 2d\Phi^2 \\ & + e^{4\Phi} da^2, \end{aligned} \quad (15)$$

where we have introduced the vector potential

$$\varphi^A = (f, \varepsilon, \psi, \chi, \Phi, a).$$

This important line element can be derived from the following Lagrangian density, which introduces the matrix  $g \in Sp(4, \mathbb{R})$  of potentials

$$\mathcal{L} = \frac{1}{4} \text{Tr}(dgg^{-1}dgg^{-1}) \quad (16)$$

in two dimensions. In terms of the complex variables  $z$  and

$\bar{z}$  this is equivalent to

$$\mathcal{L} = \frac{1}{4} \text{Tr}(g_{,z} g_{,\bar{z}}^{-1} + g_{,\bar{z}} g_{,z}^{-1}).$$

The Euler-Lagrange equations of this relation are the chiral equations

$$(g_{,z} g^{-1})_{,\bar{z}} + (g_{,\bar{z}} g)_{,z}^{-1} = 0. \quad (17)$$

The form of  $g$  can be expressed as a Gaussian decomposition of  $2 \times 2$  matrices  $P$  and  $Q$  given by

$$g = \begin{pmatrix} P^{-1} & P^{-1}Q \\ QP^{-1} & P + QP^{-1}Q \end{pmatrix}, \quad (18)$$

where  $P$  and  $Q$  are

$$\begin{aligned} P &= \begin{pmatrix} f - e^{-2\Phi} \psi^2 & -e^{-2\Phi} \psi \\ -e^{-2\Phi} \psi & -e^{-2\Phi} \end{pmatrix}, \\ Q &= \begin{pmatrix} w\psi - \varepsilon & -w \\ w & -a \end{pmatrix}. \end{aligned} \quad (19)$$

Here we have introduced the variable  $w = \chi - a\psi$ . Then, solving the quiral equation (17), we can find solutions of the EMDA theory.

### IV. THE HARMONIC MAPS ANSATZ FOR $Sp(4, \mathbb{R})$ -INVARIANT CHIRAL EQUATIONS

In this section, we apply the harmonic maps ansatz explained in the Appendix in order to solve the matrix equation (17). Metric (15) defines a target space where the covariant derivatives of the Riemann tensor are zero. Thus, following the method given in the Appendix, the Lie group element  $g \in Sp(4, \mathbb{R})$  of the topological Lie group  $Sp(4, \mathbb{R})$  can be parametrized in two variables  $\xi$  and  $\bar{\xi}$  as  $g = g(\xi, \bar{\xi})$ . We know that since  $Sp(4, \mathbb{R})$  is a linear subgroup of  $GL(n)$ , then the Maurer-Cartan form  $\omega_{MC}$  on the tangent space  $T_g(Sp(4, \mathbb{R}))$  of  $Sp(4, \mathbb{R})$ , can be defined by an element  $\mathbf{v}_g$  of  $T_g Sp(4, \mathbb{R})$  such that

$$\mathring{A} = \omega_{MC}(\mathbf{v}_g) = \mathbf{v}_g g^{-1}. \quad (20)$$

We can solve this equation to obtain

$$g_{,i} = \mathring{A}_i(g), \quad i = \xi, \bar{\xi}, \quad (21)$$

to get the matrix  $g \in G$ . It can be shown that if  $\mathring{A}$  is built as

$$\mathring{A}_i(g) = \sum_{j=1}^{\dim G_s} \mathring{\varepsilon}_i^{(k)} s_k, \quad (22)$$

being  $\mathring{\varepsilon}_i^{(k)}$  Killing vectors of the maximally symmetric space  $V_2$  with the two-dimensional metric

$$ds_{V_2}^2 = \frac{d\xi d\bar{\xi}}{V^2}, \quad (23)$$

where  $V = 1 + k\xi\bar{\xi}$  and  $s_k$  are the generators of the Lie

algebra  $\mathcal{G}_s$  of the submanifold  $G_s$  of  $Sp(4, \mathbb{R})$ . Then the element  $\mathbf{g} \in Sp(4, \mathbb{R})$  of the exponential equation (21) is a solution of the quiral equations (17) (see also the Appendix).

We find solutions of the EMDA problem, by solving Eqs. (21) in the two variables  $\xi$  and  $\bar{\xi}$ . In our present case, a representation of  $\mathbf{g}$  of the group  $Sp(4, \mathbb{R})$  is given by (18) and (19).

### V. ONE-DIMENSIONAL SUBSPACES

One-dimensional subspaces are the simplest subspaces to be handled and at the same time the richest ones. Therefore it is worth studying them with some deepness. In one dimension there is only one Killing vector, thus the Killing equation (21) reduces to solve the matrix equation

$$\mathbf{g}_{,\lambda} = \mathbf{A}\mathbf{g}, \quad (24)$$

where  $\lambda$  is the parameter solution of the Laplace equation in one dimension

$$(\rho\lambda_{,z})_{,\bar{z}} + (\rho\lambda_{,\bar{z}})_{,z} = 0 \quad (25)$$

and  $\mathbf{A} \in sp(4, \mathbb{R})$ , the corresponding Lie algebra of  $Sp(4, \mathbb{R})$ . Here, it is convenient to use the fact that the chiral equations are invariant under the left action of the group  $Sp(4, \mathbb{R})$ . Thus, if  $\mathbf{B}, \mathbf{D} \in Sp(4, \mathbb{R})$  we have that  $\mathbf{C} = \mathbf{B}\mathbf{A}\mathbf{B}^{-1}$  fulfils (24) with  $\mathbf{g}'_{,\lambda} = \mathbf{C}\mathbf{g}'$ , being  $\mathbf{g}' = \mathbf{B}\mathbf{g}\mathbf{D}$ . Then it is convenient to work with a representative of the equivalence class of  $\mathbf{A}$ . It is easy to see that there are only two independent representatives of the equivalence class such that  $\mathbf{A} \in sp(4, \mathbb{R})$ ; the first one is

$$\mathbf{A} = \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p^* & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p^* \end{pmatrix}. \quad (26)$$

With matrix  $\mathbf{A}$  the solution of Eq. (24) is

$$\mathbf{g} = \begin{pmatrix} Ae^{p\lambda} & 0 & 0 & 0 \\ 0 & \frac{1}{B}e^{q\lambda} & 0 & 0 \\ 0 & 0 & \frac{1}{A}e^{-p\lambda} & 0 \\ 0 & 0 & 0 & Be^{-q\lambda} \end{pmatrix}. \quad (27)$$

We compare (18) with (27) to get the potentials

$$\begin{aligned} f &= \frac{1}{A}e^{-p\lambda}, & \Phi &= \frac{1}{2}(q\lambda + \ln B), \\ \psi &= \epsilon = \chi = \mathbf{a} = 0. \end{aligned} \quad (28)$$

Now we give some examples. If we take the solution of the Laplace equations (25)  $\lambda = \lambda_0 \ln[1 - \frac{2m}{r}] + m_0$ , the potentials become

$$\begin{aligned} f &= \frac{1}{Ae^{pm_0}} \left(1 - \frac{2m}{r}\right)^{-p\lambda_0}, \\ \phi &= \frac{1}{2} \ln B + \frac{1}{2} q \left(m_0 + \lambda_0 \ln\left(1 - 2\frac{m}{r}\right)\right). \end{aligned} \quad (29)$$

The four-dimensional space-time metric for this solution is then

$$ds^2 = \frac{1}{f} [\hat{K} dr^2 + (r^2 - 2mr)(\hat{K} d\theta^2 + \sin^2(\theta) d\varphi^2)] - f dt^2, \quad (30)$$

where

$$\hat{K} = \left( \frac{(r-m)^2 - m^2 \cos^2(\theta)}{r^2 - 2mr} \right)^{k_0}.$$

For  $r \gg 1$  this solution has the asymptotic behavior given by

$$f \rightarrow 1 + \frac{2mp\lambda_0}{r} + 2m^2 p(1+p) \frac{1}{r^2} + \dots$$

and

$$\Phi \rightarrow \frac{1}{2} \ln B + \frac{1}{2} q m_0 - \frac{q\lambda_0 m}{r} + \dots$$

Another example is the following. We use now the harmonic map

$$\lambda = \lambda_0 \ln \left( \frac{r-m-\sqrt{m^2-\sigma^2}}{r-m+\sqrt{m^2-\sigma^2}} \right) + m_0.$$

In this case, solution (28) becomes

$$\begin{aligned} f &= \frac{1}{Ae^{pm_0}} \left( \frac{m-r+\sqrt{m^2-\sigma^2}}{m-r-\sqrt{m^2-\sigma^2}} \right)^{-p\lambda_0}, \\ \Phi &= \frac{q}{2} \left( \lambda_0 \ln \left( \frac{r-m-\sqrt{m^2-\sigma^2}}{r-m+\sqrt{m^2-\sigma^2}} \right) + m_0 \right) + \frac{1}{2} \ln B, \end{aligned} \quad (31)$$

and the four-dimensional space-time metric for this solution is

$$ds^2 = \frac{1}{f} [\hat{K} dr^2 + (r^2 - 2mr + \sigma^2)(\hat{K} d\theta^2 + \sin^2(\theta) d\varphi^2)] - f dt^2, \quad (32)$$

with

$$\hat{K} = \left( \frac{(r-m)^2 + (\sigma^2 - m^2) \cos^2(\theta)}{r^2 - 2mr + \sigma^2} \right)^{k_0}.$$

Here, the asymptotic behavior for  $r \gg 1$  is given by

$$f \rightarrow 1 + \frac{2p\lambda_0 \sqrt{m^2 - \sigma^2}}{r} + \dots$$

and

$$\Phi \rightarrow \frac{1}{2} \ln B + \frac{1}{2} q m_0 - \frac{q\lambda_0 \sqrt{m^2 - \sigma^2}}{r} + \dots,$$

where we have set  $Ae^{pm_0} = 1$ . We can use more harmonic maps in order to find more exact solutions.

In what follows, we study the second representative of  $\mathbf{A} \in sp(4, \mathbb{R})$ , given by

$$A = \begin{pmatrix} p & 1 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & -1 & -p \end{pmatrix}. \quad (33)$$

With this representative we obtain the solution

$$g = \begin{pmatrix} (a\lambda - a^2c)e^{p\lambda} & ae^{p\lambda} & 0 & 0 \\ ae^{p\lambda} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{a}e^{-p\lambda} \\ 0 & 0 & \frac{1}{a}e^{-p\lambda} & (-\frac{1}{a}\lambda + c)e^{-p\lambda} \end{pmatrix} \quad (34)$$

to obtain the potentials

$$f = \frac{e^{-p\lambda}}{a(\lambda - ac)}, \quad \phi = \frac{1}{2} \left[ p\lambda - \ln\left(\frac{1}{a}\lambda - c\right) \right], \quad (35)$$

$$\psi = -\frac{1}{\lambda - ac}, \quad \epsilon = \chi = a = 0.$$

This solution contains gravitational, dilaton, and electrostatic fields; it represents a charged, dilatonic space-time. Nevertheless, in these two solutions (28) and (35), the axion field is zero. In order to find solutions with a nonzero axion field we perform the following procedure. Because the chiral equations are invariant under the left action of the group, we can perform a rotation  $g' \rightarrow CgC^T$ , where  $C^T$  means transpose of C. We start with the matrix

$$C = \begin{pmatrix} c & 0 & -b & 0 \\ 0 & c & 0 & -d \\ \frac{1}{b} & 0 & 0 & 0 \\ 0 & \frac{1}{d} & 0 & 0 \end{pmatrix} \in Sp(4, \mathbb{R}) \quad (36)$$

and apply the left action of the group to the first representative (26). If we do so, we obtain

$$g' = \begin{pmatrix} \frac{1}{A}e^{-p\lambda}(b^2 + A^2c^2e^{2p\lambda}) & 0 & \frac{A}{b}ce^{p\lambda} & 0 \\ 0 & \frac{1}{B}e^{q\lambda}(c^2 + B^2d^2e^{-2q\lambda}) & 0 & \frac{1}{B}\frac{c}{d}e^{q\lambda} \\ \frac{A}{b}ce^{p\lambda} & 0 & \frac{A}{b^2}e^{p\lambda} & 0 \\ 0 & \frac{1}{B}\frac{c}{d}e^{q\lambda} & 0 & \frac{1}{Bd^2}e^{q\lambda} \end{pmatrix}. \quad (37)$$

With matrix  $g'$  the physical potentials are

$$f = \frac{Ae^{-p\lambda}}{A^2c^2 + b^2e^{-2p\lambda}}, \quad \epsilon = -\frac{A^2ce^{2p\lambda}}{b^3 + A^2bc^2e^{2p\lambda}},$$

$$e^{2\phi} = -\frac{1}{B}e^{q\lambda}(c^2 + B^2d^2e^{-2q\lambda}),$$

$$a = -\frac{c}{c^2d + B^2d^3e^{-2q\lambda}}, \quad w = \psi = \chi = 0. \quad (38)$$

Solution (38) represents a rotating, dilatonic solution coupled to an axion field. We show an example using the harmonic map  $\lambda = m_0 + \lambda_0 \ln(1 - \frac{2m}{r})$ . Substituting this  $\lambda$  into the solution (38), we obtain

$$f = \frac{AL_p^2}{b^2 + A^2c^2L_p^2}, \quad \epsilon = -\frac{A^2cL_p^2}{b^3 + A^2bc^2L_p^2},$$

$$e^{2\phi} = -\frac{1}{B}c^2L_q - B\frac{d^2}{L_q}, \quad a = -\frac{cL_q^2}{B^2d^3 + c^2dL_q^2},$$

$$w = \psi = \chi = 0, \quad (39)$$

where

$$L_p = e^{pm_0} \left(1 - \frac{2m}{r}\right)^{p\lambda_0}.$$

The four-dimensional space-time metric for this solution is

$$ds^2 = \frac{1}{f} [\hat{K}dr^2 + (r^2 - 2mr + \sigma^2)(\hat{K}d\theta^2 + \sin^2(\theta)d\varphi^2)] - f(dt + a \cos(\theta)d\varphi)^2, \quad (40)$$

where

$$\hat{K} = \left(\frac{(r - m)^2 - m^2\cos^2(\theta)}{r^2 - 2mr}\right)^{k_0}.$$

The asymptotic behavior for this solution ( $r \gg 1$ ) is given by

$$f \rightarrow 1 + \frac{4b^2mp\lambda_0e^{-2pm_0}}{Ar} + O(r^{-2}),$$

$$\epsilon \rightarrow -\frac{Ac}{b} - \frac{4bcmpl_0e^{-2pm_0}}{r} + O(r^{-2}),$$

$$e^{2\phi} \rightarrow -Bd^2e^{-\bar{p}m_0} - \frac{c^2e^{\bar{p}m_0}}{B} - \frac{2re^{\bar{p}m_0}m\bar{p}\lambda_0}{B} + \frac{2Bd^2e^{-\bar{p}m_0}m\bar{p}\lambda_0}{r} + O(r^{-2}), \quad (41)$$

$$a \rightarrow -\frac{ce^{2\bar{p}m_0}}{B^2d^3 + c^3de^{2\bar{p}m_0}} - \frac{4B^2cde^{2\bar{p}m_0}m\bar{p}\lambda_0}{(B^2d^2 + c^3e^{2\bar{p}m_0})^2r} + O(r^{-2}),$$

where  $\frac{Ae^{2pm_0}}{b^2 + A^2c^2e^{2pm_0}} = 1$ . If we set  $k_0 = 0$  and

$$M = -\frac{2b^2mp\lambda_0e^{-2pm_0}}{A},$$

solution (39) can be seen as a generalization of the Schwarzschild space-time with rotation, dilaton, and axion fields. Nevertheless, this solution is asymptotically flat only if  $a = 0$ , when the solution becomes static.

In the same way we can apply the left action of the group to the second representative (33). We use now the matrix

$$C = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \in Sp(4, \mathbb{R}). \quad (42)$$

After the transformation  $g' \rightarrow CgC^T$ , we obtain

$$g = \begin{pmatrix} \frac{1}{A}(ABe^{-p\lambda} - \lambda e^{-p\lambda}) & \frac{1}{A}e^{-p\lambda}(-A^2e^{2p\lambda} - 1) & 0 & -Ae^{p\lambda} \\ \frac{1}{A}e^{-p\lambda}(-Aae^{2p\lambda} - 1) & A\lambda e^{p\lambda} - A^2Be^{p\lambda} & -ae^{p\lambda} & A\lambda e^{p\lambda} - A^2Be^{p\lambda} \\ 0 & -Ae^{p\lambda} & 0 & -Ae^{p\lambda} \\ -ae^{p\lambda} & A\lambda e^{p\lambda} - A^2Be^{p\lambda} & -ae^{p\lambda} & A\lambda e^{p\lambda} - A^2Be^{p\lambda} \end{pmatrix}. \quad (43)$$

With this matrix we find the potentials

$$\begin{aligned} f &= \frac{Ae^{p\lambda}}{AB - \lambda}, \quad \epsilon = 0, \\ \psi &= \frac{1 - A^2e^{2p\lambda}}{AB - \lambda}, \quad \chi = -\frac{A^2e^{2p\lambda}}{AB - \lambda}, \\ e^{2\phi} &= \frac{(A(2AB - \lambda)\lambda - A^2(2 + AB^2))e^{2p\lambda} - A^3e^{4p\lambda} - A}{(\lambda - AB)e^{p\lambda}}, \\ a &= \frac{A^2((2AB - \lambda)\lambda - (1 + A^2B^2)) - A^4e^{2p\lambda}}{(A^4 + 1)e^{2p\lambda} - A^2((2AB - \lambda)\lambda - 2 - A^2B^2)}. \end{aligned} \quad (44)$$

Metric (44) represents a dilatonic static space-time coupled to an axion field, with electric and magnetic charges. We can see explicitly this metric using some harmonic map  $\lambda$ . Again we only give an example with the harmonic map  $\lambda = m_0 + \lambda_0 \ln(1 - \frac{2m}{r})$ , using this in the solution we find

$$\begin{aligned} f &= \frac{AL_p}{AB - m_0 - \lambda_0 \ln(1 - \frac{2m}{r})}, \\ \epsilon &= 0, \quad \psi = \frac{1 - A^2L_p^2}{AB - m_0 - \lambda_0 \ln(1 - \frac{2m}{r})}, \\ \chi &= -\frac{A^2L_p^2}{AB - m_0 - \lambda_0 \ln(1 - \frac{2m}{r})}, \\ e^{2\phi} &= \frac{(AL_x - A^2(2 + AB^2))L_p^2 - A^3L_p^4 - A}{(m_0 + \lambda_0 \ln(1 - \frac{2m}{r}) - AB)L_p}, \\ a &= \frac{A^2(L_x - (1 + A^2B^2)) - A^4L_p^2}{(A^4 + 1)L_p^2 - A^2(L_x - 2 - A^2B^2)}, \end{aligned} \quad (45)$$

where

$$\begin{aligned} L_x &= \left(2AB - m_0 - \lambda_0 \ln\left(1 - \frac{2m}{r}\right)\right) \\ &\times \left(m_0 + \lambda_0 \ln\left(1 - \frac{2m}{r}\right)\right). \end{aligned} \quad (46)$$

The asymptotic behavior for  $r \gg 1$  for this solution is as follows. It is convenient to chose  $A = \frac{m_0}{B - e^{pm_0}}$ . In this case, we have that

$$f \rightarrow 1 - 2 \frac{\lambda_0 m((m_0 p - 1)e^{pm_0} + B)}{m_0 e^{pm_0}} \frac{1}{r}.$$

Again, if we define the mass parameter  $M$  of this solution as

$$M = \frac{\lambda_0 m((m_0 p - 1)e^{pm_0} + B)}{m_0 e^{pm_0}},$$

the solution can be seen also as a generalization of the Schwarzschild space-time. In this case, this solution has an electric monopole charge  $Q$

$$\begin{aligned} Q &= 2 \frac{\lambda_0 m e^{-pm_0}}{m_0^2 (B - e^{pm_0})} [(1 + 2m_0^3 p - m_0^2) e^{2pm_0} \\ &- B((3 - m_0^2) e^{pm_0} + e^{-pm_0} B^2 - 3B)], \end{aligned} \quad (47)$$

a dilatonic charge  $Q_D$  given by

$$\begin{aligned} Q_D &= \frac{m\lambda_0}{m_0(B^2 + (2m_0e^{2pm_0} - 2e^{pm_0})B + 2m_0^2e^{4pm_0} - 2m_0e^{3pm_0} + e^{2pm_0})} \\ &\times [-e^{-pm_0}B^3 + (3 + (p - 2e^{pm_0})m_0)B^2 + (-2e^{2pm_0}pm_0^2 + (4e^{2pm_0} - 2pe^{pm_0})m_0 - 3e^{pm_0})B \\ &- 4e^{4pm_0}pm_0^3 + 2e^{3pm_0}m_0^2p + (-2e^{3pm_0} + e^{2pm_0}p)m_0 + e^{2pm_0}], \end{aligned} \quad (48)$$

and finally an axion  $Q_a$  charge such that

$$Q_a = \frac{4m_0^2 m \lambda_0 e^{pm_0}}{(e^{2pm_0} B^4 - 4e^{3pm_0} B^3 + (6e^{4pm_0} + 2m_0^2) B^2 + (-4e^{5pm_0} - 4m_0^2 e^{pm_0}) B + 2(m_0^2 + 1) e^{2pm_0} m_0^2 + e^{6pm_0})^2} \times [B^6 e^{pm_0} p + (m_0 - 6p) e^{2pm_0} B^5 + (15p - 5m_0 + m_0^2 p) e^{3pm_0} B^4 + ((10m_0 - 20p - 4m_0^2 p) e^{4pm_0} + m_0^3) B^3 + ((6m_0^2 p - 10m_0 + 15p) e^{5pm_0} - (m_0 p + 3) m_0^3 e^{pm_0}) B^2 + (- (6p - 4m_0^2 p - 5m_0) e^{6pm_0} + (3 + 2m_0 p) m_0^3 e^{2pm_0}) B + (e^{7pm_0} m_0^2 p + p - m_0) e^{7pm_0} - (1 + m_0 p) m_0^3 e^{3pm_0}]. \tag{49}$$

In order to see the physical behavior of this solution, we take the very simple choice  $m_0 = 0, B = 1$ . For this case, the asymptotic behavior of the solutions for  $r \gg 1$  is

$$\begin{aligned} f &= 1 - \frac{2\lambda_0 m (Ap + 1)}{A} \frac{1}{r} + O(r^{-2}), \\ \psi &= -\frac{(A^2 - 1)}{A} + 2 \frac{\lambda_0 m (2A^3 p + A^2 - 1)}{A^2} \frac{1}{r} + O(r^{-2}), \\ \Phi &= \frac{1}{2} \ln(1 + 2A + 2A^2) \\ &\quad - \frac{\lambda_0 m (2A^2 p + 4A^3 p + 1 + 2A - pA)}{A(1 + 2A + 2A^2)} \frac{1}{r} + O(r^{-2}), \\ a &= -\frac{A^2(1 + 2A^2)}{2A^4 + 1 + 2A^2} \\ &\quad + 4 \frac{\lambda_0 m A^2 (-A - A^3 - A^2 p + pA^4 - p)}{(1 + 2A^2 + 2A^4)^2} \frac{1}{r} + O(r^{-2}). \end{aligned} \tag{50}$$

This behavior shows the quantities we can consider to be the physical parameters of the solution. This metric is an asymptotically flat generalization of the Schwarzschild space-time, it is static and contains electromagnetic, scalar, and axion parameters.

We can choose now other harmonic maps to generate more solutions, but the important point is that we can generate solutions with physical features we want to have. The same situation happens with the two-dimensional subspaces that we will study in the next section.

### VI. TWO-DIMENSIONAL SUBSPACES: THE SUBGROUP $SO(2, 1)$

For this subgroup we start with the base of the algebra  $so(2, 1)$  given by

$$\sigma_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{51}$$

We take the following Killing basis of the maximally symmetric space  $V_2$ :

$$\begin{aligned} \overset{1}{\zeta} &= \frac{C_1}{V^2} (\chi k \bar{\xi}^2 + \bar{\chi}, \bar{\chi} k \xi^2 + \chi); & \overset{2}{\zeta} &= \frac{iC_2}{V^2} \not{d}(-\bar{\xi}, \xi); \\ \overset{3}{\zeta} &= \frac{C_3}{V^2} (\bar{\chi} k \bar{\xi}^2 + \chi, \chi k \xi^2 + \bar{\chi}), \end{aligned} \tag{52}$$

where  $\chi = \not{k} + i\not{k} \in \mathbb{C}; \not{k} \in \mathbb{R}$ . Now we choose the Maurer-Cartan form as

$$\overset{\circ}{\mathbf{A}}(\pi_3) = \overset{2}{\zeta} \sigma_1 + \overset{1}{\zeta} \sigma_2 + \overset{3}{\zeta} \sigma_3. \tag{53}$$

The integrability condition  $\mathbf{g}_{,\xi\bar{\xi}} = \mathbf{g}_{,\bar{\xi}\xi}$  for  $\mathbf{g} \in Sp(4, \mathbb{R})$  is fulfilled provided that the constants  $C_j, j = 1, 2, 3$  are restricted to

$$C_1 = -\frac{i}{\not{k}} \sqrt{\frac{k}{2}}, \quad C_2 = -\frac{2k}{\not{d}}, \quad C_3 = -\frac{i}{\not{k}} \sqrt{\frac{k}{2}}. \tag{54}$$

Thus after solving (21), we find a solution of the potential matrix  $\mathbf{g}$ , given in (18) and (19). Remember the fact that  $\mathbf{g}$  is a real and symmetric matrix, thus the conditions for symmetry  $\mathbf{g} = \mathbf{g}^T$  and reality  $\mathbf{g} \in Sp(4, \mathbb{R})$  must be taken into account. With this in mind, we get a solution for the potential matrix  $\mathbf{g}$ , to obtain

$$\mathbf{g} = \begin{pmatrix} \Xi & \Pi & 0 & \Pi \\ \Pi & \Xi & \Pi & 0 \\ 0 & \Pi & \Xi & \Pi \\ \Pi & 0 & \Pi & \Xi \end{pmatrix}, \tag{55}$$

where

$$\Xi = \frac{1 - k\xi\bar{\xi}}{1 + k\xi\bar{\xi}}$$

and

$$\Pi = \sqrt{\frac{k(1-i)\xi - (1+i)\bar{\xi}}{2(1+k\xi\bar{\xi})}}.$$

With this solution we find the following set of potentials:

$$\begin{aligned} f &= \frac{1 - \xi\bar{\xi}}{1 + \xi\bar{\xi}}, \quad \epsilon = 0, \\ \psi &= \frac{i}{\sqrt{2}} \frac{((i-1)\xi + (1+i)\bar{\xi})}{1 + \xi\bar{\xi}}, \\ \chi &= -\frac{1}{W} (\xi + \bar{\xi} + i\xi\bar{\xi}^2 + i\bar{\xi} - i\xi + \xi^2\bar{\xi} + \xi\bar{\xi}^2 \\ &\quad - i\bar{\xi}\xi^2 - i\xi^3) - \frac{1}{W} (\bar{\xi}^3\xi^2 + i\bar{\xi}^3 + \xi^3\bar{\xi}^2 - i\xi^3\bar{\xi}^2 \\ &\quad - \xi^3 - \bar{\xi}^3 + i\xi^2\bar{\xi}^3), \\ a &= -\frac{\xi^2 + \bar{\xi}^2}{1 + \xi^2\bar{\xi}^2 - i\xi^2 + i\bar{\xi}^2}, \\ \Phi &= -\frac{1}{2} \ln\left(\frac{1 - \xi^2\bar{\xi}^2}{-1 - \xi^2\bar{\xi}^2 + i\xi^2 - i\bar{\xi}^2}\right), \end{aligned} \quad (56)$$

where  $W = \sqrt{2}(-i\xi^3\bar{\xi} + i\bar{\xi}^2 - i\xi^2 + \xi^2\bar{\xi}^2 + i\xi\bar{\xi}^3 + \xi^3\bar{\xi}^3 + \xi\bar{\xi} + 1)$ . The functions  $\xi$  and  $\bar{\xi}$  are solutions of the two-dimensional harmonic maps equations, that is, of the Ernst equations [11]. We show an example using the Ernst potential for the Kerr solution, with the help of an harmonic map defined in terms of the Ernst's potential

$$\mathcal{E} = \frac{1 - \xi}{1 + \xi}, \quad (57)$$

by

$$\mathcal{E} = 1 + \frac{2q}{r - il \cos(\theta)}. \quad (58)$$

We find the solution of the EMDA field equations such that

$$\begin{aligned} ds^2 &= f^{-1} \left( dt - \frac{2q^2 l \sin(\theta)^2}{r^2 + 2qr + l^2 \cos(\theta)^2} d\varphi \right)^2 \\ &\quad - f \left\{ \left( 1 + \frac{q^2 \sin(\theta)^2}{r^2 + 2qr + l^2 \cos(\theta)^2} \right)^{-2} \right. \end{aligned} \quad (59)$$

$$\left. \left( (r+q)^2 + (l^2 - q^2) \cos(\theta)^2 \right) \left( \frac{dr^2}{\Delta} + d\theta^2 \right) + \Delta \sin(\theta)^2 d\varphi^2 \right\}, \quad (60)$$

where  $\Delta$  designs the horizon function, which is defined by

$$\Delta(r) \equiv r^2 + 2qr + l^2,$$

and  $f$  is the gravitational potential which reads

$$f = \left( 1 + \frac{2q^2}{r^2 + 2qr + l^2 \cos(\theta)^2} \right).$$

The harmonic map which is responsible of this solution is given by (57) and (58), or by

$$\xi = \frac{m}{R}, \quad (61)$$

where  $R \equiv r + q - il \cos(\theta) \in \mathbb{C}$ . This harmonic map satisfies the harmonic equations, which in complex coordinates read

$$(\rho\xi_{,z})_{,\bar{z}} + (\rho\xi_{,\bar{z}})_{,z} + 2\rho\Gamma_{\xi\xi}^{\xi} \xi_{,z}\xi_{,\bar{z}} = 0, \quad (62)$$

where  $\Gamma_{\xi\xi}^{\xi}$  are the affin connection of the auxiliar space (23).

The other interesting aspect of the solution is the electromagnetic field, that is,

$$\begin{aligned} E_r &= -\frac{q(r^2 + 2qr - 2l \cos(\theta)(r+q) - l^2 \cos^2(\theta))}{(r^2 + 2qr + 2q^2 + l^2 \cos^2(\theta))^2}, \\ E_\theta &= \frac{ql \sin(\theta) Y_+}{(r^2 + 2qr + 2q^2 + l^2 \cos^2(\theta))^2}, \quad E_\varphi = -0, \\ B_r &= \frac{(r^2 + 2qr + l^2 + 2q^2)q \sin(\theta) Y_-}{(r^2 + 2qr + 2q^2 + l^2 \cos^2(\theta))^2}, \\ B_\theta &= \frac{((r+2q)r - 2a(r+q) \cos(\theta) - l^2 \cos^2(\theta))ql \sin^2(\theta)}{(r^2 + 2qr + 2q^2 + l^2 \cos^2(\theta))^2}, \\ B_\varphi &= -0, \end{aligned} \quad (63)$$

being

$$Y_{\pm} = r^2 + 2qr + 2q^2 - l^2 \cos^2(\theta) \pm 2l(r+q) \cos(\theta).$$

Using the Komar integrals we can find the electric and the magnetic charges of the solution. If the electric charge is  $q$  we find that the magnetic monopolar charge is  $p = -q$ , which tell us that the solution is a dyon. The parameter  $l$ , which is responsible of the stationarity of the metric is the parameter of dipolar electric moment.

Another feature of this solution is that it is asymptotically flat. The Komar mass is null  $M = 0$  and its angular moment  $J = 0$  too. Of course, this analysis is valid only from the point of view of an observer that is far away from the source of the fields.

The solution contains two singularities with two regions separated in two geometric places:

- (1) the exterior singularity

$$r = q + \sqrt{q^2 - l^2 \cos(\theta)^2},$$

- (2) the interior singularity

$$r = q - \sqrt{q^2 - l^2 \cos(\theta)^2},$$

with horizons in



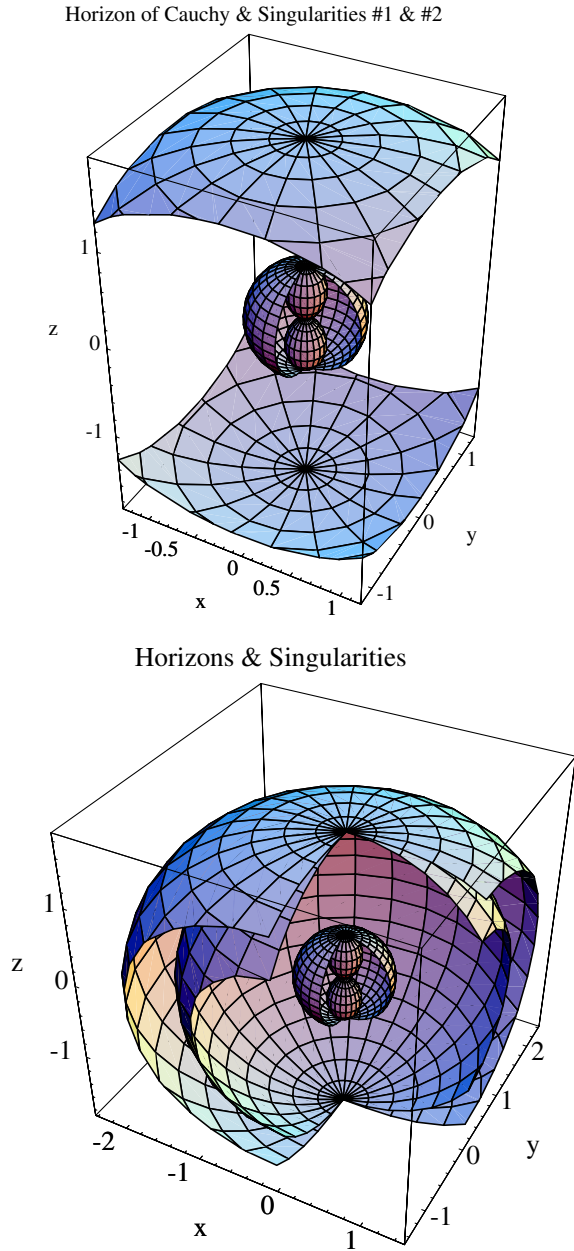


FIG. 1 (color online). Diagrams showing several views of the horizons (spherical surfaces)  $r = m \pm \sqrt{m^2 - a^2}$  and the singularities (external surface like an ellipsoid #1, and internal surface #2)  $r = m \pm \sqrt{m^2 - a^2 \cos^2(\theta)}$ , for the values  $m = 1.18$  and  $a = 1$  in spherical coordinates.

(1) the exterior horizon (events)

$$r_+ = q + \sqrt{q^2 - l^2},$$

(2) the interior horizon (Cauchy)

$$r_- = q - \sqrt{q^2 - l^2}.$$

See Fig. 1. The surface gravity of the exterior horizon is given by

$$\kappa = \frac{1}{\sqrt{2}} \frac{m^2 - a^2}{m^2 a},$$

which tell us that it is a regular events horizon.

The solution is then a dyon, which represents a collapse of electromagnetic charges. That latter fact follows from the nature of the coupling between gravity and the two scalar fields: the dilaton and the Pecci-Quinn pseudoscalar or axion.

## VII. CONCLUSIONS

The harmonic maps ansatz is an excellent tool for finding exact solutions of systems of nonlinear partial differential equations [7], in particular, this method has been very useful in solving the chiral equations derived from a nonlinear  $\sigma$  model [12]. Einstein equations in vacuum can be reduced to a nonlinear  $\sigma$  model with structural group  $SL(2, \mathbb{R})$  in the space-time and to a structural group  $SU(1, 1)$  in the potential spaces, i.e. in terms of the Ernst potentials. The electrovacuum case can also be reduced to a nonlinear  $\sigma$  model with structural group  $SU(2, 1)$  in terms of the extended Ernst potentials [6,11]. The Kaluza-Klein field equations can also be written as a  $SL(3, \mathbb{R})$  nonlinear  $\sigma$  model in the space-time as well as in the potential space [7,13]. This is possible because the corresponding potential space is a symmetric Riemannian space only for  $\alpha = 0$  and  $\alpha = \sqrt{3}$ , but this is not the case for the low energy limit in superstrings or the Maxwell-phantom theories. In [8], we extended this method [7,14] to the Einstein-Maxwell-dilaton fields with arbitrary  $\alpha$  and in this work we use this technique for the Einstein-Maxwell-dilaton-axion fields with the invariant group  $Sp(4, \mathbb{R})$ . With this method we were able to obtain exact solutions of the EMDA field equations for the one- and two-dimensional subgroups of  $Sp(4, \mathbb{R})$ . The method is very powerful, it makes possible to generalize the Schwarzschild space-time and to obtain solutions which represent magnetic and electric monopoles, dipoles, dyons, etc., coupled to gravitational monopoles, dipoles and to different multipoles of the scalar fields.

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**APPENDIX: THE HARMONIC MAPS ANSATZ**

In this appendix, we follow [12] in order to apply the method of harmonic maps to the EMDA field equations. Let  $g$  be a map defined by

$$g: \mathcal{C} \otimes \bar{\mathcal{C}} \rightarrow G \quad g \rightarrow g(z, \bar{z}) \in G, \quad (68)$$

where  $G$  is a paracompact Lie group and  $g$  fulfils the field equations derived from the Lagrangian

$$\mathcal{L} = \alpha \operatorname{tr}(g_{,z}g^{-1}g_{,\bar{z}}g^{-1}). \quad (A1)$$

The field equations are called the chiral equations for  $g$ , in explicit form they are given by

$$(\alpha g_{,z}g^{-1})_{,\bar{z}} + (\alpha g_{,\bar{z}}g^{-1})_{,z} = 0 \quad (A2)$$

where  $\alpha^2 = \det g$ .

Lagrangian (A1) represents a topological quantum field theory with gauge group  $G$ . In what follows, we give a method to find explicit expressions of the elements  $g \in G$  in terms of the local coordinates  $z$  and  $\bar{z}$ .

Let  $G_c$  be a subgroup  $G_c \subset G$  such that  $c \in G_c$  implies  $c_{,z} = 0, c_{,\bar{z}} = 0$ . Then Eq. (A2) is invariant under the left action  $L_c$  of  $G_c$  over  $G$ .

Proposition 1. Let  $\beta$  be a complex function defined by

$$\beta_{,z} = \frac{1}{4(\ln\alpha)_{,z}} \operatorname{tr}(g_{,z}g^{-1})^2, \quad g \in G \quad (A3)$$

and  $\beta_{,\bar{z}}$  with  $\bar{z}$  instead of  $z$ . If  $g$  fulfils the chiral equations, then  $\beta$  is integrable.

Proof. It is sufficient to calculate the identity  $\beta_{,z\bar{z}} = \beta_{,\bar{z}z}$ . To show this, we see that

$$\begin{aligned} \beta_{,z\bar{z}} &= \frac{1}{4} \operatorname{tr} \left[ \frac{1}{\alpha_{,z}} (\alpha g_{,z}g^{-1})_{,\bar{z}} g_{,z}g^{-1} + \frac{1}{\alpha_{,z}} \alpha g_{,z}g^{-1} g_{,z\bar{z}}g^{-1} \right. \\ &\quad \left. - \frac{1}{\alpha_{,z}} \alpha g_{,z}g^{-1} g_{,z}g^{-1} g_{,\bar{z}}g^{-1} \right. \\ &\quad \left. - \frac{\alpha_{,z\bar{z}}}{(\alpha_{,z})^2} \alpha g_{,z}g^{-1} g_{,z}g^{-1} \right] \\ &= \frac{1}{4} \frac{1}{\alpha_{,z}} \operatorname{tr} [(\alpha g_{,z}g^{-1})_{,\bar{z}} + (\alpha g_{,\bar{z}}g^{-1})_{,z} \\ &\quad - \alpha_{,z} g_{,\bar{z}}g^{-1}] g_{,z}g^{-1} \end{aligned}$$

but the matrices in the trace can be commutated. Thus, if  $g$  fulfils the chiral equations, we have

$$\beta_{,z\bar{z}} = -\frac{1}{4} \operatorname{tr}[g_{,\bar{z}}g^{-1}g_{,z}g^{-1}].$$

Let  $\mathcal{G}$  be the corresponding Lie algebra of  $G$ . The Maurer-Cartan form  $\omega_g$  of  $G$  defined by

$$\omega_g = L_{g^{-1}*}(g)$$

is a 1-form on  $G$  with values on  $\mathcal{G}$ ,  $\omega_g \in T_g^*G \otimes \mathcal{G}$ , where  $T_x^*M$  represents the tangent space of the manifold  $M$  at the point  $x$  and  $L$  is the left action of  $G$  over  $G$ ,  $L: G \otimes G \rightarrow G$ .

$L$  must be defined in a convenient manner in order to preserve the properties of the elements of  $G$ . Now we define the mappings

$$\begin{aligned} A_z: G \rightarrow \mathcal{G}, \quad g \rightarrow A_z(g) &= g_{,z}g^{-1}, \\ A_{\bar{z}}: G \rightarrow \mathcal{G}, \quad g \rightarrow A_{\bar{z}}(g) &= g_{,\bar{z}}g^{-1}. \end{aligned} \quad (A4)$$

If  $g$  is given in a representation of  $G$ , then we can write the 1-form  $\omega(g) = \omega_g$  as

$$\omega = A_z dz + A_{\bar{z}} d\bar{z}. \quad (A5)$$

We can now define a metric on  $\mathcal{G}$  in a standard manner. Since  $\omega_g$  can be written as in (A5), the tensor

$$l = \operatorname{tr}(dg g^{-1} \otimes dg g^{-1}) \quad (A6)$$

on  $G$  defines a metric on the tangent bundle of  $G$ .

Theorem 1. The submanifold of solutions of the chiral equations  $S \subset G$ , is a symmetric manifold with metric (A6).

Proof. We will only outline here the proof. We take a parametrization  $\lambda^a$   $a = 1, \dots, n$  of  $G$ . The set  $\{\lambda^a\}$  is a local coordinate system of the  $n$ -dimensional differential manifold  $G$ . In terms of this parametrization the Maurer-Cartan 1-form  $\omega$  can be written as

$$\omega = A_a d\lambda^a, \quad (A7)$$

where  $A_a(g) = (\frac{\partial}{\partial \lambda^a} g)g^{-1}$ . The chiral equations then read

$$\nabla_b A_a(g) + \nabla_a A_b(g) = 0, \quad (A8)$$

with  $\nabla_a$  the covariant derivative defined by (A6).

It follows the relation

$$\nabla_b A_a(g) = \frac{1}{2} [A_a, A_b](g). \quad (A9)$$

Thus the Riemannian curvature  $\mathcal{R}$  can be derived from (A6), their components read

$$R_{abcd} = \frac{1}{4} \operatorname{tr}(A_{[a} A_b] A_{[c} A_d]), \quad (A10)$$

where  $[a, b]$  means index commutation. This can be done, because  $G$  is a paracompact manifold. From here it follows that  $\nabla \mathcal{R} = 0$ .

Proposition 2. The function  $\alpha^2 = \det g$  is harmonic.

Proof. Using the formulas  $\operatorname{tr}(A_{,x} A^{-1}) = \ln(\det A)_{,x}$ , we can see that the trace of the chiral equations implies  $\alpha_{,z\bar{z}} = 0$ .

Let  $V_p$  be a complete totally geodesic submanifold of  $G$  and let  $\{\lambda^i\} i = 1, \dots, p$  be a set of local coordinates on  $V_p$  and suppose we completely know the submanifold  $V_p$ . It is clear that the submanifold  $V_p$  is also symmetric. The symmetries of  $G$  and  $V_p$  are in fact isometries, since both of them are paracompact manifolds, with Riemannian metrics (A6) and  $i_* l$ , respectively, where  $i$  is the restriction of  $V_p$  into  $G$ . Let us suppose that  $V_p$  possesses  $d$  isometries. The chiral equations imply

$$(\alpha \lambda^i_{,z})_{,\bar{z}} + (\alpha \lambda^i_{,\bar{z}})_{,z} + 2\alpha \sum_{ijk} \Gamma^i_{jk} \lambda^j_{,z} \lambda^k_{,\bar{z}} = 0, \quad (\text{A11})$$

$$i, j, k = 1, \dots, p,$$

where  $\Gamma^i_{jk}$  are the Christoffel symbols of  $i_*l$  and  $\lambda^i$  are the totally geodesic parameters on  $V_p$ . In terms of the parameters  $\lambda^i$  the chiral equations read

$$\nabla_j A_j(g) + \nabla_j A_i(g) = 0, \quad (\text{A12})$$

where  $\nabla_i$  is the covariant derivative of  $V_p$ . Equation (A12) is the Killing equation on  $V_p$  for the components of  $A_i$ . Since we know the manifold  $V_p$ , we know its isometries and therefore its Killing-vector space. Let  $\xi_s$ ,  $s = 1, \dots, d$ , be a base of the Killing-vector space of  $V_p$  and  $\Gamma^s$  be a base of the subalgebra corresponding to  $V_p$ . Then we can write

$$A_i(g) = \sum_s \xi_s^i \Gamma^s, \quad (\text{A13})$$

where  $\xi_s = \sum_j \xi_s^j \frac{\partial}{\partial \lambda^j}$ . The covariant derivative on  $V_p$  is given by

$$\nabla_j A_i(g) = -\frac{1}{2} [A_i, A_j](g), \quad (\text{A14})$$

where  $A_i$  fulfills the integrability conditions

$$F_{ij} = \nabla_j A_i(g) - \nabla_i A_j(g) - [A_j, A_i](g) = 0, \quad (\text{A15})$$

i.e.,  $A_i$  has a pure gauge form.

Thus, because we know  $\{\xi_s\}$  and  $\{\Gamma^s\}$  we can integrate the elements of  $S$ , since  $A_i(g) \in \mathcal{G}$  can be mapped into the group by means of the exponential map. Nevertheless it is not possible to map all the elements one by one. Fortunately we have the following proposition.

Proposition 3. The relation  $A_i^c \sim A_i$  iff there exist  $c \in G_c$  such that  $A^c = A \circ L_c$ , is an equivalence relation.

This equivalence relation separates the set  $\{A_i\}$  into equivalence classes  $[A_i]$ . Let  $TB$  be a set of representatives of each class,  $TB = \{[A_i]\}$ . Now we map the elements of  $TB \subset \mathcal{G}$  into the group  $S$  by means of the exponential map or by integration. Let us define  $B$  as the set of elements of the group, mapped from each representative

$$B = \{g \in S | g = \exp(A_i), A_i \in TB\} \subset G.$$

The elements of  $B$  are also elements of  $S$  because  $A_i$  fulfills

the chiral equations, i.e.  $B \subset S$ . For constructing all the set  $S$  we have the following theorem.

Theorem 2.  $(S, B, \pi, G_c, L)$  is a principal fibre bundle with projection  $\pi(L_c(g)) = g$ ;  $(c, g) = L_c(g)$ .

Proof. The fibres of  $G$  are the orbits of the group  $G_c$  on  $G$ ,  $F_g = \{g' \in G | g' = L_c(g)\}$  for some  $g \in B$ . The topology of  $B$  is its relative topology with respect to  $G$ . Let  $\mathcal{B}_F$  be the bundle  $\mathcal{B}_F = (G_c \times U_\alpha, U_\alpha, \pi)$ , where  $\{U_\alpha\}$  is an open covering of  $B$ . We have the following lemma.

Lemma 1. The bundle  $\mathcal{B}_F$  and

$$\mathcal{B} = (\pi^{-1}(U_\alpha), U_\alpha, \pi|_{\pi^{-1}(U_\alpha)})$$

are isomorphic.

Proof. The mapping

$$\begin{aligned} \psi_\alpha: \phi^{-1}(U_\alpha) &= \{g \in S | g' = L_c(g), g \in U_\alpha\}_{c \in G_c} \\ &\rightarrow G_c \times U_\alpha, \\ g' &\rightarrow \psi_\alpha(g') = (c, g) \end{aligned}$$

is a homeomorphism and

$$\pi|_{\pi^{-1}(U_\alpha)}(g') = g = \pi_2 \circ \psi_\alpha(g').$$

By lemma 1 the bundle  $\mathcal{B}$  is locally trivial. To end the proof of the theorem it is sufficient to prove that the  $G_c$  spaces  $(S, G_c, L)$  and  $(G_c \times U_\alpha, G_c, \delta)$ , are isomorphic, but that follows from

$$\delta \circ \text{id}|_{G_c} \times \psi_\alpha = \psi_\alpha \circ L|_{G_c \times \pi^{-1}(U_\alpha)}.$$

With this theorem it is now possible to explain the harmonic maps method as follows:

- (i) Given the chiral equations (A2), invariant under the group  $G$ , choose a symmetric Riemannian space  $V_p$  with  $d$  Killing vectors,  $p \leq n = \dim G$ .
- (ii) Look for a representation for the corresponding Lie algebra  $\mathcal{G}$  compatible with the commuting relations of the Killing vectors, via Eq. (A14).
- (iii) Write the matrices  $A_i(g)$  explicitly in terms of the geodesic parameters of the symmetric space  $V_p$ .
- (iv) Use proposition 2 for finding the equivalence classes in  $\{A_i\}$  and choose a set of representatives.
- (v) Map the Lie algebra representatives into the group.

The solutions can be constructed by means of the left action of the  $G_c$  group into  $G$ .

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