

**Dynamics of scalar field dark matter with a cosh-like potential**Tonatiuh Matos,<sup>1,\*</sup> José-Rubén Luévano,<sup>2,†</sup> Israel Quiros,<sup>3,‡</sup> L. Arturo Ureña-López,<sup>4,§</sup> and José Alberto Vázquez<sup>1,||</sup><sup>1</sup>*Departamento de Física, Centro de Investigación y de Estudios Avanzados del IPN, Apartado Postal 14-740, 07000 México, Distrito Federal, Mexico*<sup>2</sup>*Departamento de Ciencias Básicas, UAM-A, CP 02200, México, Distrito Federal, Mexico*<sup>3</sup>*Universidad Central de Las Villas, Santa Clara, CP 54830, Cuba*<sup>4</sup>*Departamento de Física, DCI, Campus León, Universidad de Guanajuato, CP 37150, León, Guanajuato, Mexico*

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The dynamics of a cosmological model of dark matter and dark energy represented by a scalar field endowed with a cosh-like potential plus a cosmological constant is investigated in detail. By studying the appropriate phase space of the equations of motion, it is shown that a standard evolution of the Universe is recovered for appropriate values of the free parameters, and that the only late-time attractor is always the de Sitter solution. We also discuss the appearance of scalar field oscillations corresponding to dark matter behavior.

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**I. INTRODUCTION**

One of the greatest mysteries of modern cosmology doubtless is the nature of dark matter (DM) [1,2]; around 23% of the matter of the Universe is attractive, but of unknown nature. The most accepted DM candidates are particles from the minimal supersymmetric standard model [3,4]. The paradigm arising from this is the cold dark matter (CDM) model that has been proved at the cosmological level with great success. For example, it predicts very well the formation and clustering of galaxies [5].

Nevertheless, some inconsistencies of the model appear when compared with observations at the galactic level in the last decade. Among other problems, the predicted number of satellite galaxies around large galaxies is much bigger than the observed one [6], and the DM density profile at the center of galaxies seems to be less steep than predicted [7].

Even though there have been several attempts to deal with these inconsistencies [8], one compelling possibility is that the DM particle is a scalar field [9]. The idea is rooted in the fact that, for the sake of mathematical and physical consistency, almost all the unified theories of physics contain scalar fields as the simplest geometrical objects; these appear with a plethora of names: Higgs, inflatons, dilatons, scalarons, radions, etc. Moreover, the discovery of the dark energy (DE) renewed the interest of cosmologists in scalar fields (see [10] for a comprehensive review).

In Ref. [9,11] some of us proposed that a scalar field rolling down a convex self-interaction potential can be a reasonable candidate for DM, calling this paradigm scalar field dark matter (SFDM). This hypothesis has some nice features. For instance, SFDM does not need extra assumptions to explain the flat DM profile in the centers of galaxies [12], or the number of satellite galaxies around the Milky Way [13].

The SFDM hypothesis has been investigated for a number of self-interaction potentials like the cosh-like [11,14,15], and the quadratic ones [15,16], with the consequent discovery of several exact solutions of cosmological interest. However, within the cosmological context, a full and detailed study of the dynamics of SFDM models, to uncover their relevant asymptotic properties, is still desirable.

The main goal of the present paper is, precisely, to study, within the cosmological context and by means of the dynamical systems tools, the asymptotic properties of the SFDM model driven by a cosh-like potential. Several relevant cosmological solutions will be correlated with concepts like past and future attractors, signaling the way the cosmic dynamics transits from early-time to intermediate, and then to late-time asymptotic states.

It is shown that the SFDM model driven by a cosh-like potential with a cosmological constant term added can also describe the cosmic dynamics of the standard model  $\Lambda$ CDM. Due attention will be paid to the late-time oscillatory solution that can be associated with CDM behavior in the model.

The paper has been organized as follows. The relevant physical features of the model are discussed in Sec. II, and then, in Sec. III, its mathematical features are specified. The details of the dynamical systems study of the SFDM model with a cosh-like potential are given in Sec. IV.

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Finally, in Sec. V, the physical discussion of the results and brief conclusions of the study are provided. We use throughout natural units  $\kappa^2 = 8\pi G = c = 1$ .

## II. SCALAR FIELD MODEL WITH A COSH-LIKE POTENTIAL

In a cosmological context, scalar fields with a cosh-like self-interaction potential have been studied in [11,13,14]. In those references the following potential was investigated,

$$V(\phi) = V_0[\cosh(\lambda\phi) - 1]^\beta, \quad (1)$$

where  $\lambda$  and  $\beta$  are free parameters; for an instance of an effective potential like this in string theory, see [17]. It was shown, see also [18], that during the oscillatory phase around the minimum of the potential in the case  $\beta > 0$ , the virial theorem gives the following expression for the mean equation of state:

$$\langle \omega_\phi \rangle = \left\langle \frac{p_\phi}{\rho_\phi} \right\rangle = \left\langle \frac{\dot{\phi}^2 - 2V}{\dot{\phi}^2 + 2V} \right\rangle = \frac{\beta - 1}{\beta + 1}. \quad (2)$$

In consequence, for  $\beta = 1$  the scalar field behaves like pressureless dust,  $\langle \omega_\phi \rangle = 0$ . A scalar field potential with this value of  $\beta$  could therefore play the role of CDM in the Universe. For  $\beta < 1/2$  this potential is a good candidate for quintessence models of DE [10,14].

In this paper we will focus our attention, precisely, on a cosh-like potential (1) with  $\beta = 1$  and a cosmological constant term ( $\Lambda$ ) added to it, i.e.,

$$V(\phi) = V_0[\cosh(\lambda\phi) - 1] + \Lambda. \quad (3)$$

The latter ingredient is necessary to include DE in the model, as long as oscillations of the scalar field around the minimum of the cosh potential play the role of the CDM.

The  $\Lambda$  term can also be thought of as part of the scalar field Lagrangian (for instance, as a way to introduce the quantum effects of the scalar field [19]), so that the resulting picture may represent the unification of DM and DE with a single field, a bit in the spirit discussed in [20]. We shall call it just the SFDM for short, implicitly assuming that DE is already part of the scalar field Lagrangian.

The expected dynamics of the model is the following [13,14]. Initially, the energy density in the scalar field was comparable to that of radiation at very early times, so that the cosh-like potential (1) behaves as an exponential one of the form  $V(\phi) \propto \exp(-\lambda|\phi|)$ . Because of the properties of the exponential potential [21], the scalar field approaches the so-called scaling solution, for which it redshifts exactly as the then dominant radiation fluid.

At later times the form of  $V(\phi)$  changes to quadratic, resulting in rapid oscillations of the scalar field around  $\phi = 0$ ; at this stage the scalar field equation of state mimics that of CDM. For reasonable values of the  $\Lambda$

term in the potential, the late-time dynamics should be that of the standard  $\Lambda$ CDM model.

## III. MATHEMATICAL FEATURES OF THE MODEL

We start with a Friedmann-Robertson-Walker spacetime with flat spatial sections, filled with a mixture of two fluids: (i) a perfect fluid of ordinary matter with density  $\rho_\gamma$  and barotropic index  $0 \leq \gamma \leq 2$  ( $\gamma = 1$  for dust,  $\gamma = 4/3$  for radiation, etc.) and (ii) a scalar field cold dark matter component with energy density  $\rho_\phi = \dot{\phi}^2/2 + V(\phi)$  and parametric pressure  $p_\phi = \dot{\phi}^2/2 - V(\phi)$ . The relevant cosmological equations of the model are the following,

$$\dot{H} = -\frac{1}{2}(\dot{\phi}^2 + \gamma\rho_\gamma), \quad (4a)$$

$$\ddot{\phi} = -3H\dot{\phi} - \frac{dV}{d\phi}, \quad (4b)$$

$$\dot{\rho}_\gamma = -3H\gamma\rho_\gamma, \quad (4c)$$

plus the Friedmann constraint:

$$H^2 = \frac{1}{3}\left(\rho_\gamma + \frac{1}{2}\dot{\phi}^2 + V\right). \quad (5)$$

As discussed in the former section, the potential  $V(\phi)$  in (3) already comprises both DM and DE. Since in this model  $\Lambda \neq 0$ , the scalar field energy density performs damped oscillations around  $V(0) = V_{\min} = \Lambda$ , meaning that the CDM energy density—accounted for by the oscillatory component—decreases until, eventually, the cosmological constant dominates.

In a natural scenario for cosmic dynamics, the scalar field  $\phi$  runs from arbitrarily large negative values ( $|\phi| \gg 1$ ) to vanishing ones ( $|\phi| \ll 1$ ). In consequence, at early times the dynamics is driven by an exponential potential

$$|\phi| \gg 1/\lambda \Rightarrow V(\phi) \approx \frac{V_0}{2} e^{-\lambda\phi}, \quad (6)$$

whereas at late times it is associated with a quadratic potential plus a cosmological constant:

$$|\phi| \ll 1/\lambda \Rightarrow V(\phi) \approx \frac{1}{2}m^2\phi^2 + \Lambda, \quad m^2 \equiv V_0\lambda^2. \quad (7)$$

## IV. DYNAMICAL SYSTEMS STUDY

The dynamical systems tools offer a very useful approach to the study of the asymptotic properties of the cosmological models [10,22]. In order to be able to apply these tools one has to (unavoidably) follow the steps enumerated below.

- (i) First: to identify the phase space variables that allow writing the system of cosmological equations in the form of an autonomous system of ordinary differential equations (ODEs). There can be several different possible choices; however, not all of them allow for

the minimum possible dimensionality of the phase space.

- (ii) Next: with the help of the chosen phase space variables, to build an autonomous system of ODEs out of the original system of cosmological equations.
- (iii) Finally (sometimes a forgotten or underappreciated step): to identify the phase space spanned by the chosen variables, which is relevant to the cosmological model under study.

After this recipe one is ready to apply the standard tools of dynamical systems.

### A. Autonomous system of ODEs

Let us to introduce the following dimensionless phase space variables in order to build an autonomous system out of the system of cosmological equations (4c) and (5) [21]:

$$x \equiv \frac{\dot{\phi}}{\sqrt{6}H}, \quad y \equiv \frac{\sqrt{V}}{\sqrt{3}H}. \quad (8)$$

After this choice of phase space variables we can write the following autonomous system of ODEs,

$$x' = -\sqrt{\frac{3}{2}} \frac{\partial_{\phi} V}{V} y^2 - 3x + \frac{3}{2} x (2x^2 + \gamma \Omega_{\gamma}), \quad (9a)$$

$$y' = \sqrt{\frac{3}{2}} \frac{\partial_{\phi} V}{V} x y + \frac{3}{2} y (2x^2 + \gamma \Omega_{\gamma}), \quad (9b)$$

where a prime denotes a derivative with respect to the time variable  $\tau \equiv \ln a$  (properly speaking, the number of e-foldings of expansion), and the dimensionless density parameter  $\Omega_{\gamma} \equiv \rho_{\gamma}/3H^2$  is given through the following expression:

$$\Omega_{\gamma} = 1 - x^2 - y^2, \quad (10)$$

which is just a rewriting of the Friedmann constraint (5).

As long as one considers just constant ( $\partial_{\phi} V = 0$ ) and exponential self-interaction potentials ( $\partial_{\phi} V/V = \text{const}$ ), Eqs. (9a) and (9b) form a closed autonomous system of ODEs. However, if one desires to go further and to consider a wider class of self-interaction potentials beyond the exponential one—as is the case in the present study—the system of ODEs (9a) and (9b) is not a closed system of equations anymore, since, in general,  $\partial_{\phi} V/V$  is a function of the scalar field itself.

A way out of this difficulty can be the method developed in [23]. In order to be able to study arbitrary self-interaction potentials, one needs to consider an extra variable  $v$  that is related to the derivative of the self-interaction potential through the following expression:

$$v\lambda \equiv -\partial_{\phi} V/V = -\partial_{\phi} \ln V. \quad (11)$$

Hence, an extra equation,

$$v' = -\sqrt{6}\lambda x v^2 (\Gamma - 1), \quad (12)$$

has to be added to the above autonomous system of equations. The quantity  $\Gamma \equiv V \partial_{\phi}^2 V / (\partial_{\phi} V)^2$  in Eq. (12) is, in general, a function of  $\phi$ . The idea behind the method in [23] is that  $\Gamma$  can be written as a function of the variable  $v$ , and, perhaps, of several constant parameters, as happens for a wide class of scalar potentials.

Let us introduce a new function,  $f(v) = v^2(\Gamma(v) - 1)$ , so that Eq. (12) can be written in the more compact form

$$v' = -\sqrt{6}\lambda x f(v). \quad (13)$$

Equations (9a), (9b), (10), and (13) form a three-dimensional—closed—autonomous system of ODEs,

$$x' = \sqrt{\frac{3}{2}} \lambda y^2 v - 3x + \frac{3}{2} x [2x^2 + \gamma(1 - x^2 - y^2)], \quad (14a)$$

$$y' = -\sqrt{\frac{3}{2}} \lambda x y v + \frac{3}{2} y [2x^2 + \gamma(1 - x^2 - y^2)], \quad (14b)$$

$$v' = -\sqrt{6}\lambda x f(v), \quad (14c)$$

that can be safely studied with the help of the standard dynamical systems tools [22]. Notice the obvious symmetry of the ODEs (14) under the simultaneous change of sign of  $\lambda$  and  $v$ :  $\lambda \rightarrow -\lambda$ ,  $v \rightarrow -v$ .

An important limitation of the approach explained above is related to the fact that, for potentials that vanish at the minimum—such as, for instance  $V = V_0(\cosh(\lambda\phi) - 1)^{\beta}$ , or  $V = m^2 \phi^2/2$ —variable  $v$  diverges at this important point of the late-time dynamics. However, thanks to the presence of  $\Lambda$  in Eq. (3), such an issue is absent in the present study, and the approach of [23] can be safely applied.

In terms of the phase space variables  $x$ ,  $y$ , and  $v$ , the following relevant magnitudes, (i) the deceleration parameter  $q \equiv -(1 + \dot{H}/H^2)$  and (ii) the effective equation of state parameter of the scalar field  $\gamma_{\phi} \equiv 2\dot{\phi}^2/(\dot{\phi}^2 + 2V)$ , can be written, respectively, as

$$q = -1 + \frac{3}{2}(2x^2 + \gamma \Omega_{\gamma}), \quad (15a)$$

$$\gamma_{\phi} = \frac{2x^2}{x^2 + y^2}. \quad (15b)$$

### B. Function $f(v)$ for the cosh-like potential (3)

We now find function  $f(v)$  in Eq. (14), for the potential (3). According to the definition (11), one has

$$v(\phi) = \frac{\sinh(\lambda\phi)}{\cosh(\lambda\phi) - 1 + \alpha}, \quad (16)$$

where  $\alpha \equiv \Lambda/V_0$ . From (16) it follows, in particular, that

$$\lim_{\phi \rightarrow \pm\infty} v(\phi) = \pm 1, \quad \lim_{\phi \rightarrow 0} v(\phi) = 0. \quad (17)$$

The value  $\alpha = 1$  is a critical one. Actually, the function

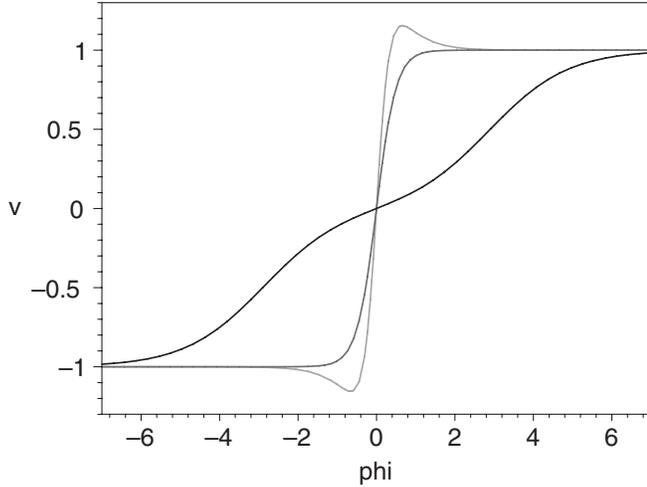


FIG. 1. A plot of function  $v(\phi)$  for the chosen values of the parameters of the potential  $\lambda = 1$ ,  $\alpha = 10$ , darker curve;  $\alpha = 1$ , dark-to-gray curve; and  $\alpha = 0.5$ , soft-gray curve.

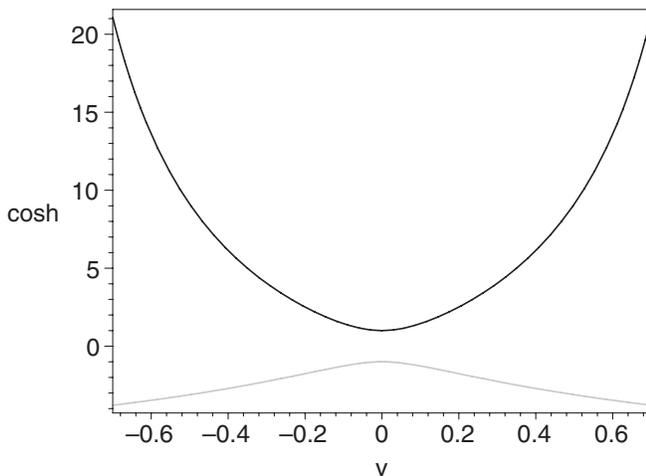
(16) has an extremum whenever

$$\cosh(\lambda\phi_{\text{ext}}) = \frac{1}{1-\alpha} \Rightarrow v_{\text{ext}} = [\alpha(2-\alpha)]^{-1/2}. \quad (18)$$

The fact that the cosh function is always equal to or bigger than unity means that  $v(\phi)$  has an extremum only if  $0 < \alpha < 1$ . It can be seen that  $v \in [-v_{\text{ext}}, v_{\text{ext}}]$ .

For  $\alpha \geq 1$  the function  $v$  is, at least, a monotonically nondecreasing function as the scalar field takes values in the interval  $-\infty < \phi < \infty$ ; then, in this case  $v \in [-1, 1]$ . We show in Fig. 1 some examples of function  $v(\phi)$  to illustrate the different behaviors just discussed above.

The function  $v(\phi)$  in Eq. (16) can be inverted to have the cosh function written in terms of  $v$ ,



$$\cosh(\lambda\phi(v))^{\pm} = \frac{(\alpha-1)v^2 \pm \sqrt{1 + \alpha(\alpha-2)v^2}}{1-v^2}, \quad (19)$$

where the “ $\pm$ ” signs refer to the two different branches of  $\cosh(\lambda\phi)$ .

In order to illustrate the discussion above, in Fig. 2 we plot the cosh function vs  $v$  for two sets of values of the parameters of the potential. For the case  $\alpha > 1$ , the positive branch actually depicts the right (whole) cosh-function behavior.

For the case  $\alpha < 1$ , instead, the whole cosh-function behavior has to be covered by a union of that part of the negative branch to the left of the vertical asymptote  $v = -1$ , starting at infinitely large values of  $\cosh(\lambda\phi)$  (i.e., infinitely large negative values of  $\phi$ ) until the gray curve meets the dark one (the positive branch) at the union point at  $v = v_{\text{min}} = -1/\sqrt{\alpha(2-\alpha)}$ .

The same is true for positive values of  $\phi$ : the right cosh-function behavior is depicted by a union of that part of the negative branch to the right of the vertical asymptote at  $v = 1$ , starting at infinitely large positive values of  $\phi$ , until it joins the positive branch at  $v = v_{\text{max}} = 1/\sqrt{\alpha(2-\alpha)}$ . The range of intermediate-to-small values of  $\phi$  (including the special point at  $\phi = 0$ ) is completely covered by the positive branch of function (19).

For the potential (3), function  $f(v)$  appearing in Eqs. (13) and (14) can be written in the following way:

$$f^{\pm}(v) = \frac{\pm(1-v^2)\sqrt{1 + \alpha(\alpha-2)v^2}}{\alpha-1 \pm \sqrt{1 + \alpha(\alpha-2)v^2}}, \quad (20)$$

where the  $\pm$  signs refer to the positive and the negative branches of function  $f(v)$  [directly related to the positive and the negative branches of the cosh function in Eq. (19)], respectively. In Eq. (20) the choice of the “+” or the “-”

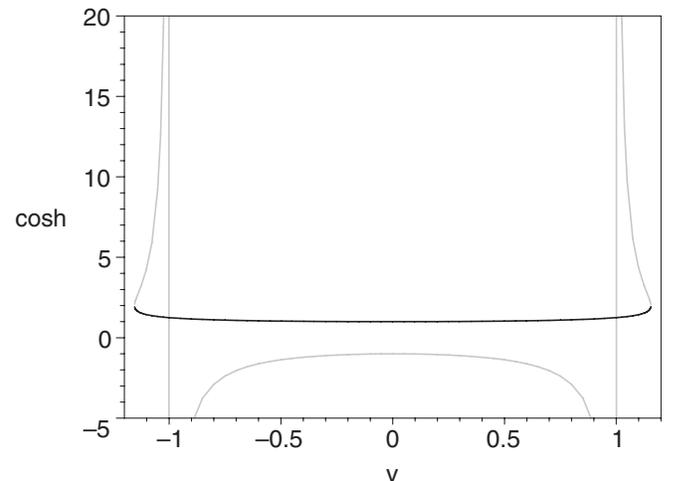


FIG. 2. A plot of function  $\cosh(\lambda\phi)$  vs  $v$ , for the chosen values of the free parameters of the potential (left,  $\lambda = 1$ ,  $\alpha = 10$ ; right,  $\lambda = 1$ ,  $\alpha = 0.5$ ). Both branches of the cosh function, the positive (dark curve) and the negative one (gray line) are shown. Notice that, for the second case ( $\alpha < 1$ ), the whole cosh-function behavior is depicted by a union of part of the negative and the entire positive branches.

sign is to be made simultaneously in the numerator and in the denominator on the right-hand side. Hence, for instance, for the positive branch of  $f(v)$  one has

$$f^+(v) = \frac{+(1 - v^2)\sqrt{\dots}}{\alpha - 1 + \sqrt{\dots}}, \text{ etc.} \quad (21)$$

It has to be emphasized that, while for values of the free parameter  $\alpha \geq 1$  the positive branch  $f^+(v)$  in Eq. (20) is enough to depict the whole dynamics, for  $0 < \alpha < 1$ , instead, one needs to take into account both branches  $f^\pm(v)$ .

Actually, in the latter case the piece of the dynamics in the  $v$  interval

$$1 < |v| \leq \frac{1}{\sqrt{\alpha(2 - \alpha)}} \quad (22)$$

is covered by the corresponding values of  $f^-(v)$  in Eq. (20):  $f^- (|v| > 1)$ . The rest of the dynamics—including the late-time behavior—is covered by the piece of the positive branch lying in the interval  $|v| \leq 1$ :  $f^+ (|v| \leq 1)$ . In consequence, for  $0 < \alpha < 1$ , the whole dynamics is covered by

$$f_{0 < \alpha < 1} = f^- (|v| > 1) \cup f^+ (|v| \leq 1). \quad (23)$$

To summarize, for  $\alpha \geq 1$  the cosmic dynamics driven by potential (3) can be associated with the following three-dimensional (compact) phase space, spanned by the variables  $x$ ,  $y$ , and  $v$  (we take into account only expanding cosmologies  $H \geq 0$ ),

$$\Psi_{\alpha \geq 1} = \{(x, y, v): 0 \leq x^2 + y^2 \leq 1, |x| \leq 1, 0 \leq y \leq 1, |v| \leq 1\}, \quad (24)$$

and we have to worry only about the positive branch of  $f(v)$  in Eq. (20). Meanwhile, for  $0 < \alpha < 1$ , the three-dimensional (compact) phase space is given by

$$\Psi_{0 < \alpha < 1} = \left\{ (x, y, v): 0 \leq x^2 + y^2 \leq 1, |x| \leq 1, 0 \leq y \leq 1, |v| \leq \frac{1}{\sqrt{\alpha(2 - \alpha)}} \right\}, \quad (25)$$

and depending on the piece of the dynamics one is interested in, one has to rely either on the negative branch of Eq. (20),  $f^- (|v| > 1)$  (early-times-to-intermediate dynamics), or on the positive branch,  $f^+ (|v| \leq 1)$  (late-time

dynamics), the whole dynamics being described by  $f_{0 < \alpha < 1}$  in Eq. (23).

In what follows, depending on the value of parameter  $\alpha$ , we shall look for fixed points of the autonomous system (14), with  $f(v)$  given, either by (i) the positive branch of Eq. (20), within the phase space  $\Psi_{\alpha \geq 1}$  defined in Eq. (24), whenever  $\alpha \geq 1$ , or (ii) by  $f_{0 < \alpha < 1}$  defined in Eq. (23), within the phase space  $\Psi_{0 < \alpha < 1}$  defined in Eq. (25), if  $0 < \alpha < 1$ .

### C. Equilibrium points and stability

The fixed points of the autonomous system (14), in the phase space  $\Psi_{\alpha \geq 1}$  defined by Eq. (24) or in  $\Psi_{0 < \alpha < 1}$  defined by Eq. (25), are listed in Table I, while the eigenvalues of the corresponding linearization matrices are shown in Table II.

In the case when the parameter  $0 < \alpha < 1$ , by looking at the definition of the function  $f(v)$  in Eq. (20), it might seem that, in addition to the critical points in Table I, there can be also equilibrium points associated with the values  $v = \pm 1/\sqrt{\alpha(2 - \alpha)}$ . However, if one looks at Fig. 3, one can see that at  $v = \pm 1/\sqrt{\alpha(2 - \alpha)}$  the derivative of the function  $f_{0 < \alpha < 1}$  in Eq. (23) is undefined, so that the linear approach undertaken in this investigation is not valid any more. At any rate, our numerical studies (see below) indicate that the above points in phase space are not critical points, so that we shall not include them in our analysis.

The main properties of the equilibrium points shown in Tables I and II can be summarized as follows.

- (i) The existence of the fluid-dominated solution, equilibrium point  $P_1$ , is independent of the value of variable  $v$ , meaning that this phase of the cosmic evolution is present at all times. However, the fluid-dominated solution is always a saddle point.
- (ii) The kinetic-energy-dominated solution, equilibrium point  $P_2$ , is decelerating. For  $\lambda \leq \sqrt{6}$  it is the past attractor for any phase space trajectory.
- (iii) The scalar-field-dominated solution, equilibrium point  $P_3$ , exists whenever  $\lambda \leq \sqrt{6}$ , and it is accelerating for  $\lambda < \sqrt{2}$  (decelerating otherwise). It is always a saddle point.
- (iv) The scaling solution, equilibrium point  $P_4$ , exists whenever  $\lambda \geq \sqrt{3\gamma}$  but it is always a saddle point. This solution is correlated with a decelerated ex-

TABLE I. Properties of the fixed points for the autonomous system (14).

$P_i$	$x$	$y$	$v$	Existence	$\Omega_\gamma$	$\Omega_\phi$	$\gamma_\phi$	$q$
$P_1$	0	0	$v$	Always	1	0	Undefined	$-1 + 3\gamma/2$
$P_2$	$\pm 1$	0	$\pm 1$	Always	0	1	2	2
$P_3$	$\lambda/\sqrt{6}$	$\sqrt{1 - \lambda^2/6}$	1	$\lambda^2 \leq 6$	0	$\pm 1$	$\lambda^2/3$	$-1 + \lambda^2/2$
$P_4$	$\sqrt{\frac{3}{2}} \frac{\gamma}{\lambda}$	$\sqrt{\frac{3\gamma(2-\gamma)}{2\lambda^2}}$	$\pm 1$	$\lambda^2 \geq 3\gamma$	$1 - 3\gamma/\lambda^2$	$3\gamma/\lambda^2$	$\gamma$	$-1 + 3\gamma/2$
$P_5$	1	1	0	Always	0	1	0	-1

TABLE II. Eigenvalues of the linearization matrices corresponding to the fixed points in Table I. We have used the following definition:  $R \equiv \sqrt{11\gamma^2 - 28\gamma + 12(1 + 2\gamma^2[3\gamma - 2 + 12(1 - \gamma)\gamma^2/\lambda^2 - \gamma^2(1 - 3\gamma^2/\lambda^2)]/\lambda^2)}$ .

$P_i$	$\lambda_1$	$\lambda_2$	$\lambda_3$
$P_1$	$-3 + 3\gamma/2$	$3\gamma/2$	0
$P_2$	$6 - 3\gamma$	$3 - \sqrt{3/2}\lambda$	$\sqrt{6}\lambda$
$P_3$	$-3 + \lambda^2/2$	$\lambda^2 - 3\gamma$	$\lambda^2$
$P_4$	$-3(2 - \gamma)/4 + \sqrt{3}R/4$	$-3(2 - \gamma)/4 - \sqrt{3}R/4$	$3\gamma$
$P_5$	$-3\gamma$	$-3(1 - \sqrt{1 - 4\lambda^2/\alpha})/2$	$-3(1 + \sqrt{1 - 4\lambda^2/\alpha})/2$

pansion of the Universe for  $\gamma \geq 2/3$  (and inflationary otherwise). The  $\phi$  field mimics matter with a barotropic equation of state  $\gamma_\phi = \gamma$ , and its energy densities scale such that

$$\frac{\rho_\phi}{\rho_\gamma} = \frac{1}{\lambda^2/3\gamma - 1}. \quad (26)$$

- (v) The late-time dynamics driven by potential (3) is correlated with the minimum  $V_{\min} = \Lambda$  at  $\phi = 0$ —and its neighborhood—which, in the phase space  $\Psi$ , is depicted by equilibrium point  $P_5$ . This point corresponds to the inflationary de Sitter solution  $3H^2 = \Lambda$ . Since the real parts of the eigenvalues of the linearization matrix for  $P_5$  are all negative (see Table II), the de Sitter solution is always the future attractor for any phase space trajectory.

Equilibrium points  $P_2$ ,  $P_3$ , and  $P_4$  are associated with early-time dynamics since, according to Eq. (16),  $v = 1$  is correlated with infinitely large values of the variable  $\phi$  for which  $V \simeq (V_0/2)e^{-\lambda\phi}$ . Indeed, we see from Eq. (14) that if we set  $v = 1$ , then the equations of motion of the exponential case [21] are exactly recovered, even though the fixed points of the present dynamical system have different properties.

Contrary to the exponential case, we have found that all critical points have been made unstable by the presence of the new variable  $v$ . Maybe not surprisingly, parameter  $\alpha$  has nothing to do with the existence of the critical points and their properties.

The only exception to the rule is point  $P_5$ , which stands as the only stable point in phase space and is also affected by the presence of parameter  $\alpha$ . If  $\lambda < \sqrt{\alpha}/2$ , all eigenvalues of point  $P_5$  are real and positive, meaning that the physical system does not perform any oscillation at late times.

A physical explanation for this is found in the following. Taking into account that the mass of the scalar field is  $m^2 = \lambda^2 V_0$ , we find

$$\frac{\lambda^2}{\alpha} = \frac{m^2}{\Lambda} \simeq \frac{m^2}{3H^2}, \quad (27)$$

where we have used the fact that point  $P_5$  corresponds to a de Sitter solution,  $3H^2 = \Lambda$ . Thus, we see that the condition  $\lambda^2 < \alpha$  is equivalent to  $m^2 < H^2$ , which is the condition for overdamped oscillations (slow-rolling) of the scalar field around the minimum of the potential.

In the opposite case,  $\lambda > \sqrt{\alpha}/2$ , corresponding to  $m^2 > H^2$ , the linear perturbations of the scalar field perform damped oscillations around the minimum of the potential

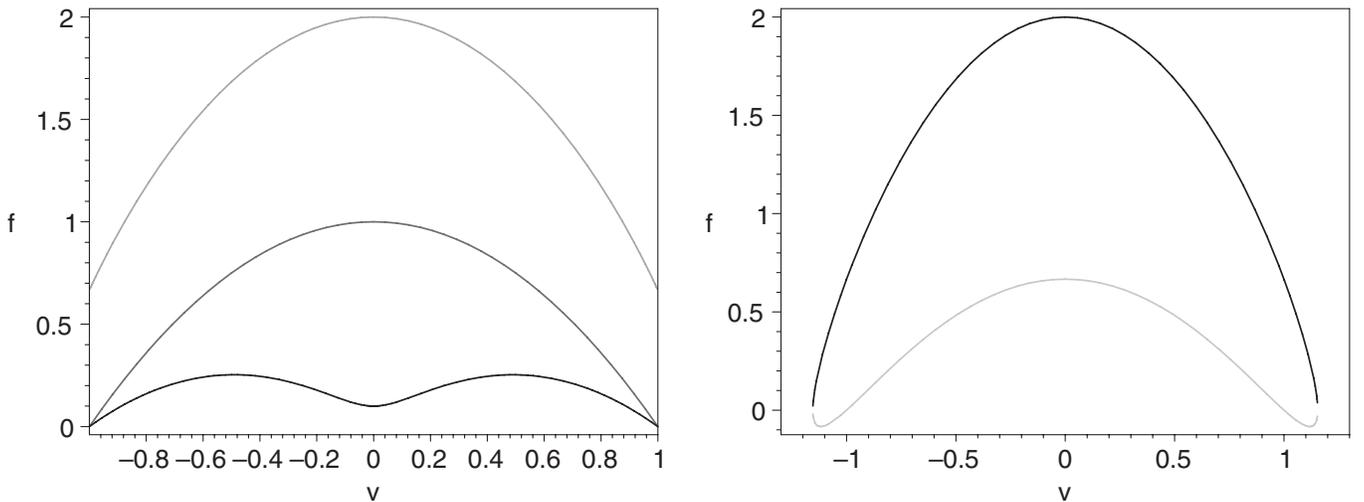


FIG. 3. (Left) A plot of function  $f^+$  in Eq. (20) vs  $v$ , for the chosen values of the free parameters in Fig. 1. (Right) A plot of  $f$  vs  $v$  for  $\alpha = 0.5 < 1$  ( $\lambda = 1$ ). Both branches of  $f(v)$  in Eq. (20), the positive (darker curve) and the negative one (gray line), are depicted.

that are characterized by a cyclic frequency

$$\omega = \frac{3}{2} \sqrt{\frac{4\lambda^2}{\alpha} - 1}. \tag{28}$$

The general solution for the evolution of linear perturbations  $\delta \mathbf{x} = (\delta x, \delta y, \delta v)$  in the neighborhood of the equilibrium point  $P_5$  can be written as

$$\begin{aligned} \delta \mathbf{x} &= C_1 \mathbf{a}_1 e^{\lambda_1 \tau} + C_2 \mathbf{a}_2 e^{\lambda_2 \tau} + C_3 \mathbf{a}_3 e^{\lambda_3 \tau} \\ &= C_1 \mathbf{a}_1 e^{-3\gamma \tau} + e^{-3\tau/2} (C_2 \mathbf{a}_2 e^{i\omega \tau} + C_3 \mathbf{a}_3 e^{-i\omega \tau}), \end{aligned} \tag{29}$$

where the  $\lambda$ 's and the  $\mathbf{a}$ 's are the eigenvalues and the eigenvectors, respectively, of the linearization matrix around  $P_5$ , and the  $C$ 's are arbitrary constants.

The oscillatory behavior with frequency  $\omega$  is what, according to Refs. [11,13,14], can be associated with CDM because the amplitudes of the perturbations decrease at a rate  $\propto \exp(-3\tau/2)$ . The corresponding mean energy density of the scalar field then dilutes with an effective equation of state  $\langle \omega_\phi \rangle = 0$  [see Eq. (2)], i.e.,  $\langle \rho_\phi \rangle \propto a^{-3}$ .

The above results are illustrated in Figs. 4 and 5 for a fixed value  $\gamma = 4/3$ . In particular, it is apparent the way the orbits, at late times, coil around the segment  $\{(x, y, v) = (0, y, 0) : 0 \leq y \leq 1\}$  until, eventually, they reach the inflationary de Sitter attractor  $P_5 = (0, 1, 0)$ . The oscillating solution arises due to the choice of the free parameters that obey the constraint  $\lambda > \sqrt{\alpha}/2$ .

For completeness, we show in Figs. 6 and 7 the late-time and the early-to-intermediate-time behaviors of the orbits of Eq. (14), respectively, for small values of the parameter  $\alpha$  ( $0 < \alpha < 1$ ).

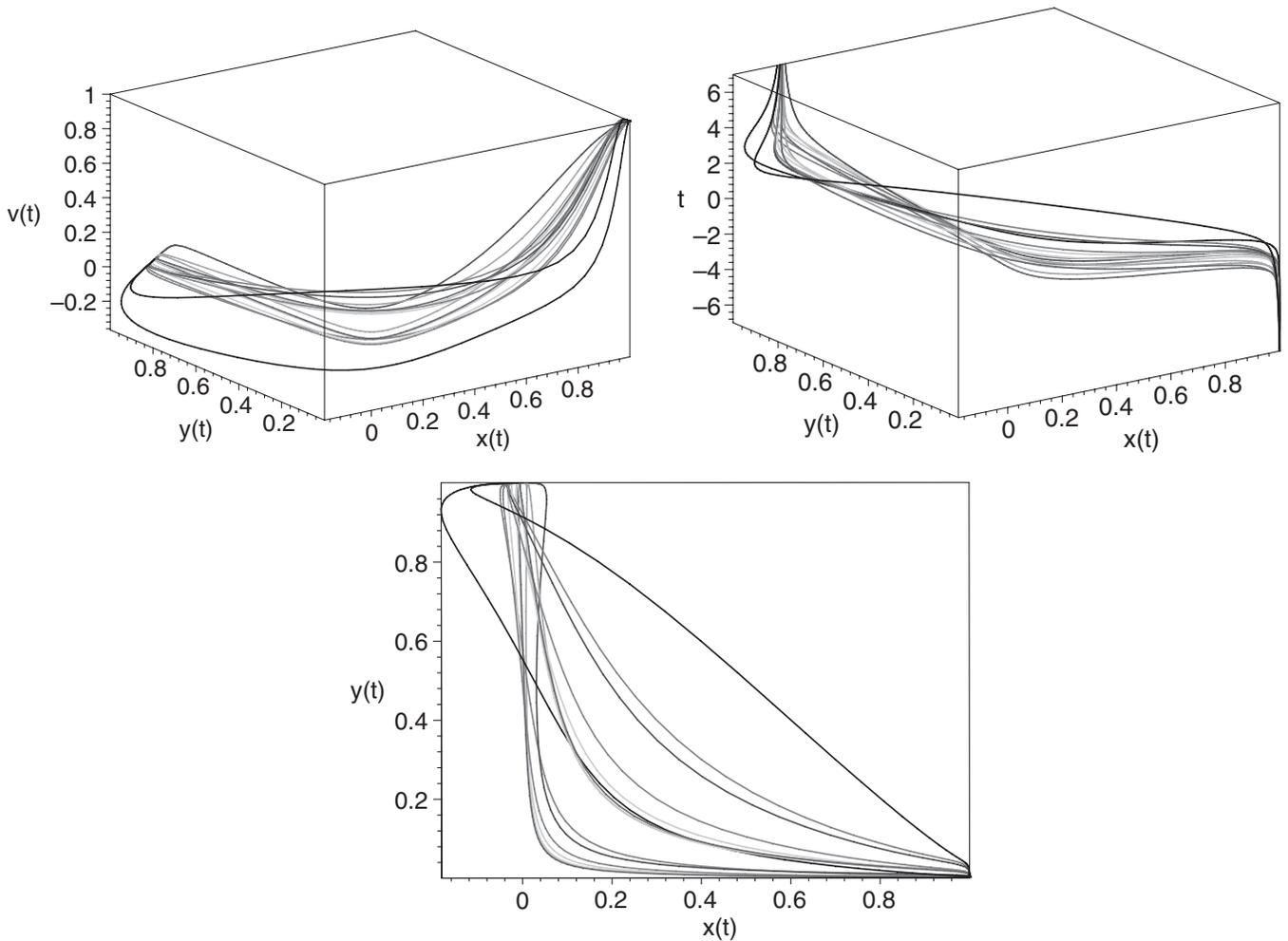


FIG. 4. (Top left) Orbits of the autonomous system (14) with function  $f(v)$  in Eq. (20), (top right) and the corresponding flux in time  $\tau$ , for the values  $\lambda = 2$ ,  $\alpha = 10$ , and  $\gamma = 4/3$ ). (Bottom) Projection of the orbits onto the phase plane  $(x, y)$ . Notice that the kinetic-energy-dominated solution  $(x, y, v) = (1, 0, 1)$  is the past attractor, while the de Sitter solution  $(0, 1, 0)$  is the future attractor. The radiation-dominated solution  $(0, 0, v)$  is always a saddle fixed point for any given  $v$ .

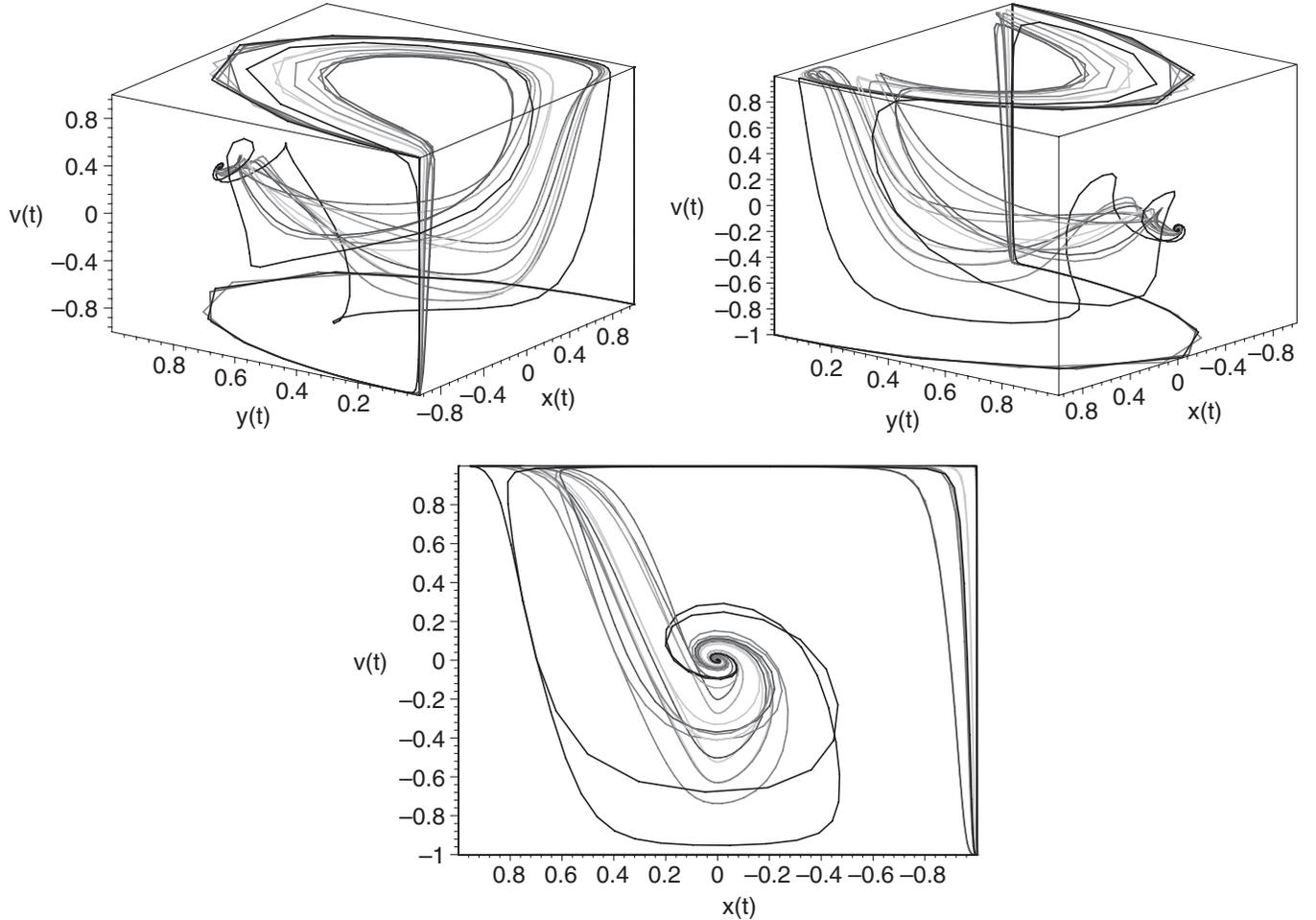


FIG. 5. (Top left and top right) Different perspectives of the local late-time behavior of the orbits of the autonomous system (14), with function  $f(v)$  in Eq. (20), for the values  $\lambda = 5$ ,  $\alpha = 3$ , and  $\gamma = 4/3$ . (Bottom) Projection of the orbits into the phase plane  $(x, v)$ . At late times the phase space trajectories coil around the de Sitter segment  $\{(0, y, 0): 0 \leq y \leq 1\}$  until, eventually, they reach the inflationary de Sitter attractor  $(0, 1, 0)$ .

Because of the choice of the values of the free parameters obeying also the condition  $\lambda^2 > \alpha/4$  as in the former case, the orbits coil around the de Sitter segment  $(x, y, v) = (0, y, 0)$  until, eventually, they reach the de Sitter solution  $3H^2 = \Lambda$  (Fig. 6). The early-times-to-intermediate behavior (Fig. 7) clearly shows that the kinetic-energy-dominated solution (equilibrium point  $P_2$ ) is the past attractor, while the scaling solution (point  $P_4$ ) is a saddle equilibrium point.

We want to emphasize that the spiral form of the orbits of Eq. (14)—see Fig. 6—is what can be properly interpreted as CDM behavior, so that, only for  $\lambda > \sqrt{\alpha}/2$ , the scalar field component in our model should be called SFDM.

## V. DISCUSSION AND CONCLUSIONS

It was in Refs. [11,13] that a careful analysis of the properties of the cosh potential as a CDM candidate was done. The main assumptions were that the scalar field

energy density was initially close to the scaling regime, and that the scalar field was massive enough to be DM at late times. The scaling regime allows the tracking behavior of the scalar field during the radiation-dominated epoch, and a large value of its mass starts the CDM behavior at early times so that we can recover a proper DM-dominated epoch at intermediate times in the evolution of the Universe. At the end, it was determined from some cosmological constraints that  $\lambda \simeq 20$  and  $m \simeq 10^{-23}$  eV.

In terms of the dynamical system analysis carried out here, the required evolution of the scalar field is recovered if the equilibrium points  $P_1$  (fluid-dominated),  $P_4$  (scaling solution), and  $P_5$  (de Sitter solution) exist in the phase space. This is the case for large values of  $\lambda$ . Moreover, we need trajectories close to point  $P_5$  to oscillate around it; for the value of the scalar field mass given above and the expected one for the cosmological constant  $\Lambda$ , we find that  $\lambda^2/\alpha \gg 1$  and then the desired CDM behavior around point  $P_5$  is guaranteed.

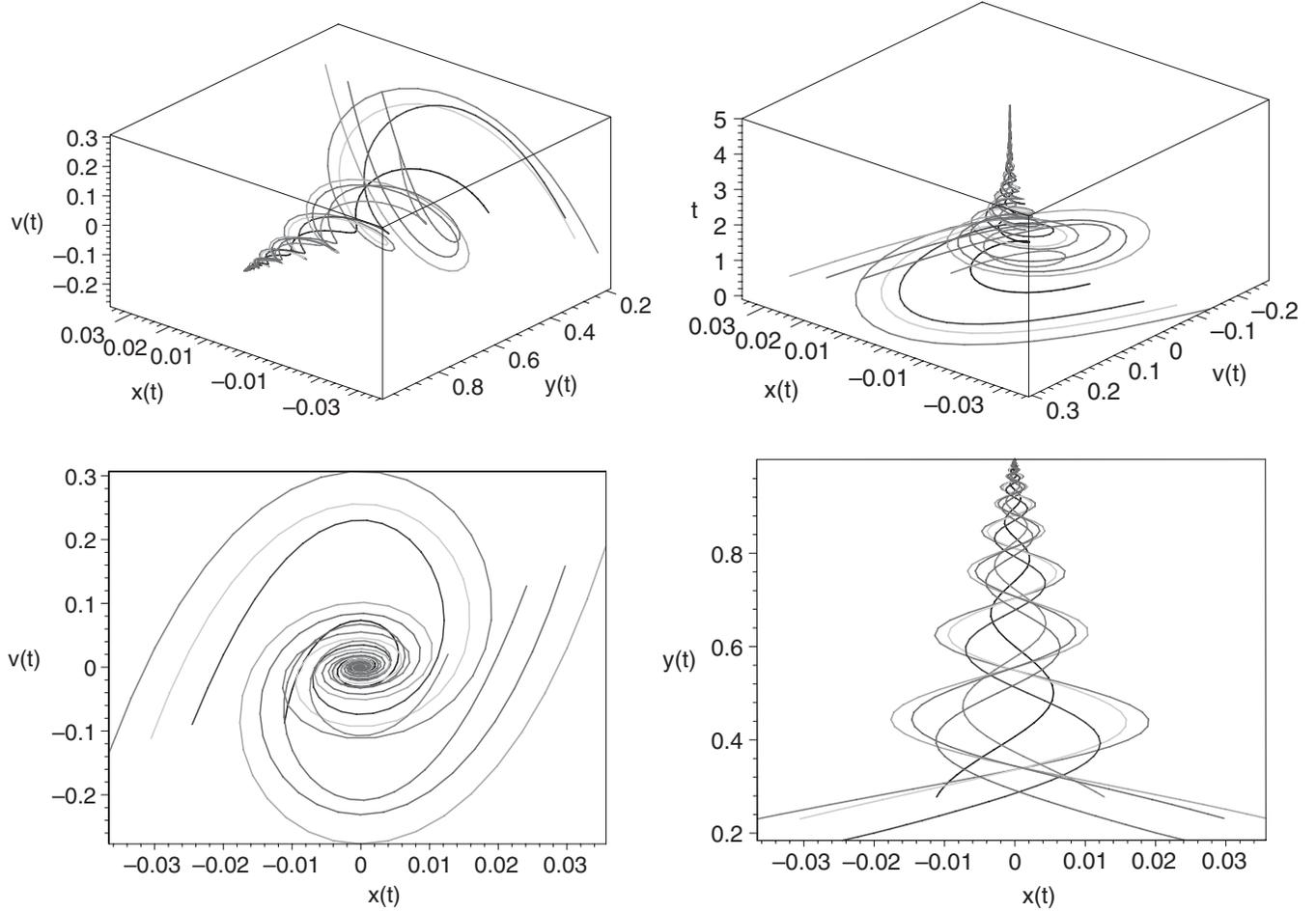


FIG. 6. (Top left) Local behavior of the orbits of the autonomous system (14) with function  $f(v)$  in Eq. (23), in the neighborhood of the point  $P_5$  associated with late-time cosmic dynamics, (top right) and the corresponding flux in time  $\tau$ , for the values  $\lambda = 2$ ,  $\alpha = 0.1$ , and  $\gamma = 4/3$ —background radiation. (Bottom left and right) Projection of the orbits into the phase planes  $(x, v)$  and  $(x, y)$ , respectively.

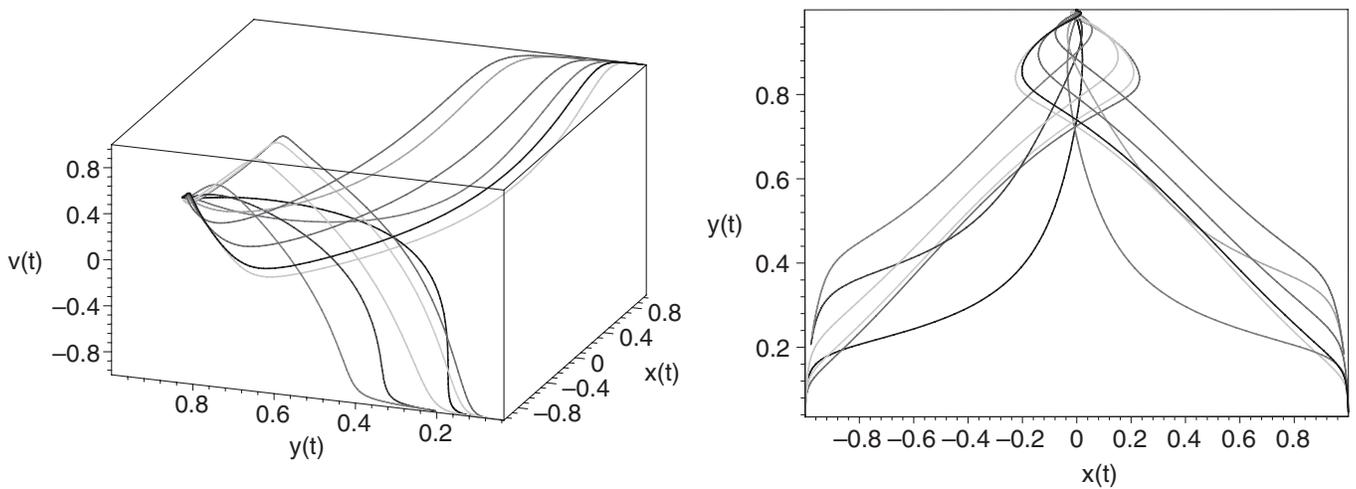


FIG. 7. (Left) Early-to-intermediate times behavior of the orbits of the autonomous system (14) with function  $f(v)$  in Eq. (23), for the values  $\lambda = 2$ ,  $\alpha = 0.1$ , and  $\gamma = 4/3$ . (Right) Projection of the orbits into the phase plane  $(x, y)$ . It is evident that the kinetic-energy-dominated solution (point  $P_2$  in Table I) is the past attractor, while the scaling solution [point  $P_4 = (0.81, 0.71, 1)$ ] is a saddle equilibrium point.

The cosmic evolution of the  $\phi$  field after the end of inflation most probably starts in the neighborhood of  $P_1$ , where there is domination of a radiation fluid ( $\gamma = 4/3$ ). However, this point is very unstable, and the field evolves towards the neighborhood of point  $P_4$  so that the scalar field tracks the dominating radiation fluid.<sup>1</sup> Meanwhile, variable  $v \rightarrow 0$ , and the scalar field evolution is then driven away from point  $P_4$  and ends up making oscillations around point  $P_5$ , which is the only stable solution.

The expected DM intermediate epoch appears once the oscillations of the scalar field start. Because the scalar field energy density redshifts slower than the radiation one, it will dominate the expansion of the Universe before the (purely) de Sitter solution shows up.

In other words, the cosmological model transits by each one of the mentioned phases, spending some time in their respective neighborhoods. This will happen for a wide range of initial conditions, specially if the equipartition of energy is achieved at the end of inflation, thanks to the early exponential behavior of the model.

As mentioned before, a simpler scalar field model for DM and DE could be a quadratic potential plus a cosmological constant [20], but this model needs some extra fine-tuning in the initial conditions to recover an appropriate

DM-dominated epoch. The cosh potential is a good example that shows how higher-order interaction terms in the potential produce a nontrivial evolution at early times and help to alleviate the fine-tuning in the initial conditions.

Our results are also strongly dependent on the assumption that the scalar field in the model does not interact with other matter fields. This is just the standard DM hypothesis, even though it is known that models with interactions in the dark sector have a very different dynamic; see Ref. [24] for an instance of an interacting scalar field endowed with an exponential potential.

As a final remark, we mention again our implicit assumption that the SFDM model with a cosh potential is indeed a (double) unification of DM and DE; actually, the inflaton field could also be united if some extra hypotheses are used (see Ref. [25]). Unification is a possibility that arises naturally in the SFDM model [20], but one that requires full functionality of the model as an alternative to CDM. This is ongoing research that will be published elsewhere.

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<sup>1</sup>The necessary requirement for the occurrence of an equilibrium point,  $x' = y' = v' = 0$ , means that the Universe may keep evolving in the neighborhood of this point for quite a long time, depending on the nature of the fixed point: if the equilibrium point is a saddle in the phase space, it is correlated with a transient stage of the cosmic evolution, but if it is an attractor, it may only represent either the starting point of the evolution—past or early-time attractor—or its end point—future or late-time attractor.

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- [1] G. Hinshaw *et al.* (WMAP Collaboration), *Astrophys. J. Suppl. Ser.* **180**, 225 (2009).
  - [2] L. Bergstrom, *New J. Phys.* **11**, 105006 (2009).
  - [3] M. Taoso, G. Bertone, and A. Masiero, *J. Cosmol. Astropart. Phys.* 03 (2008) 022.
  - [4] M. Srednicki, *Eur. Phys. J. C* **15**, 143 (2000).
  - [5] A. Del Popolo, *Astronomy Reports* **51**, 169 (2007).
  - [6] A. Burkert, *IAU Symp.* **171**, 175 (1996); *Astrophys. J.* **447**, L25 (1995).
  - [7] B. Moore, *Nature (London)* **370**, 629 (1994); W. J. G. de Blok, S. S. McGaugh, A. Bosma, and V. C. Rubin, *Astrophys. J.* **552**, L23 (2001).
  - [8] J. A. Tyson, G. P. Kochanski, and I. P. Dell’Antonio, *Astrophys. J.* **498**, L107 (1998).
  - [9] T. Matos and F. S. Guzman, *Classical Quantum Gravity* **17**, L9 (2000).
  - [10] E. J. Copeland, M. Sami, and S. Tsujikawa, *Int. J. Mod. Phys. D* **15**, 1753 (2006).
  - [11] T. Matos and L. A. Urena-Lopez, *Classical Quantum Gravity* **17**, L75 (2000).
  - [12] A. Bernal, T. Matos, and D. Nunez, *Rev. Mex. Astron. Astrofis.* **44**, 149 (2008).
  - [13] T. Matos and L. A. Urena-Lopez, *Phys. Rev. D* **63**, 063506 (2001).
  - [14] V. Sahni and L. M. Wang, *Phys. Rev. D* **62**, 103517 (2000).
  - [15] T. Matos, J. A. Vazquez, and J. Magana, arXiv:0806.0683 [Mon. Not. R. Astron. Soc. (to be published)].
  - [16] V. A. Belinsky, I. M. Khalatnikov, L. P. Grishchuk, and Y. B. Zeldovich, *Phys. Lett.* **155B**, 232 (1985); A. de la Macorra and G. Piccinelli, *Phys. Rev. D* **61**, 123503 (2000); L. A. Urena-Lopez and M. J. Reyes-Ibarra, arXiv:0709.3996.
  - [17] T. Matos, J. R. Luevano, and H. Garcia-Compean, *Int. J. Mod. Phys. A* **23**, 1949 (2008).
  - [18] M. S. Turner, *Phys. Rev. D* **28**, 1243 (1983).
  - [19] T. Matos, arXiv:0909.3634.

- [20] A. R. Liddle and L. A. Urena-Lopez, *Phys. Rev. Lett.* **97**, 161301 (2006); A. R. Liddle, C. Pahud, and L. A. Urena-Lopez, *Phys. Rev. D* **77**, 121301 (2008).
- [21] E. J. Copeland, A. R. Liddle, and D. Wands, *Phys. Rev. D* **57**, 4686 (1998).
- [22] A. A. Coley, *Dynamical Systems and Cosmology* (Kluwer, Dordrecht, Netherlands, 2003), p. 200; arXiv:gr-qc/9910074.
- [23] W. Fang, Y. Li, K. Zhang, and H. Q. Lu, *Classical Quantum Gravity* **26**, 155005 (2009).
- [24] C. G. Boehmer, G. Caldera-Cabral, R. Lazkoz, and R. Maartens, *Phys. Rev. D* **78**, 023505 (2008).
- [25] J. E. Lidsey, T. Matos, and L. A. Urena-Lopez, *Phys. Rev. D* **66**, 023514 (2002).