

Testing dark matter halos using rotation curves and lensing: A warning on the determination of the halo mass

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There are two observations of galaxies that can offer some insight into the nature of dark matter (DM), namely, the rotation curves and the gravitational lensing. While the first one can be studied using the Newtonian limit, the second one is completely relativistic. Each one separately cannot determine the nature of DM, but both together give us key information about this open problem. In this work we use a static and spherically symmetric metric to model the DM halo in a galaxy or in a galaxy cluster. The metric contains two free functions, one associated with the distribution of mass and the other one with the gravitational potential. We use galactic, typical rotation curves to uniquely determine the kinematics of the halos. We compute separately the mass functions for a perfect fluid and a scalar field, and demonstrate that both models can be fitted to the observations, though with different masses. We then employ lensing to discriminate between these models. This procedure represents a test of models using two measurements: rotation curves and lensing.

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There are at least four main groups of cosmological observations which are consistently described by the Λ CDM model, namely, the acoustic peaks in the cosmic microwave background radiation, Supernovae type Ia data, rotation curves of spirals and dynamics of galactic clusters, and their gravitational lensing [1–3]. The last two of these observations are related most significantly with the presence of dark matter (DM), where a consistent model considers a DM halo surrounding the galaxies and galactic clusters.

Using Newtonian gravity we can infer the mass of the DM halos from observations of velocity profiles in galaxies and galaxies clusters. Alternatively, one can use lensing images or distortions of background galaxies due to the space-time curvature [4,5]. This is notorious, particularly in the case of gravitational lensing, which is a general relativistic effect *par excellence*, as it strictly cannot be dealt with using Newtonian physics.

There have been many attempts to model DM halos in the general relativistic framework, and some of them have shown that general relativistic effects can be important [6–13], but in general they make use of approximations or limits that in some cases represent assumptions about DM properties. The scalar field DM model is an alternative proposed in the past [14] to fit the observed amount of substructure [15], the critical mass of galaxies [16], the rotation curves of galaxies [17], the central density profile

of low surface brightness galaxies [18], the evolution of the cosmological densities [19], among other topics.

In the present work we focus on two models for the DM halo (a perfect fluid and a scalar field) and explicitly show exactly what is determined by the observations and what are extra assumptions. This is particularly important as the unknown nature of the DM is one of the most relevant questions that one would like to solve. It is clear that to make assumptions specifically on the nature of DM, in order to obtain information on the nature of the DM, is solving the problem.

In our study we consider a static and spherically symmetric space-time in general relativity, described by the line element

$$ds^2 = -e^{2\Phi/c^2} c^2 dt^2 + \frac{dr^2}{1 - \frac{2Gm}{c^2 r}} + r^2 d\Omega^2, \quad (1)$$

where $d\Omega = d\theta^2 + \sin^2\theta d\varphi^2$. The gravitational potential $\Phi(r)$ and the mass function $m(r)$ are functions of the radial coordinate only. In fact, due to the symmetries of this space-time, all physical quantities depend only on r . The Einstein equations are the known set [20]:

$$m' = -\frac{4\pi r^2}{c^2} T^t{}_t, \quad (2)$$

$$\left(1 - \frac{2mG}{c^2 r}\right) \frac{\Phi'}{c^2} - \frac{mG}{c^2 r^2} = \frac{4\pi Gr}{c^4} T^r{}_r, \quad (3)$$

where prime $\prime \equiv \partial/\partial r$. The potentials of such space-time are determined by the DM halo. In order to see the difference between two hypothesis on the nature of DM, we will

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consider two types of composition for halos: a perfect fluid and a scalar field.

The above equations are complemented by the conservation equation of the matter-energy generating the curvature of the space-time. But given the different nature of the fluids considered here, this equation is treated separately for each fluid.

In the case of the perfect fluid, the stress-energy tensor is given by $T_{\mu\nu} = (\rho + p/c^2)u_\mu u_\nu + pg_{\mu\nu}$, where the density is $\rho = (1 + \epsilon)\rho_0$, where ρ_0 is the rest mass energy density and ϵ the internal energy per unit mass, u^μ is the comovil four velocity, normalized as $u_\mu u^\mu = -c^2$, and p is the pressure. The conservation equation, $T^\mu{}_{\nu;\mu} = 0$, implies the field equation

$$(\rho c^2 + p) \frac{\Phi'}{c^2} + p' = 0, \quad (4)$$

which can be rewritten as

$$T^{r'}{}_r + (T_r^r - T_t^t) \frac{\Phi'}{c^2} = 0. \quad (5)$$

Now, for the scalar field the stress-energy tensor is given by

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}(g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} + 2V(\phi)), \quad (6)$$

where $\phi_{,\alpha} = \partial\phi/\partial x^\alpha$; and $V(\phi)$ is the scalar potential. The components of the stress-energy tensor are

$$\begin{aligned} T^t{}_t &= -\frac{1}{2}\left(1 - \frac{2Gm}{c^2 r}\right)\phi'^2 - V(\phi), \\ T^r{}_r &= \frac{1}{2}\left(1 - \frac{2Gm}{c^2 r}\right)\phi'^2 - V(\phi), \\ T^\theta{}_\theta &= T^\varphi{}_\varphi = T^t{}_t. \end{aligned} \quad (7)$$

From the conservation equation for the scalar field, $T^\mu{}_{\nu;\mu} = 0$, one obtains a field equation, the Klein-Gordon equation,

$$\phi'' + \left(\frac{\Phi'}{c^2} - \frac{m'G}{c^2 r} + \frac{(3mG/c^2) - 2r}{r^2}\right)\phi' - \frac{\partial V}{\partial\phi} = 0, \quad (8)$$

which can be written as

$$T^{r'}{}_r + (T_r^r - T_t^t)\left(\frac{\Phi'}{c^2} + \frac{2}{r}\right) = 0, \quad (9)$$

and is remarkably similar to the field equation for the perfect fluid, Eq. (5). Given this similarity, it is convenient for our mathematical description to consider the single field equation for both types of matter

$$T^{r'}{}_r + (T_r^r - T_t^t)\left(\frac{\Phi'}{c^2} + \frac{2a}{r}\right) = 0, \quad (10)$$

in which $a = 0$ for the perfect fluid, and $a = 1$ for the scalar field. Notice that if one considers a sort of perfect fluid given by $T^\mu{}_\nu = \text{diag}(-\rho, p, p_i, p_i)$, p_i (some times

called ‘‘tangential’’ pressure) denotes a term representing the ignorance we have on the features of the fluid. This pressure p_i is related to the other fluid variables as $p_i = (1 - a)p - a\rho c^2$, [see Eq. (10)], where ‘‘ a ’’ takes, in principle, any value. There are works that have discussed this field equation considering a as a free parameter [11–13]. For the purpose of the present work we will consider only the two extremal cases, $a = 0$ and $a = 1$, but the discussion can be directly applied for these cases as well.

In this way, the system of equations which must be solved are, the Einstein equations, Eqs. (2) and (3), and the field equation, Eq. (10). In either case, there are four unknown functions, m , Φ , p and ρ , for the case where the curvature of the space-time is due to the perfect fluid, or m , Φ , ϕ and $V(\phi)$ when the curvature is caused by the scalar field. Thus, we have three equations for four unknown functions. In either case, we need only one extra data. It is important to underline this fact. Once the extra data is given, there is no room left for any other assumption; the rest of the functions are determined by the system of equations. If, for instance, we give an equation of state for the perfect fluid, $p = p(\rho)$ or, in the case of the scalar field, an explicit form for the potential, $V(\phi)$, there is no freedom left to choose the form of the rest of the functions, they will be determined by the system of equations.

Following the line of work presented in [8], we use observational results to close the system of equations. In the case of galactic halos, two main observations can serve to obtain the desired information: measurements of rotation curves in spirals and light deflection by lensing. In this work we choose the former to complement the above field equations and use the latter to discriminate between different halo type models.

Rotation curves: The motion of test particles in such space-time is determined by the geodesic equations and, for test particles in circular motion, there is a relationship between the gravitational potential, Φ , and the tangential velocity of those particles, v_c :

$$\frac{\Phi'}{c^2} = \frac{\beta^2}{r}, \quad (11)$$

where we have defined $\beta^2 = \frac{v_c^2}{c^2}$. This tangential velocity is the one measured by observations of rotation curves in galaxies. Thus, v_c is an observable function, and by means of Eq. (11), the gravitational potential can be determined. Thus, given this observable, there is no room left for an equation of state for the perfect fluid or for a given scalar field potential.

Moreover, as long as the magnitude of the observed velocities are small with respect to the speed of light, this justifies the validity of one of the weak field approximations $\Phi/c^2 \ll 1$ that one usually assumes by taking the weak field limit. Here we want to emphasize that the approximations $2Gm/c^2 r \ll 1$ and especially $p \ll \rho$ are, in general, extra hypothesis which strongly depend

upon the nature of the DM type. It is clear that if all these conditions are satisfied, then the above system of equations, Eqs. (2)–(4), reduces to the hydrodynamic set of equations for the case of the perfect fluid model. But, for example, in the case of the scalar field, there is no Newtonian limit, and one has to be careful with these approximations.

After substituting Eq. (11) into the gravity equations, Eqs. (2) and (3), we obtain an equation (with no approximations) for the mass function as the only free function

$$m' + P(r)m = Q(r), \quad (12)$$

with

$$P(r) = \frac{2r\beta^{2l} - (1 + 2\beta^2)(3 - 2a - \beta^2)}{(1 - 2a + \beta^2)r}, \quad (13)$$

$$Q(r) = \frac{c^2 r\beta^{2l} - \beta^2(2 - 2a - \beta^2)}{G(1 - 2a + \beta^2)}.$$

The functions $P(r)$ and $Q(r)$ depend on the type of fluid we are dealing with (a) and on the rotation curves profile.

The mass function can be expressed in terms of the gravitational potential, Φ , through the integral

$$m = \frac{\int e^{\int P(r')dr'} Q(r)dr + C}{e^{\int P(r')dr'}}, \quad (14)$$

where the value of the integration constant, C , is set by the appropriate boundary conditions.

For the case of the perfect fluid, the density and pressure are directly computed from Eqs. (2) and (3), respectively.

For the scalar field, using the expressions Eqs. (7), we obtain that

$$\phi'^2 = \frac{c^4}{4\pi G r^2} \left(\frac{Gm'}{c^2} - \frac{Gm}{c^2 r} + \beta^2 \right), \quad (15)$$

$$V(\phi(r)) = \frac{c^2}{8\pi r^3} [m(1 + 2\beta^2) + m'r] - \frac{c^4}{8\pi G} \left(\frac{\beta^2}{r^2} \right). \quad (16)$$

Once the function $\beta(r)$ is given, the scalar field and its potential are straightforwardly determined in terms of the radial coordinate. In order to obtain the form $V(\phi)$, one needs to invert the solution for the scalar field ($r = r(\phi)$), and to substitute it into Eq. (16). As shown below, this procedure works at least for simple β functions.

In this way, we have shown that the mass function m , associated to a galactic halo by means of the rotation velocity strongly depends on the DM model that is being considered. The single observation of the rotation curve is not sufficient to determine the nature of DM and hence the mass associated with the halo. Moreover, we have shown that the relationship between the pressure and the density, or between the scalar field and the scalar potential is fixed,

up to integration constants, once the rotation velocity is employed.

Lensing: The other observation concerns the gravitational lensing, that for the line element given by Eq. (1), the deflection of the light ray, $\Delta\varphi$, at the radius of maximal approach, r_m , is given by [21],

$$\Delta\varphi = - \int_{\infty}^{r_m} \frac{r_m dr}{r^2 \sqrt{\left(1 - \frac{2Gm}{rc^2}\right) \left[e^{-2(\Phi/c^2)} e^{2(\Phi(r_m)/c^2)} - \frac{r_m^2}{r^2} \right]}}. \quad (17)$$

Since the gravitational potentials and the fluid variables are already determined by the rotation curves of spirals, deflection angle measurements can serve to discriminate between models. Here we deal with two examples, perfect fluid and scalar field DM models. Yet, observations of spirals that lenses light are not very common; however, the first examples of them have recently appeared [22]. We remark that from the expression of the deflection angle, it is a large step to infer the mass of the DM halo based solely on the observation of the deflection angle. A supposition has to be made on the relation between the gravitational potential, Φ , and the mass function, m [21]. Such supposition, as we have shown, not only strongly depends on the type of matter, but also on the specific characteristics of the type of matter considered.

In the next section we present some examples of known rotation curve profiles to give a quantitative description to these conclusions.

Examples: The idea in this section is to stress the conclusions that we are presenting by means of considering a typical observation of rotation velocities in spirals and to directly determine the gravitational mass in each case, when the DM is a perfect fluid (dust) and when it is a scalar field.

In practice we can consider a velocity distribution, as a phenomenological model, for instance the velocity profile coming from N-body simulations given by Navarro-Frenk-White (NFW) [23,24] or a Burkert profile [25] given by the phenomenological of rotation curves [26], to determine the mass of each type of fluid. We will show that the gravitational mass inferred by the same velocity profile is strikingly different for the perfect fluid and scalar field cases.

Constant velocity profile: We will consider the simplest case of constant rotation curves as our first example. Although there are some examples of galaxies that present a constant velocity profile, for a few disk length scales [27], this is not a typical behavior, being that our own Galaxy is a good counter example [28] and, in fact, there is an important rotation curve phenomenology described by the universal rotation curve [26,29]. However, the constant velocity profile offers us the mathematical simplicity to obtain straightforward analytical results and to show the main point of our work.

For the gravitational potential, from Eq. (11), when the velocity function is a constant, β_0 , we get

$$\Phi = c^2 \ln\left(\frac{r}{r_0}\right)^{\beta_0^2}. \quad (18)$$

The mass function can be analytically obtained for any value of the parameter a as

$$m_{\beta_0} = \frac{c^2}{G} \left(\frac{\beta_0^2(2(1-a) - \beta_0^2)}{2(1 + 2(1-a)\beta_0^2 - \beta_0^4)} r + C r^{(1+2\beta_0^2)(3-2a-\beta_0^2)/(1-2a+\beta_0^2)} \right), \quad (19)$$

where C is the integration constant of Eq. (14). For the case of DM described by a perfect fluid, $a = 0$, we fix this constant to zero in order to avoid changes in the signature of the line element, Eq. (1). Thus, the mass function, and the corresponding pressure and density in the case of the perfect fluid are given by

$$m_{pf} = \frac{c^2}{2G} \frac{\beta_0^2(2 - \beta_0^2)}{1 + 2\beta_0^2 - \beta_0^4} r, \quad (20)$$

$$\rho = \frac{c^2}{4\pi G} \frac{\beta_0^2(2 - \beta_0^2)}{r^2(1 + 2\beta_0^2 - \beta_0^4)}, \quad (21)$$

$$p = \frac{c^4}{8\pi G} \frac{\beta_0^4}{r^2(1 + 2\beta_0^2 - \beta_0^4)}, \quad (22)$$

$$\frac{p}{\rho} = \frac{\beta_0^2 c^2}{2(2 - \beta_0^2)} = \text{const.} \quad (23)$$

We can see in the limit of very small velocities, $\beta_0 \ll 1$, we recover the Newtonian limit, and pressure is negligible with respect to the density as we mentioned above; however, it is not zero and, actually, we obtain a barotropic equation of state, $p = w_0 \rho$.

On the other hand, considering the DM halo due to a scalar field, the mass function is obtained from Eq. (19), with $a = 1$. In this case, the mass function has a very peculiar behavior. The first term is small, proportional to β_0^4 , but negative. The second term, proportional to the constant C , goes as $r^{-(1+2\beta_0^2)}$; thus, by choosing a positive value for the constant C , one can have a positive mass function for a large region, but this function will present a divergence at the origin. Of course this result was expected, as the space-time metric is static. In order to avoid this problem, we had to take nonstatic space-times, like the oscillations [16], but this is beyond the scope of this work. It can be shown, however, that this divergence is covered by an apparent horizon. Some features of this case of scalar field with a nonzero constant C in the mass function, have been discussed in [11]. For the purpose of this work, we only notice that the geometric functions and those of the scalar field, have a nonintuitive behavior, but are consistent with the rotational curve. Explicitly, for the case of $C = 0$, the mass function is

$$m_{sf} = -\frac{c^2}{2G} \frac{\beta_0^4}{1 - \beta_0^4} r, \quad (24)$$

and, with the geometric functions determined, the scalar field and scalar potential are completely fixed, given by

$$\phi = \pm \sqrt{\frac{c^4}{4\pi G}} \beta_0 \ln\left(\frac{r}{r_0}\right), \quad (25)$$

$$V(r) = -\frac{c^4}{8\pi G} \frac{\beta_0^2}{r^2(1 - \beta_0^2)}, \quad (26)$$

$$V(\phi) = -\frac{c^4}{8\pi G} \frac{2\beta_0^2}{(1 - \beta_0^2)r_0^2} e^{\mp 2\sqrt{(4\pi G/c^4)}(\phi/\beta_0)}, \quad (27)$$

where the expression for the scalar field, $\phi(r)$, was inverted to obtain $r(\phi)$, and then express the scalar potential in terms of ϕ , as explained previously. The “effective mass” of the scalar field, $m_{\text{eff}} \sim \frac{2}{\sqrt{1 - \beta_0^2} r_0}$, depends inversely on the characteristic distance of the halo. This distance is of the order of kilo parsecs, and $\beta_0 \sim 10^{-3}$; thus, it will turn into a typical mass for scalar field in a galaxy, that corresponds a very light boson mass $\sim 10^{-23}$ eV/ c^2 . This result is in agreement with the one obtained in previous works, see for example [15].

Going back to our previous discussion, notice how remarkably different are the mass expressions derived from each type of fluid, being both consistent with the observed rotation velocities. This is the simplest case in which we can show how the single observation of the rotation velocities in halos determines the features of the perfect fluid model or the scalar field.

Although the mass associated to the scalar field results negative, and this can be taken as a no-go result for static scalar field halos [30], rotation curves of spirals are not exactly flat (see discussion in [26,29]) and, in addition, we have to be cautious with the supposition of a static metric which is very restrictive for the scalar field. Thus, the above result should not be taken as definitive; at most, it should be taken as a remark that a static DM halo is not well described by a static scalar field. A negative mass, or positive gravitational potential, has been known for a long time [31] from the fits to rotation curves using modifications of Newtonian gravity in which a scalar field induces a Yukawa-type force. At the end, demanding a constant velocity profile all the way in the radial direction implies an effective repulsive force to be acting on test particles in the galaxy.

In any case, it emphasizes our point, which is to show how strongly the determination of the mass of the DM depends on the type of matter considered.

The deflection angle for the case of constant rotation velocity, considering the perfect fluid and the scalar field with the constant $C = 0$, implies the following expressions:

$$\Delta\varphi = \int_0^1 \frac{dx}{\sqrt{A_{\text{type}}(x^2\beta_0^2 - x^2)}}, \quad (28)$$

$$A_{pf} = \frac{1}{1 - \beta_0^2(\beta_0^2 - 2)}, \quad (29)$$

$$A_{sf} = \frac{1}{1 - \beta_0^4}, \quad (30)$$

where we have defined $x = \frac{r_m}{r}$. Since in any case the deflection angle is a constant, i.e. it does not depend of the maximal approach radius r_m , it can be evaluated for a typical value of the velocity. For comparison we take the value $\beta_0 = 1/1200$, that corresponds to a velocity of $v_c = 250$ km/s. Evaluating the deflection angle, we get

$$\alpha_{pf} = 0.899\,547, \quad (31)$$

$$\alpha_{sf} = 0.449\,546, \quad (32)$$

for both cases the deflection angle is given in arc second units. We can see that there is a difference of almost a half arc second between them, and the simultaneous observation of the rotation velocity and the deflection of light produced by the galactic halo, can teach us about the true nature of the DM.

We notice that the deflection angle for the scalar field with a nonzero value of the constant C in the mass function takes very large values, a fact which certainly allows us to discard this option as a model for the DM halo, independently of any interpretations of the mass function.

Now we study an example that is less striking though.

NFW velocity profile: Independently from its origin, the NFW profile [23,24] is considered *per se* as a viable fitting model to describe galactic kinematics. This profile has been subject to geometrical studies elsewhere [32,33]. In this example, we assume this profile as a valid phenomenological galactic profile for the galactic data. We obtain the usual expression for the mass derived within this description, and compare it with the same form of the rotation velocity, but considering that it is due to a DM halo composed of a scalar field.

The velocity profile in the NFW model [23,24] is given by

$$v_T^2 = \frac{\sigma_0^2 r_0}{r} \left(-\frac{r/r_0}{1+r/r_0} + \ln[1+r/r_0] \right), \quad (33)$$

where $\sigma_0 = 4\pi G\rho_0 r_0^2$ is a characteristic velocity of stars in the halo, given in terms of a characteristic density, and r_0 is a scale radius. Given this velocity profile, we have to solve Eq. (14) with $a = 0$ for the perfect fluid and with $a = 1$ for the scalar field. In neither case is there an analytical solution, thus we have integrated the equations numerically. We do not want to treat specific galaxies here but want to emphasize the differences between the galaxy models. Therefore, we set σ_0 and r_0 to some typical values. In our plots we assume geometric units ($G = c = 1$), and

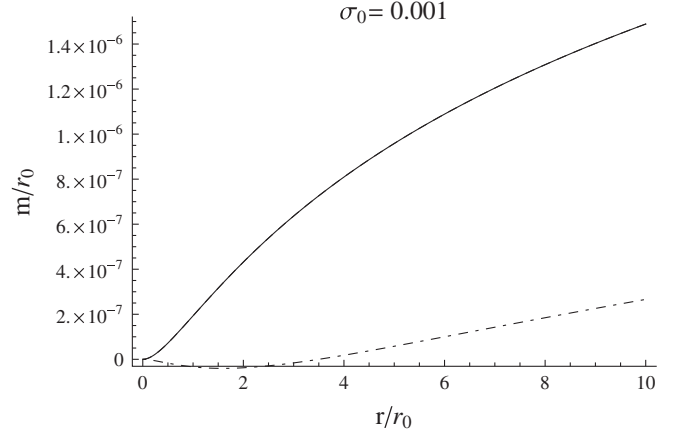


FIG. 1. Comparison of the masses using the rotation velocity profile from the NFW model with a perfect fluid (upper curve) and scalar field (lower curve).

therefore the characteristic velocity takes values, $0 < \sigma_0 < 1$, and the mass is less than the unity. For definiteness, we assume $\sigma_0 = 0.001$ and $r_0 = 1$. In Fig. 1, we plot both halo masses (perfect fluid and scalar field). Disregarding the behavior near the origin, as long as we are considering the outside region, as mentioned above, we see that the mass associated to the halo in each case is different. We now consider lensing. By integrating Eq. (11) for the given rotation velocity, Eq. (33), we obtain the following expression for the gravitational potential:

$$\Phi = -\sigma_0 r_0 \frac{\ln(1 + \frac{r}{r_0})}{r}. \quad (34)$$

We substitute this expression, together with the corresponding numerical solution one for the mass in each case, in the equation for the deflection angle, Eq. (17), and perform the integration varying the value of the radius of maximal approach, r_m . The results are plotted in Fig. 2.

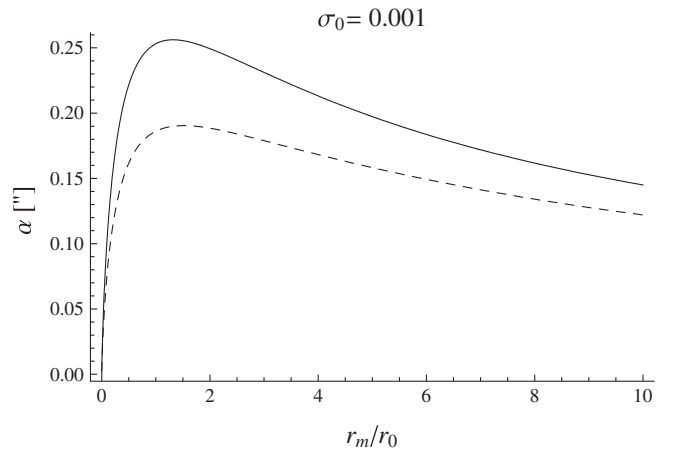


FIG. 2. Deflection angle generated by the gravitational lensing of a NFW rotation profile with a perfect fluid (upper curve) and scalar field (lower curve).

As we see, the observation of the deflection angle can determine which type of matter is actually composing the DM halo.

In this example, the mass associated to the scalar field model is essentially positive and a well-behaved function, that is, it does not follow the no-go result mentioned in the previous example. Though there is a small region where the mass becomes negative. This is due to the fact that we are demanding the velocity profile to grow in that region. In a real setting however the stellar disc adds to the velocity profile; thus, we expect that its contribution avoids negative mass regions for the scalar field.

With these examples it is clear how two different types of matter (perfect fluid and scalar field) can be consistent

with the observation of rotation curves of DM halos, though they lead to different conclusions to the mass function inferred. The deflection of light can then be used to discriminate between the two models. Even though the mass function for some model has not been an intuitively expected behavior, it is necessary to use the observation in order to discard the model, being aware of the assumptions made during the derivation of such conclusions.

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