Lectures on Numerical Relativity #5

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Outline

- Summary of Lecture #4
  - Conformal TT(Transverse-Traceless) decomp.; Physical TT decomp.
  - Thin Sandwich Decomposition (Original / Extended)
- Binary black hole initial data (Today)
- Next lectures (tentative plan)
  - Initial data: Multi-grid method (August 31st)

CTT Decomposition

- Conformal Transverse-Traceless decomposition:

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \tag{1}$$

$$K^{ij} = \psi^{-10} \tilde{A}^{ij} + \frac{1}{3} \psi^{-4} \tilde{\gamma}^{ij} K \tag{2}$$

$$\tilde{A}^{ij} = (\tilde{l}V)^{ij} + \tilde{M}^{ij} \tag{3}$$

$$\tilde{\Delta}_l V^i - \frac{2}{3} \psi^6 \tilde{D}^i K = -\tilde{D}_j \tilde{M}^{ij} + \underline{8\pi \psi^{10} j^i} \tag{4}$$

$$\tilde{D}^i \tilde{D}_i \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = \underline{-2\pi \psi^5 \rho} \tag{5}$$

- Note that

$$(\tilde{l}X)^{ij} = \tilde{D}^i X^j + \tilde{D}^j X^i - \frac{2}{3} \tilde{\gamma}^{ij} \tilde{D}_l X^l \tag{6}$$

$$\tilde{\Delta}_l V^i \equiv \tilde{D}_j (\hat{l}V)^{ij} = \tilde{D}^j \tilde{D}_j V^i + \frac{1}{3} \tilde{D}^i (\tilde{D}_j V^j) + \tilde{R}^i_j V^j \tag{7}$$

$$\rho = \psi^{-8} \tilde{\rho} \tag{8}$$

$$j^i = \psi^{-10} \tilde{j}^i \tag{9}$$

$$S = \psi^{-8} \tilde{S} \tag{10}$$

PTT Decomposition

- Physical Transverse-Traceless decomposition (in vacuum):

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \tag{11}$$

$$K^{ij} = \psi^{-4}(\tilde{A}^{ij} + \frac{1}{3}\tilde{\gamma}^{ij} K) \tag{12}$$

$$\tilde{A}^{ij} = (\hat{l}V)^{ij} + \psi^{-6}\tilde{M}^{ij} \tag{13}$$

$$\tilde{\Delta}_l V^i + 6(\tilde{l}V)^{ij}\tilde{D}_j \ln\psi = \frac{2}{3}\tilde{D}^i K - \psi^{-6}\tilde{D}_j \tilde{M}^{ij} \tag{14}$$

$$\tilde{D}^i\tilde{D}_i\psi - \frac{1}{8}\psi\tilde{R} - \frac{1}{12}\psi^5 K^2 + \frac{1}{8}\psi^5 \tilde{A}_{ij}\tilde{A}^{ij} = 0 \tag{15}$$

Thin Sandwich Decomp. (Original)

- Equations for the (conformal) thin-sandwich decomposition:

$$
\begin{aligned}
\gamma_{ij} &= \psi^4 \tilde\gamma_{ij} && (16)\\
K^{ij} &= \psi^{-10}\tilde A^{ij} + \frac{1}{3}\psi^{-4}\tilde\gamma^{ij} K && (17)\\
\tilde A^{ij} &= \frac{1}{2\tilde\alpha}\left((\tilde l\beta)^{ij} - \tilde u^{ij}\right) && (18)\\
\tilde\Delta_l \beta^i - (\tilde l\beta)^{ij}\tilde D_j \ln\tilde\alpha + \frac{4}{3}\tilde\alpha\psi^6 \tilde D^i K &= \tilde\alpha \tilde D_j\left(\frac{1}{\tilde\alpha}\tilde u^{ij}\right) + \underline{16\pi\tilde\alpha\psi^{10} j^i} && (19)\\
\tilde D^i \tilde D_i \psi - \frac{1}{8}\psi\tilde R - \frac{1}{12}\psi^5 K^2 + \frac{1}{8}\psi^{-7}\tilde A_{ij}\tilde A^{ij} &= \underline{-2\pi\psi^5\rho} && (20)
\end{aligned}
$$

- Freely specify, $\{\tilde\gamma_{ij}, \tilde u_{ij}, K, \tilde\alpha, j^i, \rho\}$.
- Solve for $\{\psi, \beta^i\}$.

Thin Sandwich Decomp. (Extended)

- Instead of choosing $\tilde{\alpha}$, use $\dot{K}$ equation from the E. Eqns (and combine it with HCE).
- Equations for the Extended Conformal Thin Sandwich (or just CTS now):

$$
\begin{aligned}
\gamma_{ij} &= \psi^4 \tilde{\gamma}_{ij} && (21)\\
K^{ij} &= \psi^{-10}\tilde{A}^{ij} + \frac{1}{3}\psi^{-4}\tilde{\gamma}^{ij} K && (22)\\
\tilde{A}^{ij} &= \frac{1}{2\tilde{\alpha}}\left((\tilde{l}\beta)^{ij} - \tilde{u}^{ij}\right) && (23)\\
\tilde{\Delta}_l \beta^i - (\tilde{l}\beta)^{ij}\tilde{D}_j \ln\tilde{\alpha} + \frac{4}{3}\tilde{\alpha}\psi^6 \tilde{D}^i K &= \tilde{\alpha}\tilde{D}_j(\frac{1}{\tilde{\alpha}}\tilde{u}^{ij}) + \underline{16\pi\tilde{\alpha}\psi^{10} j^i} && (24)\\
\tilde{D}^i\tilde{D}_i\psi - \frac{1}{8}\psi\tilde{R} - \frac{1}{12}\psi^5 K^2 + \frac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} &= \underline{-2\pi\psi^5\rho} && (25)\\
\tilde{D}^i\tilde{D}_i(\tilde{\alpha}\psi^7) - (\tilde{\alpha}\psi^7)(\frac{1}{8}\tilde{R} + \frac{5}{12}K^2\psi^4 &+ \frac{7}{8}\tilde{A}^{ij}\tilde{A}_{ij}\psi^{-8} + \underline{2\pi(\tilde{\rho} + 2\tilde{S})\psi^{-4}})\\
&+ (\dot{K} - \beta^i \tilde{D}_i K)\psi^5 = 0 && (26)
\end{aligned}
$$

where we used $\rho = \psi^{-8}\tilde{\rho}$, $S = \psi^{-8}\tilde{S}$.

- Freely specify: $\{\tilde{\gamma}_{ij}, \tilde{u}_{ij}, K, \dot{K}, \tilde{\rho}, \tilde{j}^i, \tilde{S}\}$.
- Solve for $\{\psi, \beta^i, \alpha\}$.

Extended Thin Sandwich (con't)

- For quasi-equilibrium, choose coordinate (t, x^i) adapted to the helical killing vector, $\frac{\partial}{\partial t} = \xi$. (In other words, choose corotating frame.)
- Then choosing $\partial_t \tilde{\gamma}^{ij} = \tilde{u}^{ij} = 0$ and $\dot{K} = 0$ for quasiequilibirum makes sense.
- Choose in addition $K = 0$ (maximal slicing) and conformal flatness.
- Then equations simplify:

$$\tilde{D}^i \tilde{D}_i \psi + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = \underline{-2\pi \psi^{-3} \tilde{\rho}} \quad (27)$$

$$\tilde{\Delta}_l \beta^i - (\tilde{l}\beta)^{ij} \tilde{D}_j \ln \tilde{\alpha} = \underline{16\pi \tilde{\alpha} \tilde{j}^i} \quad (28)$$

$$\tilde{D}^i \tilde{D}_i (\tilde{\alpha}\psi^7) - (\tilde{\alpha}\psi^7)(\frac{7}{8} \tilde{A}^{ij} \tilde{A}_{ij} \psi^{-8} + \underline{2\pi(\tilde{\rho} + 2\tilde{S})\psi^{-4}}) = 0 \quad (29)$$

where $\tilde{D}^i$ are flat operators.

- Once we solve for $\psi, \beta^i, \tilde{\alpha}$, we construct the initial data using

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} = \psi^4 f_{ij} \quad (30)$$

$$K^{ij} = \psi^{-10} \frac{1}{2\tilde{\alpha}} (\tilde{l}\beta)^{ij} \quad (31)$$

Black Hole Initial Data

- Canonical simplifications usually taken in the CTT context:

$$\begin{aligned} \tilde{\gamma}_{ij} &= f_{ij} \quad (\text{conformal} \quad \text{flatness}) && (32)\\ K &= 0 \quad (\text{maximal} \quad \text{slicing}) && (33)\\ \psi(r \to \infty) &= 1 \quad (\text{asymptotic} \quad \text{flatness}) && (34) \end{aligned}$$

- Then momentum contraint equations completely decouple from Hamiltonian constraint equation.
- We further choose, $\tilde{M}^{ij} = 0$. Then, MCEs become

$$\tilde{D}^j \tilde{D}_j V^i + \frac{1}{3} \tilde{D}^i (\tilde{D}_j V^j) = 0 \tag{35}$$

- A solution to this equation:

$$V^i = -\frac{1}{4r} [7P^i + n^i n_j P^j] + \frac{1}{r^2} \epsilon^{ijk} n_j S_k. \tag{36}$$

where P^i, S^i are vector parameters, n^i is the outward-pointing unit normal of a sphere in the flat conformal space ($n^i \equiv \frac{x^i}{r}$), and ϵ^{ijk} is the 3-D Levi-Civita tensor.

- Then, we obtain the solution constructed first by Bowen and York (1980).

$$\tilde{A}^{ij} = \frac{3}{2r^2}[P_i n_j + P_j n_i - (f_{ij} - n_i n_j)P^k n_k] + \frac{3}{r^3}[\epsilon_{kil} S^l n^k n_j + \epsilon_{kjl} S^l n^k n_i] \quad (37)$$

- Consider total ADM momenta on the "sphere at infinity" (asssuming $K = 0$):

$$P_i = \frac{1}{8\pi}\int_\infty A_{ij} d^2 S_j \quad (38)$$

$$S_i = \frac{1}{16\pi}\int_\infty (x^j K^{km} - x^k K^{jm}) d^s S_m \quad (39)$$

- Note that P_i, and S_i have the meaning of being *physical* linear and spin-angular momentum of the initial-data hypersurface.
- Note, furthermore, that because the momentum constraint equations are linear, we can add any number of solutions of the above form to represent a collection of linear and angular momentum sources such as multiple-black-hole solutons.

Puncture Data

- Bowen-York data first used in the context of “Misner” data.
- In Misner data, spatial hypersurface consists of two sheets (upper and lower universes) connected by two throast. Misner data assumes an isometry w.r.t the throats between upper sheet and lower sheet.
- But original BW extrinsic curvate is not inversion symmetric and does not satisfy isometry condition. There is a method (method of image) which will make the soluton inversion symmetric by adding extra terms to the eqn (37).
- “Puncture” method, on the other hand, does not require inversion symmetry allows three-sheet topology of the spiatial slice. We can use the formula, (37), directly. It is one of the most commonly used method in 3D NR mainly because of its simplicity.
- Puncture method adopts three canonical assumptions, eqns (32,33,34) and take Bowen-York form for $\tilde{A}_{ij}$.
- Then, solve Hamiltonian constraint equation,

$$\tilde{D}^i \tilde{D}_i \psi + \frac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} = 0 \qquad (40)$$

- Assume that the conformal factor takes the form

$$\psi = \frac{1}{\chi} + u \tag{41}$$

where so-called Brill-Linquist factor is given by

$$\frac{1}{\chi} \equiv \sum_{i=1}^{N} \frac{m_i}{2|\vec{x} - \vec{C}_i|} \tag{42}$$

- Here m_i is (bare) mass of each black hole and $\vec{C}_i$ location of each hole.
- HCE becomes an equation for u,

$$\tilde{D}^i \tilde{D}_i u + \eta (1 + \chi u)^{-7} = 0 \tag{43}$$

where

$$\eta = \frac{1}{8} \chi^7 \tilde{A}_{ij} \tilde{A}^{ij} \tag{44}$$

- Near each singular point or "puncture", $\chi \sim |\vec{x} - \vec{C}_i|$, so $\tilde{A}_{ij}\tilde{A}^{ij}$ behaves no worse than $|\vec{x} - \vec{C}_i|^{-6}$ and so η vanishes at least as fast as $|\vec{x} - \vec{C}_i|$ at the punctures.
- With this behavior, the existence and uniqueness of C^2 solutions can be shown (Brandt and Brugmann, 1997).
- To construct multi-hole solutions: (1) specify (bare) "masses" and (coordinate) "locations", (2) specify linear momenta, P^i, and spin momenta, S^i, and (3) Solve HCE for u.
- Compared to the Misner data where careful consideration is required on the inner boundary (at the throat), i.e. inner boundary condition needs to be imposed, puncture solutions are found on a simple Euclidean manifold. (It's OK as long as punctures are not located exactly on grid points.)
- Punctures represent asymptotic infinity compactified into a point.
- Note, however, puncture solutions and for that matter, any solutions that assume the canonical assumptions are NOT ALWAYS commensurate with the desired physical solution. For example, we can not obtain Kerr solution from the conformal flatness assumption if assuming maximal slicing at the same time.

- So conformally flat initial data for spinning holes contain some amount of unphysical "junk" radiation.
- Note that there have been a several works that tried to go beyond the conformal flat initial data for binary black holes.
- The early work that used non-conformally flatness is Matzner, Huq, and Shoemaker, PRD59, 024015 (1998).
- Basic idea of this work is to use Kerr-Schild form of metric as a base metric.
- Consider for a single black hole, the Kerr-Schild spacetime metric:

$$ds^2 = \eta_{ab} dx^a dx^b + 2H(x^\mu) l_a l_b dx^a dx^b \tag{45}$$

  where η_{ab} is flat metric, H is a scalar function of position and time and l_a is an (ingoing) null vector.
- The general Kerr-Schild black hole metric (written in Kerr's original coordinates) has

$$H = \frac{Mr}{r^2 + a^2 \cos^2 \theta} \tag{46}$$

$$l_b = \left(1, \frac{rx + ay}{r^2 + a^2}, \frac{ry - ax}{r^2 + a^2}, \frac{z}{r}\right) \tag{47}$$

- For binary black holes use the following form for the conformal metric.

$$^{(3)}\tilde{ds} = f_{ij}dx^i dx^j + {}_1H(r_1){}_1l_i{}_1l_j dx^i dx^j + {}_2H(r_2){}_2l_i{}_2l_j dx^i dx^j \tag{48}$$

where ${}_aH$ and ${}_al_i$ are the functions defined from each single black hole. This background metric has two vectors corresponding to the null vector of the Kerr-Schild form.

- Physical metric is $\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij}$.
- Momentum constraint equations are treated similar as in conformally flat cases.
- But now the derivative operators are NOT flat operators.

Quasi-Circular Orbits

- We have seen examples of how one might go about solving the constraint equations for initial data.
- For purely theoretical problems without any need to make connections to observations, obtaining mathematically consistent solutions is all we need.
- However, for certain problems, not all mathematically consistent solutions are *physically* relevant.
- For exmaple, consider last few orbits of interactions of compact binaries such binary black holes (BBHs) or binary neutron stars (BNSs).
- Well before the binaries come close to each other, orbits circularize due to gravitational radiation damping. In other words, way before the binaries reach the regime where numerical relativiy is needed, the orbits have already circularized. (*ref*)
- Therefore, quasi-circular orbit is a very good assumption to use for numerical relativity initial data.
- But how do we compute quasi-circular orbits? Or how do we discern which of the mathematical consistent data represent more physically relevant data?

Effective Potential Method

- Cook's effcctive potential method (Cook, PRD 50, 5025, 1994).
- Consider two holes with masses, M_1 and M_2. Holes are located at $\vec{C}_1$ and $\vec{C}_2$ and have linear momenta P_1^i and P_2^i respectively. (We can also think about situations where spin momenta are present, S_1^i and S_2^i. But I will not consider them here.)
- Then, the orbital angular momentum for a configuration of two black holes is given by

$$\vec{J} = \vec{C}_1 \times \vec{P}_1 + \vec{C}_2 \times \vec{P}_2 \tag{49}$$

- Define binding energy, E_b.

$$E_b \equiv E - M_1 - M_2 \tag{50}$$

where E is the ADM energy of the whole spacetime.

- Note that the effective energy contains both (1) the gravitational binding energy and (2) their orbital kinetic energies, but not the rotatonal kinetic energy of the inidividual holes.

- Mass (such as M_1, M_2) can be defined via the Christodoulou formula

$$M^2 = M_{ir}^2 + \frac{S^2}{4M_{ir}^2} \tag{51}$$

where the irreducible mass $M_{ir} \sim \sqrt{A/16\pi}$ with proper area of apparent horizon, A.

- Quasi-circular orbits can be found by computing the effective potential E_b as a function of separation l along a sequence of constant masses M_1, M_2 and angular momentum, J.

$$\frac{\partial E_b}{\partial l}|_{J,M_1,M_2} = 0 \tag{52}$$

where l is the shortest proper separation betweent the two marginally outer-trapped surfaces.

- Note that the minimum corresponds to a stable quasicircular orbit.
- For a quasicircular orbit, the binary's orbital angular velocity, Ω as measured at infinity can be determined from

$$\Omega = \frac{\partial E_b}{\partial J}|_{M_1,M_2,l} \tag{53}$$

- Effective potentail method has been used in the context of both CCT and ECTS (extended Conformal Thin Sandwich) to calculate quasicircular orbits in binary black holes.
- The results based on 2-sheeted topology (Cook) and 3-sheeted topology (Baumgarte) are very similar to each other.
- There is second method that is used to calculate quasiequilibrium states: mass comparison method.
- Basic idea is to calculate both ADM mass and Komar mass and equate them.

CTS Puncture Method

- Conformal Thin Sandwich (CTS) method uses both constraint equations and (a part of) evolution equations of the Einstein field equations to formulate initial value problem.
- CTS allows users to choose freely specifiable part of the initial data that are motivated by dynamics of the problem.
- In puncture method, one considers a hypersurface that contains N black holes with each black hole being connected through an Einstein-Rosen bridge to an additional hypersurface, giving a total of $N + 1$ hypersurfaces connected by N Einstein-Rosen bridge.
- Further, spatial infinities of N (everything other than "our" universe) hypersurfaces are represented by puncture points on R^3 grid.
- (Coordinate) singularities are factored out by closed-form function, called Brill-Lindquist conformal factor.
- Excision is not needed in the puncture method.
- (Note, for evolution, so called "moving" puncture method quickly became the most popular method in 3D NR.)
- Can we take advantage of both CTS and puncture method?

CTS Puncture (CTSP) Method

- Use the same CTS equations. Remember we solve for ψ, α, β.
- Conformal factor, ψ, is written as in puncture method.
- We don't need any special treatment for β, yet.
- New ingredient is that the conformal lapse, $\tilde{\alpha}$, is now split into analytic (closed-form) singular and unknown regular part,

$$\tilde{\alpha}\psi^7 = \alpha\psi = \sum_{a=1}^{N} \frac{c_a}{2|\vec{x} - \vec{C}_a|} + v \tag{54}$$

  where c_i is essentially free parameter to choose and v is C^2 function everywhere on R^3 as long as $\tilde{A}_{ij}$ diverges no faster than $1/r^3$ as in the original puncture method.
- Another ingredient needed for boosted black hole (BH with linear momentum) is the condition on β at the punctures. The full CTSP system of equations is solved on the full R^3 grid except at the punctures where the momentum constraint equation is not solved and instead we require that

$$\beta^i_a = B^i_a \tag{55}$$

  where freely choosable B^i_a parametrize the linear momentum.

- Now let us consider a concrete example for CTSP method.
- Here we are interesting in setting up quasiequilibrium data.
- Make the following choice for freely specifiable data: $\{\tilde{\gamma}_{ij}, K, \partial_t K, \tilde{u}_{ij}\}$.

$$\tilde{u}_{ij} = 0 \tag{56}$$

$$\partial_t K = 0 \tag{57}$$

$$\tilde{\gamma}_{ij} = f_{ij} \tag{58}$$

$$K = 0 \tag{59}$$

- For N black holes, adopting puncture method, we write

$$\psi = \sum_{a=1}^{N} \frac{m_a}{2|\vec{x} - \vec{C}_a|} + u \tag{60}$$

$$\tilde{\alpha}\psi^7 = \alpha\psi = \sum_{a=1}^{N} \frac{c_a}{2|\vec{x} - \vec{C}_a|} + v \tag{61}$$

- Put these into CTS equations.

- CTSP equations become

$$\tilde{D}^i \tilde{D}_i u = -\frac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} \quad (62)$$

$$\tilde{\Delta}_l \beta^i - (\tilde{l}\beta)^{ij}\tilde{D}_j \ln\tilde{\alpha} = 0 \quad (63)$$

$$\tilde{D}^i \tilde{D}_i v = \tilde{\alpha}\psi^7[\frac{7}{8}\psi^{-8}\tilde{A}_{ij}\tilde{A}^{ij}] \quad (64)$$

where

$$(\tilde{l}\beta)^{ij} \equiv \tilde{D}^i\beta^j + \tilde{D}^j\beta^i - \frac{2}{3}\tilde{\gamma}^{ij}\tilde{D}_k\beta^k \quad (65)$$

$$\tilde{\Delta}_l\beta^i \equiv \tilde{D}^k\tilde{D}_k\beta^i + \frac{1}{3}\tilde{D}^i\tilde{D}_j\beta^j + \tilde{R}^i_j\beta^j \quad (66)$$

and conformal flatness implies, $\tilde{R}^i_j = 0$.

- We require c_a to be positive and impose $\beta^i_a = B^i_a$ at the punctures.

- Boundary conditions (at $r \to \infty$)

$$u \sim O(r^{-1}), \quad r \to \infty \tag{67}$$

$$v \sim O(r^{-1}), \quad r \to \infty \tag{68}$$

$$\beta^i \sim O(r^{-2}) \quad \text{or} \quad \mathrm{O(r^{-3})}, \quad \mathrm{r} \to \infty \tag{69}$$

$$\tag{70}$$

- Choose parameters, $m_a, \vec{C}_a, c_a, B_a^i$, for N boosted punctures. Solve the CTSP equations given BC and with shift values specified at the punctures.
- Numerical methods typically used for solving a set of elliptic PDEs.
  - Multi-grid method on a regular grid
  - Spectral method
- I will discuss some basic ideas of Multi-Grid method and (hopefully) show you some concrete examples of it, both single-CPU and parallel implementations.

- See PPT for supp. materials