Centre de Recherches Mathématiques

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Harmonic Maps in Dilaton Gravity

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Abstract

Using a generalized potential formalism, we reduce the field equations of the action $s = \int \sqrt{-g}\, d^4x[-R + 2(\nabla\Phi)^2 + e^{-2\alpha\Phi} F_{\mu\nu}F^{\mu\nu}]$ to five non-linear differential equations. This action represents a gravitational field coupled with electromagnetism and a dilaton field Φ. This action reduces to Einstein-Maxwell-Dilaton theory for $\alpha = 0$, Kaluza-Klein theory for $\alpha = \sqrt{3}$ and to a part of low energy super strings theory for $\alpha = 1$. We suppose α arbitrary. Using the Harmonic Map Ansatz we integrate the field equations with two Killing vectors. We fiend a great amount of classes of solutions representing static gravitational fields coupled to electrical and magnetic monopoles, dipoles, quadripoles etc., and to a dilaton field. The analysis of some solutions is carry out.

1 Introduction

The goal of this work is to extract some physical information from dilaton gravity. In order to do so, we investigate the Lagrangian density

$$\mathcal{L} = \sqrt{-g}\left[-R + 2(\Delta\Phi)^2 + e^{-2\alpha\Phi}F^2\right], \tag{1}$$

where R is the scalar curvature, $F_{\mu\nu}$ is the Faraday tensor and Φ is the dilaton. The main interest of Lagrangian (1) is that it contains the most important theories unifying gravitation with electromagnetism. *Constant α det ermines the special* theories; with $\alpha = \sqrt{3}$, we derive from (1) the Kaluza–Klein field equations, for $\alpha = 1$, (1) is the Lagrangian of the low energy limit of string theory and with $\alpha = 0$, (1) represents th e Einstein–Maxwell theory minimally coupled to Φ. The field equations derived from (1) become

$$\begin{gathered}(e^{-2\alpha\Phi}F^{\mu\nu})_{;\mu} = 0, \quad \Phi^{;\mu}{}_{;\mu} + \frac{\alpha}{2}e^{-2\alpha\Phi}F_{\mu\nu}F^{\mu\nu} = 0 \\ R_{\mu\nu} = 2\Phi_{,\mu}\Phi_{,\nu} + 2e^{-2\alpha\Phi}\left(F_{\mu\lambda}F_{\nu}{}^{\lambda} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right).\end{gathered} \tag{2}$$

2 Potential Space Field Equations

In this work we assume the existence of two Killing vector fields in the space-time, $X = \partial/\partial t$ and $Y = \partial/\partial\varphi$. The line element can be written as

$$ds^2 = f(dt - \omega d\varphi)^2 - f^{-1}[e^{2k}(d\rho^2 + d\zeta^2) + \rho^2 d\varphi^2], \quad (3)$$

where f, ω and k are functions of ρ and ζ only.

In order to obtain new exact solutions of equations (2), we define an abstract potential space $V_p{}^{1,2}$ given by

$$\mathcal{L} = \frac{\rho}{2f^2}[Df^2 + (D\epsilon - \psi D\chi)^2] + \frac{\rho\kappa^2}{2f}\left(D\psi^2 + \frac{1}{\kappa^4}D\chi^2\right) + \frac{2\rho}{\alpha^2\kappa^2}D\kappa^2, \quad (4)$$

where we have used the definitions $D = (\partial/\partial\rho, \partial/\partial\zeta)$, and $\tilde{D} = (\partial/\partial\zeta, -\partial/\partial\rho)$ such that $D\tilde{G}(\rho,\zeta) = 0$. The "Coordinates" of Lagrangian (4) are defi ned by

$$\psi = 2A_t, \quad \kappa^2 = e^{-2\alpha\Phi}, \quad \tilde{D}\chi = 2\frac{f\kappa^2}{\rho}(\omega DA_t + DA_\phi)$$

$$\tilde{D}\epsilon = \frac{f^2}{\rho}D\omega + \psi\tilde{D}\chi,$$

where $A_\mu = (A_t, 0, 0, A_\varphi)$ is the electromagnetic vector potential. The functions f, ϵ, ψ, χ and κ may be interpreted as the gravitational, rotational, electric, magnetic, and scalar potentials, respectively. Variation with respect to the potentials leads to an equivalent set of field equations. They are; the Klein-Gordon equation

$$D^2\kappa + \frac{1}{\rho}D\rho D\kappa - \frac{1}{\kappa}D\kappa^2 - \frac{\alpha^2}{4f}\left(\kappa^3 D\psi^2 - \frac{1}{\kappa}D\chi^2\right) = 0,$$

the Maxwell equations

$$D^2\psi + \left(\frac{D\rho}{\rho} + 2\frac{D\kappa}{\kappa} - \frac{Df}{f}\right)D\psi - \frac{1}{\kappa^2 f}(D\epsilon - \psi D\chi)D\chi = 0$$

$$D^2\chi + \left(\frac{D\rho}{\rho} - 2\frac{D\kappa}{\kappa} - \frac{Df}{f}\right)D\chi + \frac{\kappa^2}{f}(D\epsilon - \psi D\chi)D\psi = 0$$

and the Einstein equations

$$\begin{aligned} &D^2 f + \frac{1}{f}[(D\epsilon - \psi D\chi)^2 - Df^2] + \frac{1}{2\kappa^2}(\kappa^4 D\psi^2 + D\chi^2) = 0 \\ &D^2\epsilon + D\psi D\chi + \psi D^2\chi + (D\epsilon - \psi D\chi)\left(\frac{D\rho}{\rho} - 2\frac{Df}{f}\right) = 0. \end{aligned} \quad (5)$$

(5) can be cast into first order differential equations defining

$$\begin{aligned} A_1 &= \frac{1}{2f}[f_{,z} - i(\epsilon_{,z} - \psi\chi_{,z})] & D_1 &= \frac{1}{\alpha^2}\frac{\kappa_{,z}}{\kappa} \\ B_1 &= \frac{1}{2f}[f_{,z} + i(\epsilon_{,z} - \psi\chi_{,z})] & C_1 &= (\ln\kappa)_{,z} \\ E_1 &= -\frac{1}{2}f^{-1/2}\left[\kappa\psi_{,z} - i\frac{\chi_{,z}}{\kappa}\right] & F_1 &= \frac{1}{2}f^{-1/2}\left[\kappa\psi_{,z} + \frac{i\chi_{,z}}{\kappa}\right], \end{aligned} \quad (6)$$

($z = \rho + i\zeta$) and A_2, B_2, etc. with $\bar{z}$ in place of z.[3,4] In terms of the potentials (6), the field equations (2) now read

$$
\begin{aligned}
A_{1,\bar{z}} &= A_1A_2 - A_1B_2 - \frac{1}{2}C_2A_1 - \frac{1}{2}C_1A_2 - E_1F_2\\
A_{2,z} &= A_1A_2 - A_2B_1 - \frac{1}{2}C_2A_1 - \frac{1}{2}C_1A_2 - E_2F_1\\
B_{1,\bar{z}} &= B_1B_2 - A_2B_1 - \frac{1}{2}C_2B_1 - \frac{1}{2}C_1B_2 - E_2F_1\\
B_{2,z} &= B_1B_2 - A_1B_2 - \frac{1}{2}C_2B_1 - \frac{1}{2}C_1B_2 - E_1F_2\\
E_{1,\bar{z}} &= A_1E_2 + \frac{1}{2}A_2E_1 - \frac{1}{2}B_2E_1 - \frac{1}{2}C_1E_2 - \frac{1}{2}C_2E_1 + \alpha^2D_1F_2\\
E_{2,z} &= A_2E_1 + \frac{1}{2}A_1E_2 - \frac{1}{2}B_1E_2 - \frac{1}{2}C_1E_2 - \frac{1}{2}C_2E_1 + \alpha^2D_2F_1\\
F_{1,\bar{z}} &= B_1F_2 + \frac{1}{2}B_2F_1 - \frac{1}{2}A_2F_1 - \frac{1}{2}C_1F_2 - \frac{1}{2}C_2F_1 + \alpha^2D_1E_2\\
F_{2,z} &= B_2F_1 + \frac{1}{2}B_1F_2 - \frac{1}{2}A_1F_2 - \frac{1}{2}C_1F_2 - \frac{1}{2}C_2F_1 + \alpha^2D_2E_1\\
D_{1,\bar{z}} &= -(E_1E_2 + F_1F_2) - \frac{1}{2}C_1D_2 - \frac{1}{2}C_2D_1\\
D_{2,z} &= -(E_1E_2 + F_1F_2) - \frac{1}{2}C_1D_2 - \frac{1}{2}C_2D_1,
\end{aligned}
\tag{7}
$$

which transform into 10 non-linear-first-order-differential equations in place of five of second order. There exist a Lax pair representation of (5) only for $\alpha = 0$ (see Ref. 5) and $\sqrt{3}$ (see Ref. 3). If we want to extract information of equations (5) for α arbitrary, we most fiend an other method for solving them.

3 Harmonic Maps Ansatz

In this work we will apply the harmonic maps method[6,7] to the field equations (5).[4] We shortly explain it. Let λ^i, $i = 1, \dots, p$ be harmonic maps

$$(\alpha\lambda^i_{,z})_{,\bar{z}} + (\alpha\lambda^i_{,\bar{z}})_{,z} + 2\alpha\Gamma^i_{jk}\lambda^j_{,z}\lambda^k_{,\bar{z}} = 0,$$

where Γ^i_{jk} are the Christoffel symbols of a Riemannian space V_p.

Suppose a flat Riemannian space V_2, $ds^2 = d\lambda\, d\tau$, with

$$A_i = A_i(\lambda, \tau), \quad B_i = B_i(\lambda, \tau), \quad E_i = E_i(\lambda, \tau), \quad F_i = F_i(\lambda, \tau), \quad i = 1, 2.$$

If we substitute this ansatz in (5), we obtain

$$
\begin{aligned}
&a_{i,\lambda^i} = a_i^2 - a_ib_i - e_if_i, \quad b_{i,\lambda^i} = b_i^2 - a_ib_i - e_if_i, \quad d_{i,\lambda^i} = -\frac{\alpha^2}{2}(e_i^2 + f_i^2),\\
&e_{i,\lambda^i} = \frac{3}{2}a_ie_i - \frac{1}{2}b_ie_i + d_if_i, \quad f_{i,\lambda^i} = \frac{3}{2}b_if_i - \frac{1}{2}a_if_i + d_ie_i,\\
&a_i = a_i(\lambda^i), \quad b_i = b_i(\lambda^i), \quad e_i = e_i(\lambda^i), \quad f_i = f_i(\lambda^i), \quad d_i = d_i(\lambda^i),
\end{aligned}
\tag{8}
$$

where $\lambda^i = \lambda,\ \tau$. We have now 5 non linear first order differential equations in place of 5 coupled of second order. This equations are of course much more easier to solve.

4 Exact Solutions

It is possible to fiend static exact solutions of (8). For this case the line element is

$$ds^2 = f\,dt^2 - f^{-1}[e^{2k}(d\rho^2 + d\zeta^2) + \rho^2 d\varphi^2],$$

where f and k are functions of λ and τ only. Function k can be separated into three parts[8]

$$k = k_g + k_e + k_s. \tag{9}$$

We can write the line element in terms of the harmonic maps λ and τ, that means, in terms of functions fulfilling the harmonic map equation

$$\lambda_{,\rho\rho} + \frac{1}{\rho}\lambda_{,\rho} + \lambda_{,\zeta\zeta} = 0, \qquad \tau_{,\rho\rho} + \frac{1}{\rho}\tau_{,\rho} + \tau_{,\zeta\zeta} = 0.$$

We will give electrostatic (magnetostatic) solutions. In any case they can be transformed into magnetostatic (electrostatic) ones because the field equations are invariant under the transformation

$$\Phi \to -\Phi\ , \qquad F_{\mu\nu} \to F^*_{\mu\nu} = \frac{1}{2}e^{-2\alpha\Phi}\eta_{\mu\nu\rho\sigma}F^{\rho\sigma},$$

where $\eta_{\mu\nu\rho\sigma}$ is the Levi–Civita pseudotensor. In Ref. 2 we have given some static exact solutions of (8). Here we reproduce one of them

$$\begin{aligned} f &= \frac{e^\lambda}{(a_1\tau + a_2)^\gamma}, \quad \kappa^2 = \kappa_0^2(a_1\tau + a_2)^\beta e^\lambda, \quad k_{g,z} = \frac{\rho}{2}(\lambda_{,z})^2, \\ k_{s,z} &= \frac{\rho}{2\alpha^2}(\lambda_{,z})^2, \quad k_e = 0, \quad \psi = \frac{a_3\tau + a_4}{a_1\tau + a_2}, \quad \epsilon = \chi = 0, \end{aligned} \tag{10}$$

where $a_1, \ldots, q_1, q_2$, and κ_0, satisfy the relationship

$$4a_1^2 - \kappa_0^2(1 + \alpha^2)(a_1a_4 - a_2a_3)^2 = 0.$$

It is important to note that the electric potential ψ is completely determined by the harmonic function τ. This means that the electrostatic (magnetostatic) potential is determined by τ only, so we can obtain solutions with arbitrary electromagnetic fields. The most important well-know solutions can be derived from this method. Some examples are given in Ref.[2] New solutions have been derived in Refs.[9,10]

5 Geodesic Motion

Physical information can be extracted if we study the geodesic motion in the space-times generated by these solutions. There exist a special class of solutions with a very similar behavior than the Schwarzschild space-time.[11] This class is obtained from solution (10) with $\lambda = \log(1 - 2m/r)$. In Boyer-Lindquist coordinates $\rho = \sqrt{r^2 + 2mr}\sin\theta$, $z = (r - m)\cos\theta$ this new class of solutions reads

$$dS^2 = g_{22}^{\gamma} e^{2(k_e + k_s)} \left(\frac{dr^2}{1 - 2m/r} + r^2\, d\theta^2 \right) + g_{22}^{\gamma} r^2 \sin^2\theta\, d\varphi^2 - \frac{1 - 2m/r}{g_{22}^{\gamma}}\, dt^2$$

$$g_{22} = a_1 e^{q_1 \tau} + a_2 e^{q_2 \tau}, \quad \text{or} \quad g_{22} = \tau + 1 \tag{11}$$

$$A_{3,z} = Q\rho\tau_{,z}, \quad A_{3,\bar{z}} = -Q\rho\tau_{,\bar{z}}, \quad \kappa^2 = \frac{\kappa_0^2(1 - 2m/r)^{-1}}{g_{22}^{\beta}}.$$

(11) can be interpreted as a magnetized Schwarzschild solution in dilaton gravity. τ determines the electromagnetic potential. For example, we obtain a magnetic monolpole with

$$\tau = \ln\left(1 - \frac{2m}{r}\right), \qquad A_3 = 2m(1 - \cos\theta),$$

and a magnetic dipole with

$$\tau = \frac{m^2\cos\theta}{(r - m)^2 - m^2\cos^2\theta}, \qquad A_3 = \frac{m^2(r - m)\sin^2\theta}{(r - m)^2 - m^2\cos^2\theta}.$$

In this work we investigate the geodesic motion for these cases.

6 Effective Potential

Let us suppose that the geodesic motion is around a body like the sun. Since in this case metric (11) is almost spherically symmetric, we can set $\theta = \pi/2$. The geodesic equation for solution (11) can be derived from

$$\mathcal{L} = e^{2(k_e + k_s)} g_{22}^{\gamma} \frac{(dr/ds)^2}{1 - 2m/r} + g_{22}^{\gamma} r^2 \left(\frac{d\varphi}{ds}\right)^2 - \frac{1 - 2m/r}{g_{22}^{\gamma}} \left(\frac{dt}{ds}\right)^2$$

where s is the proper time of the test particle. We have two constants of motion,

$$\frac{\delta\mathcal{L}}{\delta t} = 0 \Longrightarrow \frac{1 - 2m/r}{g_{22}^{\gamma}} \left(\frac{dt}{ds}\right) = A$$

$$\frac{\delta\mathcal{L}}{\delta\varphi} = 0 \Longrightarrow g_{22}^{\gamma} r^2 \frac{d\varphi}{ds} = B$$

so dt/ds and $d\varphi/ds$ can be put in terms of A and B. Using

$$P_\mu P^\mu = -c^2,$$

one obtains

$$-\epsilon = e^{2(k_e+k_s)} g^{\gamma}_{22} \frac{(dr/ds)^2}{1-2m/r} + g^{\gamma}_{22} r^2 \left(\frac{d\varphi}{ds}\right)^2 - \frac{1-2m/r}{g^{\gamma}_{22}} \left(\frac{dt}{ds}\right)^2,$$

with $\epsilon = c^2$, 0, $-c^2$. A more familiar form for this last equation is

$$\left(\frac{dr}{ds}\right)^2 + \frac{e^{-2(k_e+k_s)}}{(1-m/r)^2 g^{\gamma}_{22}} \left\{\frac{B^2}{r^2 g^{\gamma}_{22}} + \epsilon\right\} \left[1 - \frac{2m}{r}\right] = e^{-2(k_e+k_s)} A^2. \tag{12}$$

Let us define the effective potential

$$V_{\text{eff}} = \frac{e^{-2(k_e+k_s)}}{(1-m/r)^2 g^{\gamma}_{22}} \left\{\frac{B^2}{r^2 g^{\gamma}_{22}} + \epsilon\right\} \left[1 - \frac{2m}{r}\right]$$

and the effective energy

$$E_{\text{eff}} = e^{-2(k_e+k_s)} A^2.$$

This interpretation is suggested by performing a series expansion for $r \gg 2m$. In this case the geodesic equation (12) reads

$$\left(\frac{dr}{ds}\right)^2 + V_{\text{eff}} = E_{\text{eff}}.$$

We make the standard transformation $u(\varphi) = 1/r(\varphi(s))$. The geodesic equation (12) transforms into

$$B^2(u')^2 + \left(\frac{(1-mu)^2}{1-2mu}\right)^a [(1-2mu)(B^2u^2+\epsilon) - A^2] = 0, \tag{13}$$

where $'$ means derivative with respect to φ. Since for a trajectory around a star like the sun, the mass parameter $m \sim 1$ Km., while $r \sim 10^6$ Km., $u^3 \sim 0$ is a good approximation. We take terms till u^2. In that case the geodesic eq uation transforms into

$$u'' + \omega^2 u = \frac{m\epsilon}{B^2} + 3mKu^2, \tag{14}$$

where $\omega = \sqrt{1-(am^2/B^2)(A^2-\epsilon)}$ and $K = 1 + am^2A^2/B^2$. For the Schwarzschild geodesic equation $\omega = K = 1$. We find that the trajectories are ellipses with a perihelia precession given by

$$\Delta\varphi_p = 6\pi \frac{m^2c^2K}{B^2\omega^3} = \frac{6\pi m}{b(1-e^2)} \frac{K}{\omega}, \tag{15}$$

where b is the semimajor axis of the ellipse and e is its eccentricity. Perihelia precession is always calculated in the first approximation in m. Since ω and K depend only on m^2, there is no difference between equation (14) and the one obtained from the Schwarzschild solution for the calculation of null geodesics. For

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