Proceedings of the
First Mexican School on Gravitation and Mathematical Physics

RECENT DEVELOPMENTS IN GRAVITATION AND MATHEMATICAL PHYSICS

Guanajuato, Mexico 12–16 December 1994

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CONTENTS

OVERVIEW OF INHOMOGENEOUS COSMOLOGICAL MODELS*

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ABSTRACT

The article gives a short overview of those exact solutions of the Einstein field equations which are generalizations of the Robertson - Walker spacetimes. The total number of papers in which such solutions were derived or discussed is approximately 700, but the solutions can be organized into relatively few families, each containing limiting cases of a single parent solution.

1. Introduction

In Ref. 1 it was shown that a simple generalization of the Friedmann - Lemaitre dust solutions, found by Lemaitre himself [2] can beautifully explain, within the exact theory, several processes observed in the Universe. This example shows that, contrary to the prevailing opinion in the astronomical community, searching for exact solutions to describe structures in the Universe does make sense. Such solutions are neither too difficult to find nor too simplistic. In fact, many of them were found already (see Refs. 3 and 4) and the challenge is to understand what they are telling us about the Universe. In this article, a short selection of the most important examples is presented.

2. The Szekeres - Szafron family of models

The most important contribution to this class was published by Szekeres [5]. He found just all dust solutions of the Einstein equations with the metric:

$$ds^2 = dt^2 - e^{2\alpha}dz^2 - e^{2\beta}(dx^2 + dy^2), \tag{2.1}$$

where α and β are functions of (t, x, y, z) to be determined from the Einstein equations and the coordinates of (2.1) are assumed comoving. The Szekeres solutions were generalized by Szafron [6] to include nonzero pressure. The metric (2.1) was originally just a guessed Ansatz. Later, invariant definitions were provided, of which one, due to Szafron and Collins [7] is this:

*The text of the talk actually given at the School is published elsewhere (Ref. 1). Therefore, with the permission of the organizers, the author submitted a follow-up study of the same subject.

1. The metric obeys the Einstein equations with a perfect fluid source.
2. The flow-lines of the perfect fluid are geodesic and nonrotating.
3. The hypersurfaces orthogonal to the flow-lines are conformally flat.
4. The Ricci tensor of those hypersurfaces has two of its eigenvalues equal.
5. The shear tensor has two of its eigenvalues equal.

It follows then that the two degenerate eigenspaces coincide, and that coordinates can be chosen so that the metric has the form (2.1).

The Einstein equations have to be solved separately for the case $\beta' := \frac{\partial \beta}{\partial z} = 0$, and separately for $\beta' \neq 0$. This is because the limit $\frac{\partial \beta}{\partial z} \to 0$ taken in the second collection of solutions is singular, although well-defined solutions exist if the assumption $\beta' = 0$ is made in the field equations. We shall consider here only the solutions with $\beta' \neq 0$ because they seem to be more interesting for physics and cosmology, but the class with $\beta' = 0$ is also well investigated (see Ref. 3).

The solutions are given by the following formulae:

$$e^{\beta} = \Phi(t,z)e^{\nu(x,y,z)}, \tag{2.2}$$

where $\Phi(t,z)$ is determined by:

$$2\Phi,_{tt}/\Phi + \Phi,_t^2/\Phi^2 + k(z)/\Phi^2 + (8\pi G/c^4)p(t) = 0, \tag{2.3}$$

$k(z)$ and $p(t)$(the pressure) being arbitrary functions, the function e^{ν} is given by:

$$e^{-\nu} = A(z)(x^2+y^2) + 2B_1(z)x + 2B_2(z)y + C(z), \tag{2.4}$$

where A, B_1, B_2 and C are arbitrary functions, and:

$$e^{\alpha} = h(z)e^{-\nu}(e^{\beta}),_z, \tag{2.5}$$

where $h(z)$ obeys:

$$h^{-2}(z) + k(z) = 4(AC - B_1^2 - B_2^2). \tag{2.6}$$

It is seen from (2.5) that the limit $\frac{\partial \beta}{\partial z} \to 0$ is singular indeed. The Robertson - Walker models are all contained here as the special case:

$$\Phi(t,z) = zR(t), B_1 = B_2 = 0, C = 4A = 1, k = k_o z^2, \tag{2.7}$$

where k_0 is the R-W spatial curvature index and $R(t)$ is the R-W scale factor, the coordinate z becomes the spherical radius in this limit. (In fact, the R-W limit is invariantly defined by the assumption that shear is zero, eq. (2.7) then folows by a specialization of the coordinates). Note that eq. (2.3) is formally the same as the R-W field equation, only the arbitrary "constants" depend here on z.

The solutions (2.1) - (2.7) become spherically symmetric when $(e^{\nu}),_z = 0$ and $\epsilon := AC - B_1^2 - B_2^2 > 0$. If further $(8\pi G/c^4)p = \text{const} = \Lambda$, then the Lemaitre - Tolman model [2-3] results. With $(e^{\nu}),_z = 0$ and $\epsilon < 0$, the solutions become hyperbolically symmetric, with $(e^{\nu}),_z = 0 = \epsilon$ they are plane symmetric. All these subcases were extensively investigated in the literature, see Ref. 3. Here, we shall briefly comment on the properties of the general case which has no symmetries.

The solutions defined by (2.1) - (2.7) are called "perfect fluid solutions" just because their energy-momentum tensor has the appropriate algebraic form. This does not automatically imply that the source obeys the thermodynamics of a single-component perfect fluid; it does so only if the conserved particle-number and the entropy can be defined and obey the appropriate equations. Consequences of these requirements for the Szafron models were investigated only recently, see Ref. 8. It turned out that for the solutions with $\beta' \neq 0$ considered here a self-consistent thermodynamical interpretation in terms of a single-component perfect fluid is possible only if $(e^{\nu}),_z = 0$, i.e. only if the spacetime acquires a 3-dimensional symmetry group acting on 2-dimensional orbits. (For the other collection of solutions, with $\beta' = 0$, the consequences of the thermodynamical interpretation are less severe: there exist subcases with no symmetry).

Most of the work on physical interpretation was done for the Szekeres subcase $p = 0$. Eq. (2.1) then has the first integral:

$$\Phi,_t^2 = -k(z) + 2M(z)/\Phi, \tag{2.8}$$

where $M(z)$ is another arbitrary function (it becomes $M_0 z^3$ in the R-W limit, where M_0 is the Friedmann mass integral), and the matter density ρ is:

$$\frac{8\pi G}{c^2}\rho = 2(Me^{3\nu}),_z / [e^{2\beta}(e^{\beta}),_z]. \tag{2.9}$$

Note that the integral of (2.8) (which is elementary and of the same form as in the Friedmann - Lemaitre models) contains an additional arbitrary function $t_0(z)$ such that the solution has the Big Bang singularity at $t = t_0(z)$. In a special case, $t_0(z)$ may be constant, but if it is not, then the Big Bang is not simultaneous in the comoving time: it is a process extended in time rather than a single event. Thus, the Szekeres model is a good example for studying more general kinds of singularity.

Other more important properties are these:

The Szekeres solution contains no gravitational waves. This was shown by Bonnor [9] by demonstrating that the Szekeres spacetime, in spite of lack of symmetry, can be matched to the Schwarzschild spacetime which evidently has no gravitational waves in it, and also by Covarrubias [10] by the classical method of Einstein.

The Szekeres solution can be reparametrized in such a way that it has the explicit form of an inhomogeneous perturbation superimposed on the Friedmann - Lemaitre background [11]. This Goode - Wainwright representation allows for very interesting insights, but is somewhat complicated, so it will not be presented here (see Ref. 3). For example, the perturbation obeys an equation that has the same analytic form as the equation governing the dust perturbation of density in the linear approximation. The growing and the decaying mode of perturbation are readily indentified, the former is generated by inhomogeneities in the initial density distribution, the latter by nonsimultaneities in the Big Bang.

The perturbation of the F-L background represented by the Szekeres solution with $\epsilon > 0$ was interpreted by Szekeres [12] and de Souza [13] as generated by mass dipoles placed on each sphere $z =$ const, the axis of the dipole is in general different on each sphere.

Several subcases of the $\beta' \neq 0$ collection of Szafron were separately derived as solutions of the Einstein and Einstein - Maxwell equations, see Ref. 3 for their detailed description and classification. A few examples are described below.

Ellis [14] found the plane- and hyperbolically symmetric counterparts of the Lemaitre - Tolman solution from Ref. 2, all with nonzero cosmological constant. Bronnikov and Pavlov [15] and Bronnikov [16] found the generalizations of all the solutions of Ellis to the case when the dust carries an electric and a magnetic charge, and the charges obey the Maxwell equations. The spherically symmetric subcase of this was found by Markov and Frolov [17] for $\Lambda = 0$ and generalized by Vickers [18] to the case $\Lambda \neq 0$.

As a curiosity, let us note that the plane symmetric counterpart of the Lemaitre - Tolman solution with $\Lambda = 0$ was rediscovered in 5 papers. The $\Lambda = 0$ subcase of the Lemaitre - Tolman model was derived from the Einstein equations in 20 papers and books, all of them later than the paper by Lemaitre [2]. The "self-similar" subcase of the L-T model was rediscovered in 11 papers. See Ref. 3 for details.

3. The Stephani - Barnes family of models

This is the family of perfect fluid solutions with zero shear, zero rotation and nonzero expansion. It consists of two collections of solutions:

I. The conformally flat solution

The most general solution here is:

$$ds^2 = D^2 dt^2 - V^{-2}(t, x, y, z)(dx^2 + dy^2 + dz^2), \tag{3.1}$$

where:

$$D = F(t)V_{,t}/V, \tag{3.2}$$

$$V = R^{-1}\{1 + \frac{1}{4}k(t)[(x - x_0(t))^2 + (y - y_0(t))^2 + (z - z_0(t))^2]\}, \tag{3.3}$$

$F(t), R(t), k(t), x_0(t), y_0(t)$ and $z_0(t)$ are arbitrary functions of time, F is related to the expansion scalar θ by $\theta = 3/F$. The matter denstity and pressure are given by:

$$(8\pi G/c^2)\rho = 3kR^2 + 3/F^2 := 3C^2(t), \tag{3.4}$$

$$(8\pi G/c^4)p = -3C^2(t) + 2CC_{,t}V/V_{,t}\,. \tag{3.5}$$

This solution was found by Stephani [19]; it is the most general conformally flat solution with a perfect fluid source and nonzero expansion. As seen from (3.4), the matter density in it depends only on the comoving time, while the pressure depends on all the coordinates. In general, the solution has no symmetry. In Ref. 8 it was shown that the source has the thermodynamics of a single-component perfect fluid only if the metric (3.1) - (3.3) is specialized so that it acquires an at least 3-dimensional symmetry group acting on at least 2-dimensional orbits.

The Robertson - Walker limit follows when the functions k, x_0, y_0 and z_0 are all constant. The function k then becomes the spatial curvature index. As can be seen from (3.3), in the Stephani solution k is not constant and can change its sign during the evolution (see Ref. 3).

The arbitrary functions of time cause that the evolution of the spacetime is not determined. This is because no equation of state was imposed on (3.1) - (3.3). Unfortunately, the two types of equations of state that are most often used in cosmology and astrophysics (dust, $p = 0$, and a barotropic equation of state, $f(p, \rho) = 0$) both reduce (3.1) - (3.5) to a Robertson - Walker model. Other than these, no equation of state seems realistic enough. It is fair to say that, apart from being an interesting geometric example, the Stephani solution found no application in cosmology.

II. The type D solutions

Eqs. (3.1) and (3.2) still apply here, but now $V(t, x, y, z)$ is determined by the following equation (resulting from the Einstein equations):

$$w_{uu}/w^2 = f(u), \tag{3.6}$$

where $f(u)$ is an arbitrary function. The variable u and the function w are related to the coordinates x, y, z, and to the function $V(t, x, y, z)$ differently for each of the following cases:

IIA. The spherically symmetric models

$$u = x^2 + y^2 + z^2 \equiv r^2, w(t, u) = V. \tag{3.7}$$

IIB. The plane symmetric models

$$u = z, w(t, z) = V. \tag{3.8}$$

IIC. The hyperbolically symmetric models

$$u = x/y, w(t, u) = V/y. \tag{3.9}$$

The formulae for matter density and pressure are, of course, known (see Ref.3), but are somewhat complicated, so we will not quote them.

These three classes of models were found by Barnes [20], but the spherically symmetric case was known much earlier. The first attempt that would have lead to it if properly finished was undertaken by Dingle [21], and the Einstein equations were reduced to the form (3.6) - (3.7) by Kustaanheimo and Qvist [22]. With $f(u) = 0$, the Barnes models all become conformally flat and thus are then subcases of the Stephani solution (3.1) - (3.3). In general, the solutions of eq. (3.6) are not elementary functions (they may be, for example, the elliptic functions, see many examples in Ref. 3). However, with some forms of $f(u)$, elementary solutions do exist, and a great number of them was discussed in the literature. The case discussed most often is $f(u) = (au^2 + bu + c)^{-5/2}$, where a, b and c are constants. All these particular solutions form a complicated interconnected network, see Ref. 3.

The Barnes models suffer from the same defect as the Stephani solution: with $p = 0$ they reduce to the Friedmann solutions, and with the barotropic equation of state $f(p, \rho) = 0$ they become either the Robertson - Walker metrics or the very special and not very well understood solutions found by Wyman [23] in the spherically symmetric case, and by Collins and Wainwright [24] in the plane symmetric case. Therefore, they were not very often exploited as cosmological models, and their cosmological interpretation is not yet satisfactorily worked out. Instead, particular solutions of the spherical case (3.7) were often interpreted as models of stellar collapse (see Ref. 3). The three cases (3.7) -(3.8) - (3.9) are coordinate transforms of various subcases of the single family of solutions in which:

$$u = \frac{1}{2}(x^2 + y^2 + z^2 - a)/(z + b), w(t, u) = V/(z + b), \tag{3.10}$$

where a and b are arbitrary constants. The correspondence is then this:

When $a < b^2$, the solution is spherically symmetric and reducible, by coordinate transformations, to (3.7).

When $a = b^2$, it is plane symmetric and reducible to (3.8).

When $a > b^2$, it is hyperbolically symmetric and reducible to (3.9).

All these transformations are given in Ref. 25.

Many papers were published in which the source in the Einstein equations was generalized for electric charge on the fluid particles or for heat-flow, see Ref. 3. Again, several of the solutions were multiply rediscovered. The extreme cases were the Kustaanheimo - Qvist models (3.6) - (3.7) and the spherically symmetric subcase of the Stephani solution (3.1) - (3.3) (resulting when x_0, y_0 and z_0 are constant). The first was derived in 18 papers (including one by this author, who shamefully confesses to this), the second one - in 23 papers (see Ref. 3).

One solution in the Kustaanheimo - Qvist family is important for historical reasons. It was found by McVittie [26] already in 1933 and represents a simple superposition of the Schwarzschild and Robertson - Walker solutions. In a slightly modified notation it is:

$$ds^2 = \left[\frac{1 - \mu(t, r)}{1 + \mu(t, r)}\right]^2 dt^2 - R^2(t)\frac{[1 + \mu(t, r)]^4}{(1 + \frac{1}{4}kr^2)^2}[dr^2 + r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2)], \tag{3.11}$$

where:

$$\mu(t, r) = \frac{m}{2rR}(1 + \frac{1}{4}kr^2)^{1/2}, \tag{3.12}$$

m and k being arbitrary constants and $R(t)$ being an arbitrary function. The Robertson - Walker metric results when $m = 0$, the Schwarzschild solution in the isotropic coordinates results when $k = 0$ and $R = 1$. Because of these limiting cases, the Mc Vittie solution invites physical considerations, but it was not really exploited for cosmology, the reason being the same problem with the equation of state that was mentioned above (see Ref. 3).

In contrast to the Szekeres - Szafron models, not much is known about the physical interpretation of the Stephani - Barnes models. Research in this direction should be encouraged. However, it is not productive to try to find new explicit solutions of (3.6). So many are known already that the probability of yet another rediscovery is just too high.

4. Other models

The two families presented so far are the largest and the most clearly connected to the physics of the real world. In addition, many more solutions were derived whose physical interpretation is unknown or not as well worked out. They are listed and classified in Ref. 3; their known properties are also described there. The short overview below should give the reader an idea about the wealth of existing material.

4.1. Models with null radiation

These are superpositions of the Robertson - Walker models with the most important vacuum solutions, like those of Schwarzschild, Kerr, Kerr - Newman, etc. The superpositions are not perfect fluid solutions, and their energy-momentum tensors were interpreted ex post as mixtures of perfect fluid with null radiation (whose energy-momentum tensor is $T_{\mu\nu} = \tau k_\mu k_\nu$ with $k^\mu k_\mu = 0$), sometimes also with the electromagnetic field. The solutions were in fact guessed in the course of exercises in metric-building and interpreting. As a result, the different contributions to the source are coupled through common constants so that, for example, the null radiation can in some cases vanish only if either the perfect fluid component or the inhomogeneity on the R-W background goes away. In particular, the superposition of the Schwarzschild and R-W solutions in this family is different from the McVittie solution of Ref. 26. Still, the composites reached an impressive sophistication. This activity was started by Vaidya [27] who found a superposition of the Kerr and R-W solutions, and the probably most sophisticated composite was found by Patel and Koppar [28]; it is an infinite sequence of perturbations of the flat R-W background whose first-order term is the Kerr solution.

4.2. The "stiff-fluid" models

These are solutions of the Einstein equations in which the perfect fluid source obeys the "stiff equation of state", energy density = pressure (it can be alternatively interpreted as a massless scalar field). All the solutions found have a two-dimensional Abelian symmetry group with spacelike orbits, and in the limit of homogeneous matter distribution reproduce the Robertson - Walker "stiff fluid" models. Participants in this research program claim that the models apply to the early Universe, but the real reason behind the popularity of this activity is that such solutions can be relatively simply generated from vacuum solutions with the same symmetry, of which many are known. Also, the interpretation in terms of soliton waves is carried over from the vacuum case to this case. This activity began with the paper by Tabensky and Taub [29], and the probably most sophisticated example of an explicit solution was given by Belinskii [30]. In addition, many papers were written about the algorithms to integrate the Einstein equations with this kind of source, see Ref. 3 for a review.

4.3. Perfect fluid solutions with a 2-dimensional Abelian symmetry group

Several solutions, unrelated to each other, were published. The ones most extensively discussed are those by Ruiz and Senovilla [31]. They are described in the contribution by J. M. M. Senovilla to this volume.

4.4. Other solutions

Examples of several other kinds of solutions are known:

1. The Petrov type N perfect fluid solutions of Oleson [32].

2. The type D solution with a 1-dimensional symmetry group by Martin and Senovilla [33].

3. A few simple examples of spherically symmetric perfect fluid solutions with shear, expansion and acceleration being all nonzero (see Ref. 3).

4. Examples of algebraically special solutions defined by special properties imposed on the degenerate principal null congruence of the Weyl tensor (see Ref. 3).

5. Soliton-like anisotropic perturbations propagating on the flat R-W background. They are anisotropic in the sense that pressure in the energy-momentum tensor has different values for different directions. The most elaborate example of an explicit solution was given by Diaz, Gleiser and Pullin [34].

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