AN ALGORITHM FOR GENERATING ROTATING BRANS-DICKE WORMHOLE SOLUTIONS

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Abstract

Using the ansatz of Matos and Núñez, the present article proposes an algorithm for generating several classes, not all independent, of asymptotically flat rotating wormhole solutions in the Brans-Dicke Theory. The algorithm allows us to associate a real number n with each static and generated rotating solution. We shall also demonstrate how to match a rotating wormhole to a flat spacetime at the matter boundaries. The vacuum string extensions of the solutions are straightforward. The physical interpretations of the solutions are deferred.

I. Introduction

Recently, there is a revival of interest in the Brans-Dicke Theory (BDT) due principally to the following reasons: The theory occurs naturally in the low energy limit of the effective string theory in four dimensions or the Kaluza-Klein theory. It is found to be consistent not only with the weak field solar system tests but also with the recent cosmological observations. Moreover, the theory accommodates Mach's principle. (It is known that Einstein's General Relativity (EGR) can not accommodate Mach's principle satisfactorily). All these are well known.

A less well known yet an important arena where BDT has found immense applications is the field of wormhole physics, a field recently re-activated by the seminal work of Morris, Thorne and Yurtsever (MTY) [1]. Conceptual predecessors of MTY wormholes could be traced to the geometry of Flamm paraboloid, Wheeler's concept of "charge without charge", Klein bottle or the Einstein-Rosen bridge model of a particle [2]. Wormholes are topological handles that connect two distant regions of space. These objects are invoked in the investigations of problems ranging from local to cosmological scales, not to mention the possibility of using these objects as a means of interstellar travel [1]. Wormholes require for their construction what is called "exotic matter" - matter that violate some or all of the known energy conditions, the weakest being the averaged null energy condition. Such matters are known to arise in quantum effects. However, the strongest theoretical justification for the existence of exotic matter comes from the notion of dark energy or phantom energy that are necessary to explain the present acceleration of the Universe. Some classical fields can be conceived to play the role of exotic matter. They are known to occur, for instance, in the $R + R^2$ theories [3], Visser's thin shell geometries [4] and, of course, in scalar-tensor theories [5] of which BDT is a prototype. There are several other situations where the energy conditions could be violated [6].

BDT describes gravitation through a metric tensor ($g_{\mu\nu}$) and a massless scalar field (ϕ). The BD action for the coupling parameter $\omega = -1$ can be obtained in the Jordan frame from the vacuum linear string theory in the low energy limit. The action can be conformally rescaled into what is known as the Einstein frame action in which the scalar field couples minimally to gravity. The last is referred to as the Einstein minimally coupled scalar field theory (EMS). Several static wormhole solutions in EMS and BDT have been widely investigated in the literature [7]. However, exact rotating wormhole solutions are relatively scarce, especially, in the BDT except a recent one in EMS discussed by Matos and Núñez [8]. In this context, we recall the well known fact that the formal independent solutions of BDT are *not* unique. (Of course, the black hole solution *is* unique for which the BD/EMS scalar field is trivial in virtue of the so called "no scalar hair" theorem.) Four classes of static BDT solutions were derived by Brans [9] himself way back in 1962, and the corresponding four classes of EMS solutions are also known [10]. But recently it has been shown that only two of the four classes of Brans' solutions are independent [11]; the other two can be derived from them. However, the forms of all the original four classes of Brans' or EMS solutions are suggestive in their own right and we shall consider all of them as seed solutions.

In this article, we shall derive three classes of asymptotically flat rotating wormholes in the EMS and BDT. The remaining class of solutions (class III) is not asymptotically flat and hence will not be discussed here. Our strategy is to start from the static EMS solutions and then generate rotating solutions in the EMS since they involve less number of identified constants than in the BDT. We shall then transfer them back into those of the BDT. The BDT solutions can further be rephrased as solutions of the vacuum 4-dimensional low energy string theory ($\omega = -1$). Throughout the article, we take units such that $8\pi G = c = 1$.

II. The action, ansatz and the algorithm

Let us start from the 4-dimensional, low energy effective action of heterotic string theory compactified on a 6-torus. The tree level string action, keeping only linear terms in the string tension α' and in the curvature $\widetilde{R}$, takes the following form in the matter free region ($S_{matter} = 0$):

$$S_{string} = \frac{1}{\alpha'} \int d^4x \sqrt{-\widetilde{g}} e^{-2\widetilde{\Phi}} \left[\widetilde{R} + 4\widetilde{g}^{\mu\nu} \widetilde{\Phi}_{,\mu} \widetilde{\Phi}_{,\nu} \right], \tag{1}$$

where $\widetilde{g}^{\mu\nu}$ is the string metric and $\widetilde{\Phi}$ is the dilaton field. Note that the zero values of other matter fields do not impose any additional constraints either on the metric or on the dilation [12]. Under the substitution $e^{-2\widetilde{\Phi}} = \phi$, the above action reduces to the BD action

$$S_{BD} = \int d^4x\sqrt{-\widetilde{g}}\left[\phi\widetilde{R} + \frac{1}{\phi}\widetilde{g}^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\right], \tag{2}$$

in which the BD coupling parameter $\omega = -1$. This particular value is actually model independent and it actually arises due to the target space duality. It should be noted that the BD action has a conformal invariance characterized by a constant gauge parameter ξ [13]. Arbitrary values of can actually lead to a shift from the value $\omega = -1$, but we fix this ambiguity by choosing $\xi = 0$. Under a further substitution

$$\begin{aligned} g_{\mu\nu} &= \phi\widetilde{g}_{\mu\nu} \\ d\varphi &= \sqrt{\frac{2\omega+3}{2\alpha}}\frac{d\phi}{\phi}; \alpha \neq 0; \omega \neq \frac{3}{2}, \end{aligned} \tag{3}$$

in which we have introduced, on purpose, a constant parameter α that can have any sign. Then the action (2) goes into the form of EMS action

$$S_{EMS} = \int d^4x\sqrt{-g}\left[R + \alpha g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}\right]. \tag{4}$$

The EMS field equations are given by

$$\begin{aligned} R_{\mu\nu} &= -\alpha\varphi_{,\mu}\varphi_{,\nu} \\ \varphi^{;\mu}_{;\mu} &= 0 \end{aligned} \tag{5}$$

We shall choose $\alpha = +1$, $\varphi = \varphi(l)$ in what follows, and adopt the Matos-Núñez ansatz [8]:

$$ds^2 = -f(l)\left(dt + a\cos\psi d\psi\right)^2 + \frac{1}{f(l)}\left[dl^2 + (l^2 + l_0^2)\left(d\theta^2 + \sin^2\theta d\psi^2\right)\right], \tag{6}$$

where l_0 is an arbitrary constant, a is identified as the rotational parameter and $f(l)$ is a solution of the field equations

$$\left[\left(l^2 + l_0^2\right)\frac{f'}{f}\right]' + \frac{a^2f^2}{l^2 + l_0^2} = 0, \tag{7}$$

$$\left(\frac{f'}{f}\right)^2 + \frac{4l_0^2 + a^2f^2}{\left(l^2 + l_0^2\right)^2} - 2\varphi'^2 = 0, \tag{8}$$

where the prime denotes differentiation with respect to l.

Algorithm:

Let $f_0 \equiv f(l; p, q, a = 0)$ be a given solution of the static configuration in which p, q are arbitrary constants in the solution. (Combinations of these constants can be interpreted as the mass and scalar charge of the configuration.) Then the rotating solution is

$$f(l;p,q,a) = \frac{2npq\delta f_0^{-1}}{a^2 + n\delta^2 f_0^{-2}} \tag{9}$$

where n is a real number specific to a given static solution and δ is a free parameter allowed by the rotating solution. The scalar field φ is remarkably given by the static solution of the massless Klein-Gordon equation $\varphi_{;\mu}^{;\mu} = 0$. The static solution ($a = 0$) following from Eq.(9) gives $\delta = 2pq$. For the rotating solution, the value of δ may be fixed either by the condition of asymptotic flatness or via the matching conditions at specified boundaries. *Eq.(9) is the algorithm we propose.* Matos and Núñez [8] defined the free parameter as $\delta = \sqrt{D}$. The difficulty is that the field Eqs.(7) and (8) then identically fix $\delta^2 = D = 0$ giving $f = 0$ which obviously yields a meaningless solution when put in the metric (6). When we put Eq.(9) in Eqs. (7) and (8), we find that they reduce to differential equations for $a = 0$ and $f = f_0$. This fact completely justifies our algorithm.

III. BD Class I rotating solution

Let us now consider the Class I EMS solution due to Buchdahl [14]:

$$\begin{aligned} ds^2 &= -\left(\frac{1-\frac{m}{2r}}{1+\frac{m}{2r}}\right)^{2\beta} dt^2 + \left(1-\frac{m}{2r}\right)^{2(1-\beta)} \left(1+\frac{m}{2r}\right)^{2(1+\beta)} \times \\ & \quad [dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\psi^2] \end{aligned} \tag{10}$$

$$\varphi(r) = \sqrt{\frac{2(\beta^2-1)}{\alpha}} \ln\left[\frac{1-\frac{m}{2r}}{1+\frac{m}{2r}}\right], \tag{11}$$

where m and β are two arbitrary constants. The metric (6) can be expanded to give

$$\begin{aligned} ds^2 &= -\left[1 - \frac{2M}{r} + \frac{2M^2}{r^2} + O\left(\frac{1}{r^3}\right)\right] dt^2 + \left[1 + \frac{2M}{r} + O\left(\frac{1}{r^3}\right)\right] \times \\ & \quad [dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\psi^2], \end{aligned} \tag{12}$$

from which one can read off the Keplerian mass

$$M = m\beta \tag{13}$$

The metric has a naked singularity at $r = m/2$. For $\beta = 1$, it reduces to the Schwarzschild solution in isotropic coordinates. For $\alpha = +1$ and $\beta > 1$, it represents a traversable wormhole. It is symmetric under inversion of the radial coordinate $r \to \frac{1}{r}$ and we have two asymptotically flat regions (at $r = 0$ and $r = \infty$) connected by the throat occurring at $r_0^+ = \frac{m}{2}\left[\beta + \sqrt{\beta^2 - 1}\right]$. Thus real throat is guaranteed by $\beta^2 > 1$. For the choice $\alpha = +1$, the quantity $\sqrt{2(\beta^2-1)}$ is real such that there is a real scalar charge σ from Eq.(10) given by

$$\varphi = \frac{\sigma}{r} = -\frac{2m}{r}\sqrt{\frac{\beta^2-1}{2}} \tag{14}$$

But, in this case, we have violated almost all energy conditions in importing *by hand* a negative sign before the kinetic term in Eq.(5). Alternatively, we could have chosen $\alpha = -1$ in Eq.(10), giving an imaginary charge σ. In both cases, however, we end up with the same equation $R_{\mu\nu} = -\varphi_{,\mu}\varphi_{,\nu}$. There is absolutely no problem in accommodating an imaginary scalar charge in the wormhole configuration [15,16].

Using the transformation $l = r + \frac{m^2}{4r}$, the solution (10) and (11) can be expressed as

$$\begin{aligned} ds^2 &= -f_0(l)dt^2 + \frac{1}{f_0(l)}\left[dl^2 + (l^2-m^2)\left(d\theta^2 + \sin^2\theta d\psi^2\right)\right], \\ f_0(l) &= \left(\frac{l-m}{l+m}\right)^{\beta}, \end{aligned} \tag{15}$$

$$\varphi_0(l) = \sqrt{\frac{\beta^2-1}{2}}\ln\left[\frac{l-m}{l+m}\right]. \tag{16}$$

For a detailed analysis of this wormhole solution, see [16]. Eq.(15) can be identified with Eq.(6) with $f = f_0$ provided we allow a further identification $m \to im$. The variable $l \in (-\infty, +\infty)$ gives two asymptotic flat solutions corresponding to $r = 0$ and $r = \infty$. The coordinate has a minimum value at the throat l_0^+ given by $l_0^+ = r_0^+ + \frac{m^2}{4r_0^+} = m\beta$ corresponding to $r = r_0^+$. Thus the minimum surface area at the throat is $4\pi m^2\beta^2$. For this solution, $n = 4$, $p = m$, $q = \beta$, and using the algorithm (9), the corresponding rotating EMS solution is

$$f(l;m,\beta,a) = \frac{8m\beta\delta f_0^{-1}}{a^2 + 4\delta^2 f_0^{-2}}; \varphi(l) = \sqrt{\frac{\beta^2-1}{2}}\ln\left[\frac{l-m}{l+m}\right]. \tag{17}$$

To achieve asymptotic flatness, that is, $f(l) \to 1$ as $l \to \pm\infty$, we note that $f_0(l) \to 1$ as $l \to \pm\infty$. Therefore, we must fix

$$\delta = \frac{2M \pm \sqrt{4M^2 - a^2}}{2}. \tag{18}$$

In the above, we should retain only the positive sign before the square root. The reason is the following: For $a = 0$, the negative sign gives $\delta = 0$ which implies $f = 0$ which is meaningless. On the other hand, the positive root gives $\delta = 2M$ and $f = f_0$, as desired.

For the special case $\beta = 1$, we have $\delta = \frac{2m\pm\sqrt{4m^2-a^2}}{2}$, and

$$f(l;m,a) = \frac{8m\delta\left(\frac{l-m}{l+m}\right)^{-1}}{a^2 + 4\delta^2\left(\frac{l-m}{l+m}\right)^{-2}}. \tag{19}$$

This is an asymptotically flat rotating solution without a scalar field, $\varphi = 0$, but could it be interpreted as Kerr solution in some *other* coordinates? The answer is not immediately evident. The coordinate system in the metric (6) or (15) is itself rotating in the asymptotic region ($l \to \pm\infty$) where it is represented by

$$ds^2 = -\left(dt + a\cos\psi d\psi\right)^2 + dl^2 + l^2\left(d\theta^2 + \sin^2\theta d\psi^2\right). \tag{20}$$

We can retain terms to first order in a and using Eq.(19) in Eq.(6), we get the cross term as $\left(1 - \frac{2m}{l}\right) \times 2a\cos\theta dt d\psi$. Now, we can subtract the rotational part of the coordinate system which is given by $2a\cos\theta dt d\psi$. Then we are left with $\frac{4am}{l}\cos\theta$ which is not quite the Lense-Thirring term $\frac{4am}{l}\sin^2\theta$. However, such a direct subtraction is not a valid procedure as the field equations are essentially nonlinear. Still, the Eq.(19) is a formal solution of the field equations (7) and (8) under the ansatz (6), though the question above does need a more detailed investigation. This will be a separate task to be undertaken elsewhere.

To obtain the rotating BD solution, we follow the following steps: Note from Eq.(3) that

$$\sqrt{\frac{2\omega+3}{2}}\ln\phi = \varphi = \ln\left[\frac{1-\frac{m}{2r}}{1+\frac{m}{2r}}\right]^{\sqrt{2(\beta^2-1)}} \Rightarrow \phi = \left[\frac{1-\frac{m}{2r}}{1+\frac{m}{2r}}\right]^{\sqrt{4(\beta^2-1)/(2\omega+3)}}. \tag{21}$$

Now using the constraint from the BD field equations [9], viz.,

$$4(\beta^2-1) = -(2\omega+3)\frac{C^2}{\lambda^2}, \tag{22}$$

where C, λ are two new arbitrary constants and ω is the coupling parameter, we get

$$\phi = \left[\frac{1-\frac{m}{2r}}{1+\frac{m}{2r}}\right]^{\frac{C}{\lambda}} = \left[\frac{l-m}{l+m}\right]^{\frac{C}{2\lambda}}. \tag{23}$$

The Eq.(22) can be rephrased in the familiar form [9]:

$$\lambda^2 = (C+1)^2 - C(1 - \frac{\omega C}{2}). \tag{24}$$

However, the wormhole condition $\beta^2 > 1$ requires that the right hand side of Eq.(22) be positive. This is possible if either $\omega < -\frac{3}{2}$ or λ be imaginary. Let us first consider $\omega < -\frac{3}{2}$ so that the exponents are real. Then, the final step consists in using the relation $\widetilde{g}_{\mu\nu} = \phi^{-1}g_{\mu\nu}$ together with replacing β in the exponents in the $g_{\mu\nu}$ by [7]

$$\beta = \frac{1}{\lambda}\left(1 + \frac{C}{2}\right). \tag{25}$$

This means, from Eq.(17), we have the BD rotating wormhole class I solution for $\omega < -\frac{3}{2}$ as follows:

$$ds^2 = -\widetilde{f}_1(l)dt^2 + \widetilde{f}_2(l)\left[dl^2 + (l^2 - m^2)\left(d\theta^2 + \sin^2\theta d\psi^2\right)\right], \qquad (26)$$

$$\widetilde{f}_1(l) \equiv \widetilde{f}_1(l; m, C, \lambda, a) = f(l; m, \beta, a)\phi^{-1}$$

$$= \frac{8m\delta\left[\frac{1}{2\lambda}(C+2)\right]\left[\frac{l-m}{l+m}\right]^{-\frac{1}{\lambda}\left(1+\frac{C}{2}\right)}}{a^2 + 4\delta^2\left[\frac{l-m}{l+m}\right]^{-\frac{2}{\lambda}\left(1+\frac{C}{2}\right)}} \times \left[\frac{l-m}{l+m}\right]^{-\frac{C}{2\lambda}}, \qquad (27)$$

$$\widetilde{f}_2(l) \equiv f_2(l; m, C, \lambda, a) = f^{-1}(l; m, \beta, a)\phi^{-1}$$

$$= \frac{a^2 + 4\delta^2\left[\frac{l-m}{l+m}\right]^{-\frac{2}{\lambda}\left(1+\frac{C}{2}\right)}}{8m\delta\left[\frac{1}{2\lambda}(C+2)\right]\left[\frac{l-m}{l+m}\right]^{-\frac{1}{\lambda}\left(1+\frac{C}{2}\right)}} \times \left[\frac{l-m}{l+m}\right]^{-\frac{C}{2\lambda}}, \qquad (28)$$

$$\phi(l) = \left[\frac{l-m}{l+m}\right]^{\frac{C}{2\lambda}}. \qquad (29)$$

It can be verified that the BD field equations again yield the expression (23). Using the relation $l = r + \frac{m^2}{4r}$, it can be easily expressed in the familiar (t, r, θ, ψ) coordinates with the value of δ given by Eq.(18) in which β should have the value as in Eq.(25). For instance, when $a = 0$, we have $\delta = \frac{m}{\lambda}(C+2)$ and identifying $\frac{m}{2} = B$, one retrieves the static BD metric in the original notation:

$$ds^2 = -\left(\frac{1-\frac{B}{r}}{1+\frac{B}{r}}\right)^{\frac{2}{\lambda}} dt^2 + \left(1+\frac{B}{r}\right)^4 \left(\frac{1-\frac{B}{r}}{1+\frac{B}{r}}\right)^{\frac{2(\lambda-C-1)}{\lambda}} \times$$
$$[dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\psi^2] \qquad (30)$$

$$\phi(r) = \left(\frac{1-\frac{B}{r}}{1+\frac{B}{r}}\right)^{\frac{C}{\lambda}}. \qquad (31)$$

The condition for the above solution to represent a traversable wormhole is [6]

$$(C+1)^2 > \lambda^2. \qquad (32)$$

For $\beta^2 > 1$, and $\alpha = +1$, the negative kinetic term in the field equations (5) shows that the energy density is negative violating the Weak Energy Condition (WEC) so that the Eqs.(6), (15) and (17) provide a class of rotating EMS wormhole solution. This solution is then mapped into the BD regime given by the Eqs.(24)-(27) for the range of coupling values $\omega < -\frac{3}{2}$. Same classes of solutions will be obtained by alternative calculations with $\beta^2 > 1$ and $\alpha = -1$ (imaginary scalar charge). One could also consider Eq.(11) with the values $\alpha = -1$ (positive kinetic term) and $\beta^2 < 1$. Then the above procedure would produce a rotating naked singularity in EMS and BD theory. The above calculations represent the basic scheme to be followed in other classes of EMS or BD solutions.

IV. BD Class II rotating solution

Next consider the imaginary value of λ in Eq.(21) with $\omega > -\frac{3}{2}$. Let us take $\lambda \equiv -i\Lambda$. We prefer to start with the BD solutions (30), (31) and then obtain therefrom the EMS seed solution. The reason is to show that BD class I and II solutions are not independent. Thus, to make Eq.(30) real, it is now necessary to take $B \equiv ib$ and use the identity

$$\arctan(x) = \frac{i}{2} \ln \left(\frac{1 - ix}{1 + ix} \right). \tag{33}$$

where x is real. Using further $r \to \frac{1}{r}$, we can finally rewrite (30) and (31) as

$$\begin{aligned} ds^2 &= -Exp\left[2\alpha_0 + \frac{4}{\Lambda}\arctan\left(\frac{r}{b}\right)\right]dt^2 + \\ &\quad Exp\left[2\beta_0 - \frac{4(C+1)}{\Lambda}\arctan\left(\frac{r}{b}\right) - 2\ln\left(\frac{r^2}{r^2+b^2}\right)\right] \times \\ &\quad [dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\psi^2], \end{aligned} \tag{34}$$

$$\phi(r) = Exp\left[\frac{2C}{\Lambda}\arctan\left(\frac{r}{b}\right)\right], \tag{35}$$

where α_0 and β_0 are two adjustable constants to be determined by the asymptotic flatness. Defining $\gamma = \frac{1}{\Lambda}\left(1 + \frac{C}{2}\right)$, using the relation $\widetilde{g}_{\mu\nu} = \phi^{-1} g_{\mu\nu}$ and the redefinition $\phi \to \varphi$ via Eq.(3), we obtain the exact EMS class II solution:

$$\begin{aligned} ds^2 &= -Exp\left[2\alpha_0 + 4\gamma\arctan\left(\frac{r}{b}\right)\right]dt^2 + \\ &\quad \left(1 + \frac{b^2}{r^2}\right)^2 Exp\left[2\beta_0 - 4\gamma\arctan\left(\frac{r}{b}\right)\right] \times \\ &\quad [dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\psi^2], \end{aligned} \tag{36}$$

$$\varphi(r) = \sqrt{8(1+\gamma^2)}\arctan\left(\frac{r}{b}\right). \tag{37}$$

Asymptotic flatness requires that $\alpha_0 = -\pi\gamma$, $\beta_0 = \pi\gamma$. This also represents a traversable wormhole with the throat appearing at $r_0^+ = b\left[\gamma + \sqrt{1+\gamma^2}\right]$. The metric functions expand exactly as in Eq.(12) and the mass of the wormhole is $M = 2b\gamma$. The solution can be rephrased using $l = r - \frac{b^2}{r}$ as the seed solution like in Eqs.(15), (16):

$$ds^2 = -f_0(l)dt^2 + \frac{1}{f_0(l)}\left[dl^2 + (l^2 + 4b^2)\left(d\theta^2 + \sin^2\theta d\psi^2\right)\right], \tag{38}$$

$$f_0(l) = Exp\left[4\gamma\left\{-\frac{\pi}{2} + \arctan\left(\frac{l + \sqrt{l^2 + 4b^2}}{2b}\right)\right\}\right], \tag{39}$$

$$\varphi_0(l) = \sqrt{8(1+\gamma^2)}\arctan\left(\frac{l + \sqrt{l^2+4b^2}}{2b}\right), \tag{40}$$

where b,γ are the constant parameters of the solution. The minimum surface area in the l coordinate is $4\pi(2b\gamma)^2$. The field Eqs.(7) and (8) are satisfied by the solution (39), (40) and we have also seen how the class II solution can be derived from the class I solution. For this solution $n=\frac{1}{4}, p=2b, q=4\gamma$ and using the algorithm (9), the corresponding rotating EMS solution is Eq.(6) with $l_0=2b$ and

$$f(l;b,\gamma,a)=\frac{16b\gamma\delta f_0^{-1}}{4a^2+\delta^2 f_0^{-2}};\varphi(l)=\sqrt{8(1+\gamma^2)}\arctan\left(\frac{l+\sqrt{l^2+4b^2}}{2b}\right). \quad (41)$$

To achieve asymptotic flatness, that is, $f(l)\to 1$ as $l\to\pm\infty$, we note that $f_0(l)\to 1$ as $l\to\pm\infty$. This information immediately gives us

$$\delta=2\left[2M+\sqrt{4M^2-a^2}\right]. \quad (42)$$

This case represents wormhole spacetime in which exotic matter is spread throughout the spacetime though with asymptotically decaying density. A more realistic situation is to confine the rotating matter within fixed limits $l=l_{B\pm}$ while allowing for a flat spacetime beyond those limits. Then, we need to have a matching on two sides at $l=l_{B\pm}$ and obtain values $\delta_\pm$, one for each side. We first match the static case for $l\in[l_{B+},l_{B-}]$ by introducing a constant ε in Eq.(39) as

$$\begin{aligned} f_0(l) &= Exp\left[4\gamma\left\{\varepsilon+\arctan\left(\frac{l+\sqrt{l^2+4b^2}}{2b}\right)\right\}\right], \\ \varphi_0(l) &= \sqrt{8(1+\gamma^2)}\arctan\left(\frac{l+\sqrt{l^2+4b^2}}{2b}\right). \end{aligned} \quad (43)$$

The constant can be chosen as follows: $\varepsilon=+\frac{\pi}{2}$ for $l\in[l_{B+},\infty)$ and $\varepsilon=-\frac{\pi}{2}$ for $l\in[l_{B-},-\infty)$. Thus at the upper boundary $l=l_{B+}$,

$$\begin{aligned} f_{0+} &= f_0(l_{B+})=Exp\left[4\gamma\left\{+\frac{\pi}{2}+\arctan\left(\frac{l_{B+}+\sqrt{l_{B+}^2+4b^2}}{2b}\right)\right\}\right], \\ \varphi_{0+} &= \varphi_0(l_{B+})=\sqrt{8(1+\gamma^2)}\arctan\left(\frac{l_{B+}+\sqrt{l_{B+}^2+4b^2}}{2b}\right), \end{aligned} \quad (44)$$

and at the lower boundary $l=l_{B-}$,

$$\begin{aligned} f_{0-} &= f_0(l_{B-})=Exp\left[4\gamma\left\{-\frac{\pi}{2}+\arctan\left(\frac{l_{B-}+\sqrt{l_{B-}^2+4b^2}}{2b}\right)\right\}\right], \\ \varphi_{0-} &= \varphi_0(l_{B-})=\sqrt{8(1+\gamma^2)}\arctan\left(\frac{l_{B-}+\sqrt{l_{B-}^2+4b^2}}{2b}\right). \end{aligned} \quad (45)$$

Now, to match the rotating EMS solution to those boundaries, we need:

$$f_{\pm} \equiv f(l_{B\pm}, b, \gamma, a, \delta_{\pm}) = f_{0\pm}, \tag{46}$$

in which case the constants are determined by

$$\delta_{\pm} = 2\left[2M + \sqrt{4M^2 - a^2 f_{0\pm}^2}\right]. \tag{47}$$

These reduce to the same expression for δ as in Eq.(42) above if $l_{B\pm} \to \infty$. In the intermediate region , we have

$$f = f(l; b, \gamma, a, \varepsilon, \delta) \tag{48}$$

where $\delta_- \le \delta \le \delta_+$, $-\frac{\pi}{2} \le \varepsilon \le \frac{\pi}{2}$.

To obtain the rotating BD wormhole class II solution, one has to replace γ by $\gamma = \frac{1}{\Lambda}\left(1 + \frac{C}{2}\right)$ in the solution (38) and use $\Lambda^2 = C(1 - \frac{\omega C}{2}) - (C+1)^2$. The last relation follows directly from Eq.(22) when $\lambda \equiv -i\Lambda$. The complete formal solution can be obtained by using Eqs.(34), (38) and the first equation in (41) as

$$\begin{aligned}
ds^2 &= -\widetilde{f}_1(l)\left(dt + a\cos\psi d\psi\right)^2 + \widetilde{f}_2(l)\left[dl^2 + (l^2 + 4b^2)\left(d\theta^2 + \sin^2\theta d\psi^2\right)\right], \\
\widetilde{f}_1(l) &\equiv \widetilde{f}_1(l; b, C, \Lambda, a) = f(l; b, \gamma, a)\phi^{-1} \\
&= \frac{2b\delta\left[\frac{1}{\Lambda}(C+2)\right] Exp\left[-\frac{2\pi}{\Lambda} - \frac{4}{\Lambda}\arctan\left(\frac{l+\sqrt{l^2+4b^2}}{2b}\right)\right]}{a^2 + \frac{1}{4}\delta^2 Exp\left[-\frac{4\pi}{\Lambda} - \frac{8}{\Lambda}\arctan\left(\frac{l+\sqrt{l^2+4b^2}}{2b}\right)\right]} \times \\
&\quad Exp\left[-\frac{2C}{\Lambda}\arctan\left(\frac{l+\sqrt{l^2+4b^2}}{2b}\right)\right],
\end{aligned} \tag{49}$$

$$\widetilde{f}_2(l) \equiv \widetilde{f}_2(l; b, C, \Lambda, a) = f^{-1}(l; b, \gamma, a)\phi^{-1}, \tag{50}$$

$$\phi(l) = Exp\left[\frac{2C}{\Lambda}\arctan\left(\frac{l+\sqrt{l^2+4b^2}}{2b}\right)\right], \tag{51}$$

where δ is given by Eq.(42) to ensure asymptotic flatness.

V. BD Class IV rotating solution

We start with the static EMS class IV solution as the seed solution [10]:

$$ds^2 = -f_0(l)dt^2 + \frac{1}{f_0(l)}\left[dl^2 + l^2\left(d\theta^2 + \sin^2\theta d\psi^2\right)\right], \tag{52}$$

$$f_0(l) = Exp\left[-\frac{\gamma}{bl}\right], \phi_0(l) = -\frac{\gamma}{\sqrt{2}bl}, M = \frac{\gamma}{2b}. \tag{53}$$

For this solution, $n = 4$, $p = \gamma$, $q = \frac{1}{2b}$. This solution represents an asymptotically flat traversable wormhole with the minimum surface area $4\pi\left(\frac{\gamma}{2b}\right)^2$. Thus the rotating EMS solution is:

$$f(l;b,\gamma,a)=\frac{4\gamma\delta e^{\frac{\gamma}{bl}}}{b\left(a^2+4\delta^2 e^{\frac{2\gamma}{bl}}\right)};\varphi(l)=-\frac{\gamma}{\sqrt{2}bl}. \tag{54}$$

Asymptotic flatness fixes $\delta=\frac{2M+\sqrt{4M^2-a^2}}{2}$ and redefine the EMS constants γ, b as

$$\frac{\gamma}{b}=\frac{C+2}{B} \tag{55}$$

and then use the BD constraint [$\lambda=0$ in Eq.(24)]:

$$C=\frac{-1\pm\sqrt{-2\omega-3}}{\omega+2}, \tag{56}$$

such that, following similar arguments surrounding Eqs.(21)-(25), for $\omega<-\frac{3}{2}$, we have real values for C and

$$\phi(l)=Exp\left[-\frac{C}{Bl}\right]. \tag{57}$$

The rotating BD IV metric is:

$$\begin{aligned}\widetilde{f}_1(l) &\equiv \widetilde{f}_1(l;B,C,a)=f(l;b,\gamma,a)\phi^{-1}\\ &= \frac{\frac{4\delta}{B}(C+2)Exp\left[\frac{C+2}{Bl}\right]}{a^2+4\delta^2 Exp\left[\frac{2(C+2)}{Bl}\right]}\times Exp\left[\frac{C}{Bl}\right], \qquad (58)\\ \widetilde{f}_2(l) &\equiv \widetilde{f}_2(l;B,C,a)=f^{-1}(l;b,\gamma,a)\phi^{-1} \qquad (59)\end{aligned}$$

There is not much to say about the class III solution. The EMS solution can be obtained from the class IV EMS solution under the same constraint (56). All that one has to do is invert $l\to\frac{1}{l}$ so that

$$\begin{aligned}ds^2 &= -f_0(l)dt^2+g_0(l)\left[dl^2+l^2\left(d\theta^2+\sin^2\theta d\psi^2\right)\right], \qquad (60)\\ f_0(l) &= Exp\left[-\frac{\gamma l}{b}\right], g_0(l)=\left(\frac{l}{b}\right)^{-4}Exp\left[\frac{\gamma l}{b}\right]\phi_0(l)=\frac{\gamma l}{\sqrt{2}b}. \qquad (61)\end{aligned}$$

The above solution is not asymptotically flat though it is flat at $l=0$. Therefore, it does not meet the requirement of the asymptotic "flaring out" condition for traversable wormholes. Hence, we do not discuss the solution further including its rotating BD version.

VI. Rotating wormholes in string theory

Formal rotating solutions in the string theory can also be obtained via the conformal transformation

$$h_1(l)=\phi^{-1}f(l)=e^{-2\varphi}f(l), h_2(l)=\phi^{-1}f^{-1}(l)=e^{-2\varphi}f^{-1}(l), \tag{62}$$

where $\phi(l)$ and $f(l)$ are the BD rotating scalar and EMS rotating metric solutions respectively. The complete solution is

$$ds^2 = -h_1(l)\left(dt + a\cos\psi d\psi\right)^2 + h_2(l)\left[dl^2 + (l^2 + l_0^2)\left(d\theta^2 + \sin^2\theta d\psi^2\right)\right] \quad (63)$$

$$\widetilde{\Phi} = -\frac{1}{\sqrt{2}}\varphi. \quad (64)$$

The values of l_0^2 are $-m^2$, $4b^2$ and 0 for classes I, II and IV string solutions respectively. The corresponding values for $f(l)$ and φ can be taken from the Eq.(17) above. We display only class I rotating string solution here:

$$h_1(l) = = \frac{8m\beta\delta\left(\frac{l-m}{l+m}\right)^{-(\beta+\sqrt{\beta^2-1})}}{a^2 + 4\delta^2\left(\frac{l-m}{l+m}\right)^{-2\beta}}; h_2(l) == \frac{a^2 + 4\delta^2\left(\frac{l-m}{l+m}\right)^{-2\beta}}{8m\beta\delta\left(\frac{l-m}{l+m}\right)^{-\beta+\sqrt{\beta^2-1}}} \quad (65)$$

$$\widetilde{\Phi}(l) = -\frac{1}{2}\sqrt{\beta^2 - 1}\ln\left[\frac{l-m}{l+m}\right], \quad (66)$$

where δ is given by Eq.(18) with $M = m\beta$. Consider the static case, $a = 0$. Using the identities

$$l^2 - m^2 \equiv r^2\left(1 - \frac{m^2}{4r^2}\right)^2; \frac{l-m}{l+m} \equiv \left[\frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}}\right]^2, \quad (67)$$

we may retrieve the solution for static stringy wormholes derived in Ref.[12]. Identifying the exponents as $E = \pm\beta$ and $F = \pm\sqrt{\beta^2 - 1}$, we have $E^2 - F^2 = 1$. This is actually the constraint provided by the string field equations in the wormhole case. In the case of naked singularity, $E = \pm\beta$ and $F = \pm\sqrt{1 - \beta^2}$ and in this case, the string field equations give $E^2 + F^2 = 1$, as can be verified from the works in Ref.[12].

Other classes of solutions can be derived likewise. The rotating solution for naked singularity can be obtained simply by choosing $\beta^2 < 1$ without any extra effort.

VII. Conclusions

Asymptotically flat rotating solutions are rather rare in the literature, be it of a wormhole or naked singularity. The present article has provided *formal* asymptotically flat rotating solutions in the EMS and BD theories with extensions to string theory. There is however a caveat here. Do the solutions really represent a physically rotating configuration? We can only give a partial answer. Certainly, the solutions are not the result of just writing down a known static metric in rotating coordinate systems, but more – they *do* contain information about the rotation of the physical configurations in question.

The solutions represent mathematically interesting features of EMS and BD field equations. The string solutions are just the BD solutions with $\omega = -1$. As we saw, the solutions admit two arbitrary parameters a and δ. The quantity a has been interpreted by Matos and Núñez [8] as a rotation parameter of the gravity field. However, we think that their ansatz, that we have used here, represents a *nonlinear mixture* of the rotation of coordinate frame and the rotation of the gravity field due to wormhole or naked singularity. The other parameter δ is fixed either by asymptotic flatness or by the desired matching conditions. Further investigations into the nature of solutions with a view to separating the real rotational effects from the fictitious effects arising out of the coordinate rotation might be rewarding.

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