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2008 Class. Quantum Grav. 25 165020

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Class. Quantum Grav. **25** (2008) 165020 (19pp) doi:10.1088/0264-9381/25/16/165020

New features of extended wormhole solutions in the scalar field gravity theories

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Received 18 October 2007, in final form 17 June 2008
Published 5 August 2008
Online at stacks.iop.org/CQG/25/165020

Abstract

This paper reports new interesting features characteristic of wormhole solutions in the scalar field gravity theories. To demonstrate these, using a slightly modified form of the Matos–Núñez algorithm, we obtain an extended class of asymptotically flat wormhole solutions belonging to the Einstein minimally coupled scalar field theory. Generally, solutions in these theories do not represent traversable wormholes due to the occurrence of curvature singularities. However, the Ellis I solution of the Einstein minimally coupled theory, when Wick rotated, yields an Ellis class III solution representing a singularity-free traversable wormhole. We see that Ellis I and III are not essentially independent solutions. The Wick-rotated seed solutions, extended by the algorithm, contain two new parameters a and δ. The effect of the parameter a on the geodesic motion of test particles reveals some remarkable features. By arguing for Sagnac effect in the extended Wick-rotated solution, we find that the parameter a can indeed be interpreted as a rotation parameter of the wormhole. The analysis reported here has wide applicability, for it can be adopted in other scalar field theories, including string theory.

PACS numbers: 04.20.−q, 98.62.Ai, 98.80.−k, 95.30.Sf

1. Introduction

Recently, there has been a revival of interest in the scalar field gravity theories including the Brans–Dicke theory principally due to the following reasons: such theories naturally occur in the low energy limit of the effective string theory in four dimensions or the Kaluza–Klein theory. It is found to be consistent not only with the weak field solar system tests but also with

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recent cosmological observations. Moreover, the theory accommodates Mach's principle. All the information is well known.

A less well known yet an important arena where the Brans–Dicke theory has found immense application is the field of wormhole physics, a field recently re-activated by the seminal work of Morris, Thorne and Yurtsever (MTY) [1]. The conceptual predecessors of modern day wormholes could be traced to the geometry of Flamm paraboloid, Wheeler's concept of 'charge without charge', the Klein bottle or the Einstein–Rosen bridge model of a charged particle [2]. Wormholes are topological handles that connect two distant regions of space. These objects are invoked in the investigation of problems ranging from local to cosmological scales, not to mention the possibility of using these objects as a means of interstellar travel [1]. Wormholes require for their construction a type of matter—called 'exotic matter'—that violates some or all of the known energy conditions, the weakest being the averaged null energy condition. Such matters are known to arise in quantum effects (Casimir effect, for example). However, the strongest theoretical justification for the existence of exotic matter comes from the notion of dark energy or phantom energy that is necessary to explain the present acceleration of the universe. Some classical fields can play the role of exotic matter. They are known to occur, for instance, in the $R + R^2$ theories [3], Visser's thin shell geometries [4] and, of course, in scalar–tensor theories [5] of which the Brans–Dicke theory is a prototype. There are several other situations where the energy conditions could be violated [6].

The Brans–Dicke theory describes gravitation through a metric tensor ($g_{\mu\nu}$) and a massless scalar field (ϕ). The Brans–Dicke action for the coupling parameter $\omega = -1$ can be obtained in the Jordan frame from the vacuum linear string theory in the low energy limit. The action can be conformally rescaled into what is known as the Einstein frame action in which the scalar field couples minimally to gravity. The latter is referred to as the Einstein minimally coupled scalar field theory. Several static wormhole solutions in the Einstein minimally coupled scalar field theory and the Brans–Dicke theory have been investigated in the literature [7]. However, to our knowledge, exact rotating wormhole solutions are relatively scarce except for a recent one in the Einstein minimally coupled scalar field theory discussed by Matos and Núñez [8]. In this context, we recall the well known fact that the formal independent solutions of the Brans–Dicke theory are *not* unique. (Of course, the black hole solution *is* unique for which the Brans–Dicke or minimal scalar field is trivial in virtue of the so-called no scalar hair theorem.) Four classes of static Brans–Dicke theory solutions were derived by Brans [9] himself way back in 1962, and the corresponding four classes of Einstein minimally coupled field theory solutions are also known [10]. But recently it has been shown that only two of the four classes of Brans' solutions are independent [11]; the other two can be derived from them. However, although all the original four classes of Brans or Einstein minimally coupled solutions are important in their own right, we shall here consider, for illustrative purposes, only one of them (Ellis I) as seed solution. The same procedure can be easily adopted in the other three classes.

The general motivation in the present paper is to frame a proper algorithm for generating singularity-free asymptotically flat rotating wormhole solutions from the Ellis seed solutions and investigate the role of new parameters in the extended solutions. The analysis also answers a certain long-standing query about wormhole solutions in the Brans–Dicke theory.

In this paper, using a slightly modified algorithm of Matos and Núñez [8], we shall provide a method for generating wormhole solutions from the known static seed solutions belonging to the Einstein minimally coupled scalar field theory. The solutions can be transferred to those of the Brans–Dicke theory via inverse Dicke transformations. For illustration of the method, only Ellis I seed solution is considered here, others are left out because they can be dealt with similarly. The Brans–Dicke solutions can be further rephrased as solutions of the vacuum four-dimensional low-energy string theory ($\omega = -1$) and section 2 shows how to do that. In sections 2–5, we shall analyze and compare the behavior of the Ellis III and the Wick-rotated Ellis I solution pointing out certain interesting differences between these two geometries. In section 6, the study of the geodesic motion in the extended Wick-rotated Ellis I solution reveals the role of the Matos–Núñez parameter a. Section 7 shows, via consideration of the Sagnac effect, that a can indeed be accepted as a rotation parameter. Finally, in section 8, we shall summarize the results. Throughout the paper, we take the signature $(-, +, +, +)$ and units so that $8\pi G = c = 1$, unless restored specifically. Greek indices run from 0 to 3 while Roman indices run from 1 to 3.

2. The action, ansatz and the algorithm

Let us start from the four-dimensional, low energy effective action of heterotic string theory compactified on a 6-torus. The tree level string action, keeping only linear terms in the string tension α' and in the curvature $\widetilde{R}$, takes the following form in the matter free region ($S_{\text{matter}} = 0$):

$$S_{\text{string}} = \frac{1}{\alpha'}\int \mathrm{d}^4x\sqrt{-\widetilde{g}}\,\mathrm{e}^{-2\widetilde{\Phi}}[\widetilde{R} + 4\widetilde{g}^{\mu\nu}\widetilde{\Phi}_{,\mu}\widetilde{\Phi}_{,\nu}], \tag{1}$$

where $\widetilde{g}^{\mu\nu}$ is the string metric and $\widetilde{\Phi}$ is the dilaton field. Note that the zero values of other matter fields do not impose any additional constraints either on the metric or on the dilaton [12]. Under the substitution $\mathrm{e}^{-2\widetilde{\Phi}} = \phi$, the above action reduces to the Brans–Dicke action

$$S_{BD} = \int \mathrm{d}^4x\sqrt{-\widetilde{g}}\left[\phi\widetilde{R} + \frac{1}{\phi}\widetilde{g}^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\right], \tag{2}$$

in which the Brans–Dicke coupling parameter $\omega = -1$. This particular value is actually model independent and it actually arises due to the target space duality. It should be noted that the Brans–Dicke action has a conformal invariance characterized by a constant gauge parameter ξ [13]. Arbitrary values of ξ can actually lead to a shift from the value $\omega = -1$, but we fix this ambiguity by choosing $\xi = 0$. Under a further substitution

$$\begin{aligned} &g_{\mu\nu} = \phi\widetilde{g}_{\mu\nu} \\ &\mathrm{d}\varphi = \sqrt{\frac{2\omega+3}{2\alpha}}\frac{\mathrm{d}\phi}{\phi}; \quad \alpha \neq 0; \quad \omega \neq \frac{3}{2}, \end{aligned} \tag{3}$$

in which we have introduced, on purpose, a constant parameter α that can have any sign. Then the action (2) becomes that of Einstein minimally coupled scalar field theory

$$S_{\text{EMS}} = \int \mathrm{d}^4x\sqrt{-g}[R + \alpha g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}]. \tag{4}$$

The field equations are given by

$$R_{\mu\nu} = -\alpha\varphi_{,\mu}\varphi_{,\nu} \tag{5}$$

$$\varphi_{;\mu}^{;\mu} = 0. \tag{6}$$

We shall choose $\alpha = +1$, $\varphi = \varphi(l)$ in what follows. The negative sign on the right-hand side of equation (5) implies that the source stress–energy violates some energy conditions. The ansatz we take is the following:

$$\mathrm{d}s^2 = -f(l)\,(\mathrm{d}t + a\cos\theta\,\mathrm{d}\psi)^2 + f^{-1}(l)\left[\mathrm{d}l^2 + \left(l^2 + l_0^2\right)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\psi^2)\right], \tag{7}$$

where l_0 is an arbitrary constant, the constant a is interpreted in [8] as a rotational parameter of the wormhole. We call it the Matos–Núñez parameter. The ansatz in (7) is actually a subclass of the more general class of stationary metrics given by [14, 15]

$$ds^2 = -f(dt - \omega_i dx^i)^2 + f^{-1}h_{ij}\, dx^i\, dx^j, \tag{8}$$

where the metric function f, the vector potential ω_i and the reduced metric h_{ij} depend only on space coordinates x^i. We shall see below that the parameter a can be so adjusted as to make a symmetric wormhole out of an asymmetric one. Note also that the replacement of $a \to -a$ does not alter the field equations.

The function $f(l)$ of the ansatz (7) is a solution of the field equations (5) and (6) if it satisfies the following:

$$\left[\left(l^2+l_0^2\right)\frac{f'}{f}\right]' + \frac{a^2 f^2}{l^2+l_0^2} = 0, \tag{9}$$

$$\left(\frac{f'}{f}\right)^2 + \frac{4l_0^2 + a^2 f^2}{\left(l^2+l_0^2\right)^2} - 2\varphi'^2 = 0, \tag{10}$$

where the prime denotes differentiation with respect to l.

Algorithm. Let $f_0 \equiv f_0(l;\, p, q;\, a = 0)$ and $\varphi_0 = \varphi_0(l;\, p, q;\, a = 0)$ be a known seed solution set in which p, q are arbitrary constants interpreted as the mass and scalar charge of the static configuration. Then the new generated (or extended) solution set (f, φ) is

$$f(l;\, p, q;\, a) = \frac{2npq\delta f_0^{-1}}{a^2 + n\delta^2 f_0^{-2}}, \qquad \varphi(l;\, p, q;\, a) = \varphi_0, \tag{11}$$

where n is a natural number and the parameters p, q are specific to a given seed solution set (f_0, φ_0) while δ is a free parameter allowed by the generated solution in the sense that it cancels out of the nonlinear field equations. The scalar field φ_0 is remarkably given by the *same* static solution of the massless Klein–Gordon equation $\varphi^{;\mu}_{;\mu} = 0$. The seed solution ($a = 0$) following from equations (9) and (10) gives $\delta = 2pq$. For the generated solution ($a \neq 0$), the value of δ may be fixed either by the condition of asymptotic flatness or via the matching conditions at specified boundaries. Equation (11) is the algorithm we propose. This is similar to, but not quite the same as, the Matos–Núñez [8] algorithm. The difference is that they defined the free parameter as $\delta = \sqrt{D}$. The difficulty in this case is that, for our seed solution set (f_0, φ_0) below, the field equations (9) and (10) identically fix $\delta^2 = D = 0$ giving $f = 0$, which is obviously meaningless. The other difference is that we have introduced a real number n that now designates each seed solution f_0 and likewise the corresponding new solution f. With the known parameters n, p and q plugged into the right-hand side of equation (11), the new solution set (f, φ) identically satisfies equations (9) and (10). One also sees that the algorithm can be applied with the set (f, φ) as the new seed solution and the process can be indefinitely iterated to generate *any* number of new solutions. This is a notable generality of the algorithm.

3. Ellis I solution and its geometry

The study of the solutions of Einstein minimally coupled scalar field system has a long history. Static spherically symmetric solutions have been independently discovered in different forms by many authors and their properties are well known [16–18]. We start from the following form of class I solution, due to Buchdahl [16][3], of the Einstein minimally coupled theory,

$$\mathrm{d}s^2 = -\left(\frac{1-\frac{m}{2r}}{1+\frac{m}{2r}}\right)^{2\beta}\mathrm{d}t^2 + \left(1-\frac{m}{2r}\right)^{2(1-\beta)}\left(1+\frac{m}{2r}\right)^{2(1+\beta)} \times [\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\psi^2] \tag{12}$$

$$\varphi(r) = \sqrt{\frac{2(\beta^2-1)}{\alpha}}\ln\left[\frac{1-\frac{m}{2r}}{1+\frac{m}{2r}}\right], \tag{13}$$

where m and β are two arbitrary constants. The same solution, in harmonic coordinates, has been obtained and analyzed also by Bronnikov [17][4].

The metric (12) can be expanded which gives

$$\mathrm{d}s^2 = -\left[1-\frac{2M}{r}+\frac{2M^2}{r^2}+O\left(\frac{1}{r^3}\right)\right]\mathrm{d}t^2 + \left[1+\frac{2M}{r}+O\left(\frac{1}{r^2}\right)\right] \times [\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\psi^2], \tag{14}$$

from which one can read off the Keplerian mass

$$M = m\beta. \tag{15}$$

Solution (12) describes all weak field tests in the solar system because it exactly reproduces the post-Newtonian parameters. (Higher order post-Newtonian effects are analyzed in [19].) For $\beta = 1$, it reduces to the Schwarzschild black hole solution in isotropic coordinates. For $\alpha = +1$ and $\beta > 1$, it represents a naked singularity. The metric is invariant in form under inversion of the radial coordinate $r \to \frac{m^2}{4r}$ and consequently we have two asymptotically flat regions (at $r = 0$ and $r = \infty$), the minimum area radius (throat) occurring at $r_0 = \frac{m}{2}\left[\beta+\sqrt{\beta^2-1}\right]$. Thus, a real throat is guaranteed by the condition $\beta^2 > 1$, which might be called here the wormhole condition. However, despite these facts, since a naked singularity occurs at $r = m/2$, it is not traversable and so Visser [4] called it a 'diseased' wormhole. (See the appendix for the definition of traversability.) For the choice $\alpha = +1$, the quantity $\sqrt{2(\beta^2-1)}$ is real such that there is a real scalar charge σ from equation (13) given by

$$\varphi = \frac{\sigma}{r} = -\frac{2m}{r}\sqrt{\frac{\beta^2-1}{2}}. \tag{16}$$

But, in this case, we have violated almost all energy conditions importing *by hand* a negative sign before the kinetic term in equation (5). Alternatively, we could have chosen $\alpha = -1$ in equation (13), for which the stress tensor would satisfy all energy conditions but then we would have had to allow an imaginary scalar field φ. In either case, however, we end up with the same equation $R_{\mu\nu} = -\varphi_{,\mu}\varphi_{,\nu}$. There is absolutely no problem in accommodating an imaginary scalar charge in lieu of having a configuration with violating energy conditions [20, 21].

The Ricci scalar R for the solution (12) is

$$R = \frac{2m^2r^4(1-\beta^2)}{(r-m/2)^{2(2-\beta)}(r+m/2)^{2(2+\beta)}}, \tag{17}$$

which diverges at $r = m/2$ showing a curvature singularity there. For $\beta \geqslant 2$, the divergence in the Ricci scalar is removed, but then the metric becomes singular. However, metric singularity is often removable when one redefines the metric in better coordinates and parameters.

[3] The original solution was discovered in Fisher I Z 1948 *Zh. Eksp. Teor. Fiz.* **18** 636 (*Preprint* gr-qc/991108.) These solutions have been independently rediscovered in different forms afterwards.

[4] The form of the solution here is in 'harmonic' coordinates (t, u, θ, φ). It can be easily transferred to the Janis–Newman–Winnicour (JNW) 'standard' form $(t, \rho, \theta, \varphi)$ which, in turn, is equivalent to the Buchdahl 'isotropic' form used in equation (12) in the text.

Using the coordinate transformation $l = r + \frac{m^2}{4r}$, solutions (12) and (13) can be expressed as

$$\begin{aligned} ds^2 &= -f_0(l)\,dt^2 + \frac{1}{f_0(l)}[dl^2 + (l^2 - m^2)(d\theta^2 + \sin^2\theta\,d\psi^2)], \\ f_0(l) &= \left(\frac{l-m}{l+m}\right)^{\beta}, \end{aligned} \tag{18}$$

$$\varphi_0(l) = \sqrt{\frac{\beta^2-1}{2}}\ln\left[\frac{l-m}{l+m}\right]. \tag{19}$$

In this form, it is exactly the Ellis I solution that has also been discussed by Bronnikov and Shikin [22]. Equations (18) and (19) identically satisfy the field equations (9) and (10) for $a = 0$. This is our seed solution set, but we still need to redefine it because of the appearance of a naked singularity at $l = m$. The throat l_0 appears at $l_0 = r_0 + \frac{m^2}{4r_0} = m\beta > m$ corresponding to $r = r_0$. Thus the minimum surface area now has a value $4\pi m^2\beta^2$. For this solution, $p = m, q = \beta$, and the seed solutions (18), (19) turn out to correspond to $n = 4$ so that the generated solution in the Einstein minimally coupled theory identically satisfying the field equations (9) and (10) for $a \neq 0$ is

$$f(l; m, \beta, a) = \frac{8m\beta\delta f_0^{-1}}{a^2 + 4\delta^2 f_0^{-2}}; \qquad \varphi(l) = \sqrt{\frac{\beta^2-1}{2}}\ln\left[\frac{l-m}{l+m}\right]. \tag{20}$$

To achieve asymptotic flatness at both sides, that is, $f(l) \to 1$ as $l \to \pm\infty$, we note that $f_0(l) \to 1$ as $l \to \pm\infty$. Therefore, we must fix

$$\delta = \frac{2M \pm \sqrt{4M^2 - a^2}}{2}. \tag{21}$$

In the above, we should retain only the positive sign before the square root. The reason is that, for $a = 0$, the negative sign gives $\delta = 0$ implying $f = 0$, which is meaningless. On the other hand, the positive root gives $\delta = 2M$ and $f = f_0$, as desired. Note that $\beta = 1$ does not lead to Kerr black hole solution from the generated metric. However, solution (20) might represent the spacetime of a rotating wormhole [8].

Let us now examine wormhole geometries in the static and generated solutions in the Einstein minimally coupled theory.

(a) Static seed case ($a = 0$)

The first observation is that the metric functions in equation (18) diverge at the singularity $l = \pm m$ as does the Ricci scalar. The next observation relates to the behavior of the area radius. It exhibits certain peculiar properties for the metric in (18) for $\beta > 1$. For the segments $l \geqslant m$ and $l \leqslant -m$, we have the area radius $\rho_0^I(l) = \sqrt{f_0^{-1}(l^2 - m^2)}$. Then, the area $4\pi\rho_0^2(l)$ decreases from $+\infty$ at one asymptotic flat end to a minimum value $\rho_{0\,\mathrm{min}} = \rho_0(l_0)$ at the throat $l = l_0 = m\beta$, and then becomes asymptotically large, but *not* flat, at a radial point $l = m$. In the remaining segment, we have $|l| < m$, and the area now has to be redefined as $\rho_0^{II}(l) = \sqrt{f_0^{-1}|m^2 - l^2|}$. It then decreases to zero at $l = -m$ (another throat, zero radius!) and then opens asymptotically out to $-\infty$ at the other asymptotic flat end. Though in the r-coordinate version, the metric is inversion symmetric, there are now two asymptotically flat, isometric universes with their own throats and they are actually *disconnected* by the naked singularity at $l = m$. They are also *asymmetric* around $l = 0$ due to the fact that the throat radii in the two universes are different.

(b) Generated case $(a \neq 0)$

Here again, the metric function given by equation (20) diverges at $l = \pm m$. The throat of the rotating wormhole can be found from the roots of the equation for l,

$$a^2\left[l\left(f_0^2 - 1\right) + m\beta\left(\left(f_0^2 + 1\right)\right] + 4m\beta(l - m\beta)\left(2m\beta + \sqrt{4m^2\beta^2 - a^2}\right) = 0. \tag{22}$$

They can be computed only numerically for given values of the parameters m and β. However, the area radius $\rho(l) = \sqrt{f^{-1}(l^2 - m^2)}$ for the solution f shows that, for $a \neq 0$, the area jumps to infinity for $\beta > 1$ at $l = \pm m$ but flares out asymptotically to $\pm\infty$ on both sides. Now one has three disconnected universes, that is, a one-sided asymptotically flat universe, a both-sided non-flat but asymptotically large 'sandwich universe' and another one-sided asymptotically flat universe. For the extreme case $a = 2m\beta$, the picture is the same except that the behavior of $\rho(l)$ is now symmetric around $l = 0$. A natural question arises if metric (20) can be made free of the singularity manifesting itself in the infinite jump in the area at $l = \pm m$ as well as in the curvature. We shall address this question in the following section.

4. Ellis III solution via Wick rotation

We can remove the aforementioned singularities by analytical continuation of Ellis I solutions (f_0, φ_0) using Wick rotation of the parameters, while maintaining the real numerical value of the throat radius. In the solution set (f_0, φ_0), we choose

$$m \to -\mathrm{i}m, \qquad \beta \to \mathrm{i}\beta, \tag{23}$$

so that $l_0 = m\beta$ is invariant in sign and magnitude.

(a) $a = 0$

Then the metric resulting from the seed equation (18) is our redefined seed solution,

$$\mathrm{d}s^2 = -f_0'(l)\,\mathrm{d}t^2 + \frac{1}{f_0'(l)}[\mathrm{d}l^2 + (l^2 + m^2)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\psi^2)] \tag{24}$$

$$f_0'(l) = \exp\left[-2\beta\,\mathrm{arccot}\left(\frac{l}{m}\right)\right] \tag{25}$$

$$\varphi_0'(l) = \left[\sqrt{2}\sqrt{1 + \beta^2}\right]\mathrm{arccot}\left(\frac{l}{m}\right). \tag{26}$$

This is not a new solution but just the Ellis III solution [18], which can be obtained in the original form by using the relation[5]

$$\mathrm{arccot}(x) + \arctan(x) = +\frac{\pi}{2}; \qquad x > 0 \tag{27}$$

$$= -\frac{\pi}{2}; \qquad x < 0 \tag{28}$$

and the function on the left shows a finite jump (of magnitude π) at $x = 0$. Thus, from equations (25) and (26), we get two branches, the +ve sign corresponds to the side $l > 0$ and

[5] We thank an anonymous referee for pointing this out.

the $-$ve sign to $l < 0$,[6]

$$f_{0\pm}^{\text{Ellis}}(l) = \exp\left[-2\beta\left\{\pm\frac{\pi}{2} - \arctan\left(\frac{l}{m}\right)\right\}\right] \tag{29}$$

$$\varphi_{0\pm}^{\text{Ellis}}(l) = \left[\sqrt{2}\sqrt{1+\beta^2}\right]\left(\pm\frac{\pi}{2} - \arctan\left(\frac{l}{m}\right)\right). \tag{30}$$

We have thus demonstrated that the Ellis I and IIII solutions are *not* essentially independent as one can be derived from the other. We might study the solutions (25) and (26) *per se*, or equivalently, study the two restricted branches taken together, while allowing a discontinuity at the origin $l = 0$. Alternatively, we might disregard (25), (26) and treat *each* of the $\pm$ set in equations (29) and (30) as independently derived exact solution valid in the unrestricted range of l with no discontinuity at $l = 0$. The two alternatives are not quite the same. In fact, each individual branch represents a geodesically complete, asymptotically flat wormhole (termed as 'drainholes' by Ellis) having different masses, one positive and the other negative, at two mouths respectively. The known Ellis III solution is the +ve branch which is continuous over the entire interval $l \in (-\infty, +\infty)$. The $-$ve branch can be similarly interpreted.

It is of interest to compare the behavior of the Ellis III solution (29) with the Wick rotated Ellis I solution (25): (i) the Ellis III metric function $f_{0+}^{\text{Ellis}}(l) \to 1$ as $l \to +\infty$ but $f_{0+}^{\text{Ellis}}(l) \to \mathrm{e}^{-2\pi\beta}$ as $l \to -\infty$. These two limits correspond to a Schwarzschild mass M at one mouth and $-M\mathrm{e}^{\pi\beta}$ at the other. There is no discontinuity at the origin because $f_{0+}^{\text{Ellis}}(l) \to \mathrm{e}^{-\pi\beta}$ as $l \to \pm 0$. In the solution (25), on the other hand, there is a discontinuity at the origin because $f_0(l) \to \mathrm{e}^{\pm\pi\beta}$ as $l \to \pm 0$, while there is no asymptotic mass jump since $f_0(l) \to 1$ as $l \to \pm\infty$. The curvature scalars for both (25) and (29) are formally the same and given by

$$R_0' = -\frac{2m^2(1+\beta^2)}{(l^2+m^2)^2}\exp\left[-2\beta\,\text{arccot}\left(\frac{l}{m}\right)\right] \tag{31}$$

$$R_{0+}^{\text{Ellis}} = -\frac{2m^2(1+\beta^2)}{(l^2+m^2)^2}\exp\left[-2\beta\left\{\frac{\pi}{2} - \arctan\left(\frac{l}{m}\right)\right\}\right] \tag{32}$$

which go to zero as $l \to \pm\infty$. That means, the spacetime is flat on two sides for both the solutions. Next, we verify what happens to these scalars at the singular coordinate radius $(r = m/2)$ that has now been shifted to the origin $l = r - \frac{m^2}{4r} = 0$. (ii) The Ellis curvature scalar $R_{0+}^{\text{Ellis}} \to -\frac{2(1+\beta^2)}{m^2}\mathrm{e}^{-\pi\beta}$ as $l \to \pm 0$ whereas the curvature scalar R_0' exhibits a finite jump from $-\frac{2(1+\beta^2)}{m^2}\mathrm{e}^{-\pi\beta}$ to $-\frac{2(1+\beta^2)}{m^2}\mathrm{e}^{+\pi\beta}$ as $l \to \pm 0$. (iii) The area radius $\rho_{0+}^{\text{Ellis}}(l) = \sqrt{f_{0+}^{-1(\text{Ellis})}(l^2+m^2)} \to m\sqrt{\mathrm{e}^{\pi\beta}}$ as $l \to \pm 0$ whereas $\rho_0'(l) = \sqrt{f_0'^{-1}(l^2+m^2)}$ shows a finite jump from $m\sqrt{\mathrm{e}^{\pi\beta}}$ to $m\sqrt{\mathrm{e}^{-\pi\beta}}$ as $l \to \pm 0$. All the above shows that the behavior of (29) is better than that of (25) at the origin. However, for both the solutions, the throat appears at the same radius $l_0 = M = m\beta$. Similar considerations apply for the $-$ve branch.

Ellis III static wormhole (29) can be straightforwardly extended to its rotating form via algorithm (11) and this has actually been done in [8]. Hence, we would concentrate on the Wick rotated Ellis I solution (25) rather than the solution (29) and see if we can remove the discontinuity in it by using the new parameter a. Let us examine the case when $a \neq 0$.

[6] An extra factor $\sqrt{2}$ appears in equations (26) and (30) because we took $\alpha = +1$ in our equation (5) instead of +2. Our parameters are to be identified with those appearing in the Ellis solution as follows: (m, a, n) of [18] are $(M, m, m\sqrt{1+\beta^2})$ of the present paper.

(b) $a \neq 0$

The minimum area radius of the extended solution is obtained from the equation $\frac{d\rho'}{dl} = 0$, where $\rho'(l) = \sqrt{f'^{-1}(l^2 + m^2)}$ is the area radius. From numerical study of the resulting equation we find that the minimum area occurs at $l < m\beta$ and it decreases with the increase of a for fixed values of m and β. We also note that the finite jump persists in the area radius of the extended solution $f'(l; m, \beta, a)$, δ being still given by equation (21). Surprisingly however, when $a = 2m\beta$, the area function $\rho'(l)$ decreases from $+\infty$ to the minimum value at the throat, then increases to a finite value at $l = 0$, undergoes no jump at $l = 0$ but passes continuously, though not with C^2 smoothness, across $l = 0$ on to $-\infty$. The Ricci scalar R' for $f'(l; m, \beta, a)$ is given by

$$R' = \frac{8\beta(1+\beta^2)\left(2m\beta + \sqrt{4m^2\beta^2 - a^2}\right)\exp\left[2\beta\,\mathrm{arccot}\left(\frac{l}{m}\right)\right]}{m\left(1+\frac{l^2}{m^2}\right)^2\left[a^2 + \left(2m\beta + \sqrt{4m^2\beta^2 - a^2}\right)^2\exp\left[4\beta\,\mathrm{arccot}\left(\frac{l}{m}\right)\right]\right]} \tag{33}$$

and it approaches the value $\frac{4(1+\beta^2)e^{\pi\beta}}{m^2(1+e^{2\pi\beta})}$ as $l \to \pm 0$, that is, no jump occurs in it.

The area behavior shows that, for the extreme case $a = 2m\beta$ we do have a wormhole with a *single* metric covering both the asymptotically flat universes ($l \to \pm\infty$) connected by a finite wedge-like protrusion in the shape function at $l = 0$. This wedge prevents C^2 continuity across $l = 0$ in the area function but sews up two exactly symmetrical asymptotically flat universes on both sides. The numerical values of the free parameters m and β can always be suitably controlled to make the curvature tensor and hence the tidal force finite.

For $a \neq 2m\beta$, a single coordinate chart cannot cover the entire spacetime as the area radius has a jump at $l = 0$. However, we can artificially circumvent this jump by multiple metric choices on different segments with C^0 continuity at the junctions. We can get a cue for this construction from the static case. Consider the Wick-rotated metric (25) on the right segment (AB) and the metric form (18) on the left segment (BC) so that the areas match at a radial point $l = l_1$. The radius $l = l_1$ is a root of the equation (area from right (AB) = area from left (BC))

$$(m^2 - l^2) \times \left(\frac{l-m}{l+m}\right)^{-\beta} = (m^2 + l^2) \times \exp\left[2\beta\,\mathrm{arccot}\left(\frac{l}{m}\right)\right]. \tag{34}$$

By numerical computation, we find that $0 < l_1 < m$ such that the two otherwise disjoint universes, one represented by the branch AB and the other by BC, can be connected at $B(l = l_1)$. The point $l = 0$ is covered by the segment BC which has no jump there. At the joining point B, there is C^0 continuity in the area function and the tidal forces can be shown to be finite throughout the generator curve ABC. Similar arguments hold true in the rotating case. Branch AB belongs to the Wick-rotated solution (f'), while the sector BC belongs to the original solution (f). Numerical calculations show that the matching occurs at either of the two points: $B(l_1)$ or $B(l_2)$ such that $-m < l_1, l_2 < m$.

Wormholes can also be constructed by employing the 'cut-and-paste' procedure [4]. One takes two copies of the static wormholes and joins them at a radius $l = l_b > l_0$. The interface between the two copies will then be described by a thin shell of exotic matter. The shape functions on both sides will be symmetric. However, when rotation is introduced, numerical calculation shows that the throat radius decreases from the static value while the flaring out occurs faster. It is of some interest to note that Crisóstomo and Olea [23, 24] developed a Hamiltonian formalism to obtain the dynamics of a massive rotating thin shell in (2+1) dimensions. There, the matching conditions are understood as continuity of the Hamiltonian functions for an ADM foliation of the metric. Of course, this procedure can be trivially extended to deal with axially symmetric solutions in (3+1) dimensions. For soliton solutions, see [25].

5. Extended Brans–Dicke I solution

To obtain the rotating Brans–Dicke solution, we pursue the following steps: note from equation (3) that

$$\sqrt{\frac{2\omega+3}{2}}\ln\phi = \varphi = \ln\left[\frac{1-\frac{m}{2r}}{1+\frac{m}{2r}}\right]^{\sqrt{2(\beta^2-1)}} \Rightarrow \phi = \left[\frac{1-\frac{m}{2r}}{1+\frac{m}{2r}}\right]^{\sqrt{4(\beta^2-1)/(2\omega+3)}}. \tag{35}$$

Now using the constraint from the Brans–Dicke [9] field equations, namely,

$$4(\beta^2-1) = -(2\omega+3)\frac{C^2}{\lambda^2}, \tag{36}$$

where C, λ are two new arbitrary constants and ω is the coupling parameter, we get

$$\phi = \left[\frac{1-\frac{m}{2r}}{1+\frac{m}{2r}}\right]^{\frac{C}{\lambda}} = \left[\frac{l-m}{l+m}\right]^{\frac{C}{2\lambda}}. \tag{37}$$

Equation (36) can be rephrased in the familiar form [9]

$$\lambda^2 = (C+1)^2 - C\left(1-\frac{\omega C}{2}\right). \tag{38}$$

However, the minimum area condition $\beta^2 > 1$ requires that the right-hand side of equation (36) be positive. This is possible if either $\omega < -\frac{3}{2}$ or λ be imaginary. Let us first consider $\omega < -\frac{3}{2}$ so that the exponents are real. Then, the final step consists in using the relation $\widetilde{g}_{\mu\nu} = \phi^{-1}g_{\mu\nu}$ together with replacing β in the exponents in the $g_{\mu\nu}$ by [7]

$$\beta = \frac{1}{\lambda}\left(1+\frac{C}{2}\right). \tag{39}$$

Thus the extended Brans–Dicke class I solution for $\omega < -\frac{3}{2}$ becomes

$$ds^2 = -\widetilde{f}_1(l)\,dt^2 + \widetilde{f}_2(l)[dl^2 + (l^2-m^2)(d\theta^2+\sin^2\theta\,d\psi^2)], \tag{40}$$

$$\begin{aligned}\widetilde{f}_1(l) &\equiv \widetilde{f}_1(l;m,C,\lambda,a) = f(l;m,\beta,a)\phi^{-1}\\ &= \frac{8m\delta\left[\frac{1}{2\lambda}(C+2)\right]\left[\frac{l-m}{l+m}\right]^{-\frac{1}{\lambda}\left(1+\frac{C}{2}\right)}}{a^2+4\delta^2\left[\frac{l-m}{l+m}\right]^{-\frac{2}{\lambda}\left(1+\frac{C}{2}\right)}}\times\left[\frac{l-m}{l+m}\right]^{-\frac{C}{2\lambda}},\end{aligned} \tag{41}$$

$$\begin{aligned}\widetilde{f}_2(l) &\equiv f_2(l;m,C,\lambda,a) = f^{-1}(l;m,\beta,a)\phi^{-1}\\ &= \frac{a^2+4\delta^2\left[\frac{l-m}{l+m}\right]^{-\frac{2}{\lambda}\left(1+\frac{C}{2}\right)}}{8m\delta\left[\frac{1}{2\lambda}(C+2)\right]\left[\frac{l-m}{l+m}\right]^{-\frac{1}{\lambda}\left(1+\frac{C}{2}\right)}}\times\left[\frac{l-m}{l+m}\right]^{-\frac{C}{2\lambda}},\end{aligned} \tag{42}$$

$$\phi(l) = \left[\frac{l-m}{l+m}\right]^{\frac{C}{2\lambda}}. \tag{43}$$

It can be verified that the Brans–Dicke field equations again yield the expression (38). Using the relation $l = r + \frac{m^2}{4r}$, it can be easily expressed in the familiar (t, r, θ, ψ) coordinates with the value of δ given by equation (21) in which β should have the value as in equation (39). For instance, when $a = 0$, we have $\delta = \frac{m}{\lambda}(C+2)$ and identifying $\frac{m}{2} = B$, one retrieves the static Brans–Dicke metric in the original notation,

$$ds^2 = -\left(\frac{1-\frac{B}{r}}{1+\frac{B}{r}}\right)^{\frac{2}{\lambda}} dt^2 + \left(1+\frac{B}{r}\right)^4 \left(\frac{1-\frac{B}{r}}{1+\frac{B}{r}}\right)^{\frac{2(\lambda-C-1)}{\lambda}} \times [dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\psi^2] \tag{44}$$

$$\phi(r) = \left(\frac{1-\frac{B}{r}}{1+\frac{B}{r}}\right)^{\frac{C}{\lambda}}. \tag{45}$$

The solution (44) represents a naked singularity at $r = B$ and the condition for the existence of a minimum area is [6, 7]

$$(C+1)^2 > \lambda^2. \tag{46}$$

For $\beta^2 > 1$, and $\alpha = +1$, the negative kinetic term in the field equations (5) shows that the energy density is negative, which violates at least the weak energy condition. The solution (40) still does not represent a rotating wormhole in the Brans–Dicke theory. For this purpose, one has first to make a change

$$m \to \mathrm{i}m, \qquad \lambda \to -\mathrm{i}\lambda \tag{47}$$

in equations (41)–(43) to get the Wick-rotated counterpart. The next step is to use the relations (27) and (28) obtaining two branches as obtained in section 4. Either of the branches would then represent rotating wormholes in the Brans–Dicke theory for the range of coupling parameter $\omega < -\frac{3}{2}$. The same class of solutions may be obtained by alternative calculations with $\beta^2 > 1$ and $\alpha = -1$ with an imaginary scalar charge. The above steps represent the basic scheme that can be followed in other classes of solutions in the Einstein minimally coupled or Brans–Dicke theory.

The discussion about wormholes in the Einstein minimally coupled theory can be transferred almost *in verbatim* into that of the Brans–Dicke theory, once we use the crucial relation (39) connecting the parameters in both the theories.

6. Geodesic motion in the extended solution

As an illustration, we consider the class of solution (25) generated from the Wick-rotated seed solution (18) of the Einstein minimally coupled scalar field theory. Thus our solution set is given by (dropping primes)

$$f(l; m, \beta, a) = \frac{8m\beta\delta \exp\left[2\beta \operatorname{arccot}\left(\frac{l}{m}\right)\right]}{a^2 + 4\delta^2 \exp\left[4\beta \operatorname{arccot}\left(\frac{l}{m}\right)\right]} \tag{48}$$

$$\varphi(l) = \left[\sqrt{2(1+\beta^2)}\right] \operatorname{arccot}\left(\frac{l}{m}\right), \tag{49}$$

so that the metric components $g_{\nu\lambda}$ are

$$g_{00} = -f, \tag{50}$$

$$g_{11} = f^{-1}, \tag{51}$$

$$g_{22} = f^{-1}\left(l^2 + l_0^2\right), \tag{52}$$

$$g_{33} = f^{-1}\left(l^2 + l_0^2\right)\sin^2\theta - fa^2\cos^2\theta, \tag{53}$$

$$g_{03} = g_{30} = -fa\cos\theta. \tag{54}$$

The four velocity is defined by

$$U^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}p}, \qquad x^\mu \equiv (t, l, \theta, \phi), \qquad \mathrm{d}p = m_0\,\mathrm{d}s \tag{55}$$

in which p is the new affine parameter and m_0 is the invariant rest mass of the test particle. The geodesic equations are given by

$$\frac{\mathrm{d}U_\mu}{\mathrm{d}p} - \frac{1}{2}\frac{\partial g_{\nu\lambda}}{\partial x^\mu}U^\nu U^\lambda = 0, \qquad g_{\nu\lambda}U^\nu U^\lambda = m_0^2 = \epsilon. \tag{56}$$

Since the metric functions $g_{\nu\lambda}$ do not contain t and ϕ, the corresponding momenta are conserved, that is, the $\mu = 0$ and $\mu = 3$ equations give, respectively,

$$U_0 = -f.U^0 - faU^3\cos\theta = k \tag{57}$$

$$U_3 = -faU^0\cos\theta + U^3\left[f^{-1}\left(l^2 + l_0^2\right)\sin^2\theta - fa^2\cos^2\theta\right] = h, \tag{58}$$

where k and h are arbitrary constants. From the above, it follows that the 'angular momentum' of the particle is

$$r^2(l)U^3 = \frac{h - ka\cos\theta}{\sin^2\theta} \equiv \eta \tag{59}$$

$$r^2(l) \equiv f^{-1}\left(l^2 + l_0^2\right). \tag{60}$$

The $\mu = 2$ component of the geodesic equation gives

$$\frac{\mathrm{d}}{\mathrm{d}p}\left(r^2(l)\frac{\mathrm{d}\theta}{\mathrm{d}p}\right) + \frac{1}{2}faU^0U^3\sin\theta - (U^3)^2[r^2(l) + fa^2]\cos\theta\sin\theta = 0. \tag{61}$$

Instead of the $\mu = 1$ equation, we take the second of equation (56) which gives

$$\epsilon = -f[U^0 + U^3 a\cos\theta]^2 + f^{-1}(U^1)^2 + r^2(l)[(U^2)^2 + (U^3)^2\sin^2\theta]. \tag{62}$$

Looking at equation (61), we see that two solutions are possible. One is

$$U^3 = 0 \Rightarrow \theta = \arccos\left(\frac{h}{ka}\right) = \text{const.} \Rightarrow U^2 = 0 \tag{63}$$

which means θ can assume any constant value depending on the independent values of h, k and a. But such motions will only be radial since $U^2 = 0$ and $U^3 = 0$. In other words the gravitating source acts like a *radial sink*! The radial equation of motion (62) reduces to

$$\left(\frac{\mathrm{d}l}{\mathrm{d}p}\right)^2 = \epsilon f(l; m, \beta, a) + k^2. \tag{64}$$

This can be rewritten very succinctly as

$$\frac{\mathrm{d}^2 l}{\mathrm{d}p^2} = \frac{\epsilon}{2}\frac{\mathrm{d}f}{\mathrm{d}l}. \tag{65}$$

The other solution of equation (61) is $\theta = 0$, which implies that the test particle motion is restricted to polar planes. However, we can always choose the pole perpendicular to this plane so that the angle φ varies in that plane with the particle motion. Moreover, from equation (59), we must have $h = ka$ so that $r^2(l)U^3 = \eta \neq 0$. Then, we end up again with the same metric function but without the explicit appearance of a as an arbitrarily regulated free parameter. Thus, in the equation of motion, we have $f = f(l; m, \beta, h/k)$ so that a, which is a parameter of the gravitating source, is obtained from the *motional characteristics* like h and k of the test particle itself. This intriguing feature is somewhat analogous to the fact that the mass of the gravitating Sun can be determined from the motion of a test particle (planet) around it. As a special case, the parameter a can be so chosen as to completely mask the angular momentum η of the test particle, that is, as $h \to ka$, there is a possibility that $\eta \to 0$ too. Physically, it is like choosing the parameter a to coincide with the orbital φ-angular momentum of the test particle. The equation of motion is again exactly the same as equation (62) since $U^2 = 0$ even though $U^3(\neq 0)$ does not appear explicitly. This is due to the fact that, in the last term of equation (62), $(U^3)^2 \sin^2\theta = 0$ but the signature of orbiting (non-radial) test particle is manifest in the presence of h,

$$\left(\frac{\mathrm{d}l}{\mathrm{d}p}\right)^2 = \epsilon f(l; m, \beta, h/k) + k^2. \tag{66}$$

The turning points of the orbit will occur when $\epsilon f = -k^2$ and $\frac{\mathrm{d}f}{\mathrm{d}l} \neq 0$. From these conditions, we have the turning points occurring at

$$l = l_0 = m \cot\left(\frac{\ln x_0}{\beta}\right) \tag{67}$$

where

$$x_0 = \frac{-2m\beta\epsilon \pm \sqrt{4m^2\beta^2\epsilon^2 - k^4a^2}}{2k^2\delta}. \tag{68}$$

Circular orbits will occur if $\epsilon f = -k^2$ and $\frac{\mathrm{d}f}{\mathrm{d}l} = 0$ and they will be stable if $\frac{\mathrm{d}^2 f}{\mathrm{d}l^2} < 0$ and unstable if $\frac{\mathrm{d}^2 f}{\mathrm{d}l^2} > 0$.

7. Sagnac effect

The best way to assess the effect of a nonzero U^3 (or $\mathrm{d}\varphi \neq 0$) and of the Matos–Núñez parameter a is through the Sagnac effect [26] analyzing the orbit in the plane $\theta = 0$. The effect stems from the basic physical fact that the round trip time of light around a closed contour, when the source is fixed on a turntable, depends on the angular velocity, say Ω, of the turntable. Using special theory of relativity and assuming $\Omega r \ll c$, one obtains the proper time $\delta\tau_s$, when the two beams meet again at the starting point as

$$\delta\tau_s \cong \frac{4\Omega}{c^2} S, \tag{69}$$

where $S\ (\equiv \pi r^2)$ is the projected area of the contour perpendicular to the axis of rotation.

Without any loss of rigor, we take $a \to -a$ for notational convenience although it is not mandatory. Suppose that the source/receiver of two oppositely directed light beams is moving along a circumference $l = R =$ constant. Suitably placed mirrors reflect both beams back to their origin after a circular trip about the central rotating wormhole. (The motion is thus *not* geodesic or force free!) Let us further assume that the source/receiver is moving with uniform orbital angular speed ω_0 with respect to distant stars such that the rotation angle is

$$\varphi_0 = \omega_0 t. \tag{70}$$

Under these conditions, the metric becomes

$$d\tau^2 = -f(R; m, \beta, a)[1 - a\omega_0]^2\, dt^2. \tag{71}$$

The trajectory of a light ray is $d\tau^2 = 0$, which gives

$$[1 - a\omega]^2 = 0 \tag{72}$$

where ω is the angular speed. The roots of the above equation coincide so that

$$\omega_{1\pm} = \frac{1}{a}. \tag{73}$$

Therefore, the rotation angle for the light rays is

$$\varphi = \omega_{1\pm} t = \omega_{1\pm}\frac{\varphi_0}{\omega_0}. \tag{74}$$

The first intersection of the world lines of the two light rays with the world line of the orbiting observer after emission at time $t = 0$ occurs when

$$\varphi_+ = \varphi_0 + 2\pi, \qquad \varphi_- = \varphi_0 - 2\pi, \tag{75}$$

so that

$$\frac{\varphi_0}{\omega_0}\omega_{1\pm} = \varphi_0 \pm 2\pi \tag{76}$$

where $\pm$ refer to co-rotating and counter-rotating beams respectively. Solving for φ_0, we get

$$\varphi_{0\pm} = \pm\frac{2\pi\omega_0}{\omega_{1\pm} - \omega_0}. \tag{77}$$

The proper time as measured by the orbiting observer is found from equation (71) by using $dt = d\varphi_0/\omega_0$ and integrating between φ_{0+} and φ_{0-}. The final result is the Sagnac delay given by

$$\begin{aligned}|\delta\tau_s| &= \sqrt{f(R; m, \beta, a)}\left[\frac{1 - a\omega_0}{\omega_0}\right](\varphi_{0+} - \varphi_{0-}) \\ &= \sqrt{f(R; m, \beta, a)}\left[\frac{1 - a\omega_0}{\omega_0}\right]\left[\frac{4\pi\omega_0}{\frac{1}{a} - \omega_0}\right]. \\ &= \sqrt{f(R; m, \beta, a)}(4\pi a). \end{aligned} \tag{78}$$

This shows remarkably that the Sagnac delay depends only on the Matos–Núñez parameter a. Interestingly, the result is independent of ω_0 meaning that it is independent of the motional state of the source/receiver, be it static with respect to the frame of distant stars or moving with regard to it. When $a = 0$, there is no delay because the wormhole spacetime is then nonrotating (no turntable!). The above result supports the conclusion of [8] from an altogether different viewpoint that a can indeed be interpreted as a rotational parameter of the wormhole.

8. Summary

Asymptotically flat rotating solutions are rather rare in the literature, be they of wormholes or naked singularities. Algorithm (11) together with some operations provides a method for generating new wormhole solutions in the Einstein minimally coupled theory and, further on, in the Brans–Dicke theory. Ellis III wormhole or its rotating counterpart is more elegant than the matched solutions due to the lack of C^∞ continuity at the junctions. Nevertheless, the present study opens up possibilities to explore in more detail new solutions in other theories too. For instance, the string solutions are just the Brans–Dicke solutions with $\omega = -1$. As we saw, the asymptotically flat wormhole solutions admit two arbitrary parameters a and δ. The Matos–Núñez parameter a has been interpreted by the authors [8] as a rotation parameter. Here we have shown how the parameter makes its appearance in the Sagnac effect.

Static Ellis I wormholes do not appear traversable due to the singularity manifested in the behavior of the area function and curvature. To tackle this problem, we analytically continued it via Wick rotation and rederived singularity free, asymptotically flat, Ellis III traversable wormholes as one of its branches. This showed, contrary to the general belief, that the two classes of solutions are not independent. Comparative features of the Wick-rotated and Ellis III solutions are pointed out. In the extended Wick rotated solution, numerical graphics show that the wormhole can be covered by a single metric with C^0 continuity. That is, the jumps can be sewed up at the origin for the extreme value of a $(=2m\beta)$ or can be avoided by choosing multiple metric patches for nonextreme values of a, the junctions again having only C^0 continuity. In either case, the tidal forces can depend on adjustable free parameters m, β and a. As an illustrative seed solution, we considered only the Ellis I solution in the Einstein minimally coupled scalar field theory and finally mapped the extended solution into that of Brans–Dicke theory.

In connection with the static Brans–Dicke I solution, we recall a long-standing query by Visser and Hochberg [27] which has been the guiding motivation in sections 2–5: 'It would be interesting to know a little bit more about what this region actually looks like, and to develop a better understanding of the physics on the other side (that is, across the naked singularity) of this class of Brans–Dicke wormholes.' The above analyses answer how one could achieve a both-sided asymptotically flat traversable singularity free wormhole via Wick rotation of the Ellis I solution, which is merely the conformally rescaled Brans–Dicke I solution. Because of the fact that Ellis III is rederivable from Ellis I, and the former is a traversable wormhole (see the appendix), we can say that the Brans–Dicke I solution is also a traversable wormhole, but *only* in its Ellis III reincarnation. We believe that this argument provides some understanding of the 'other side': the mathematical operations of conformal rescaling plus Wick rotation, plus a trigonometric relation eases out the naked singularity and converts the Brans–Dicke I solution into a traversable wormhole.

The next task was to investigate the extended solutions containing new parameters. Accordingly, in section 6, we studied the geodesic motion in the extended geometry and obtained remarkable results in the Einstein minimally coupled theory: for nonzero values of constant θ, the spacetime acts like a radial sink. For $\theta = 0$, the spacetime allows nonradial motions ($U^3 \neq 0$) but the Matos–Núñez parameter a can be entirely expressed in terms of the constants of motion. In section 7, we calculated the Sagnac effect in the extended spacetime and found that the delay depends on a. If $a = 0$, the delay is zero implying that a could be interpreted as a rotation parameter thus supporting the conclusion of Matos and Núñez [8] from an altogether different viewpoint.

A pertinent question is whether the wormholes under consideration are stable. It is not unlikely that the passage of a hypothetical traveler could destabilize them. We do not deal with this issue here.

Acknowledgments

The authors are deeply indebted to Guzel N Kutdusova of Scientific Research Administration of BSPU for efficient assistance. We thank three anonymous referees for useful suggestions.

Appendix

A wormhole is defined to be traversable by a hypothetical traveler if it satisfies some general constraints [1]. We shall demonstrate that the Ellis III wormhole satisfies all of these. It can be made traversable even by a human traveler under suitable choices of constants m and β. Accordingly, let us put one branch, say, the +ve branch of equation (29) in the metric (24). That is, let us put $f_0'(l) = f_{0+}^{\mathrm{Ellis}}(l)$ in metric (24) and rewrite it in the standard MTY form [1] by defining a radial variable ρ as

$$\rho^2 = (l^2 + m^2)\exp\left[2\beta\left\{\frac{\pi}{2} - \arctan\left(\frac{l}{m}\right)\right\}\right]. \tag{A.1}$$

(Note that $l \to \pm\infty$ implies $\rho \to \pm\infty$ and $l \to \pm 0$ implies $\rho \to \pm m\mathrm{e}^{\pi\beta}$.) Then the metric (24) in the coordinates (t, ρ, θ, ψ) becomes

$$\mathrm{d}s^2 = -\mathrm{e}^{2\Phi(\rho)}\,\mathrm{d}t^2 + \frac{\mathrm{d}\rho^2}{1 - \frac{b(\rho)}{\rho}} + \rho^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\psi^2), \tag{A.2}$$

where $\Phi(\rho)$ is the redshift function given by

$$\Phi(\rho) = \beta\left[\arctan\left\{\frac{l(\rho)}{m}\right\} - \frac{\pi}{2}\right] \tag{A.3}$$

and $b(\rho)$ is the shape function given by

$$b(\rho) = \rho\left[1 - \frac{[l(\rho) - m\beta]^2}{\rho^2}\exp\left[2\beta\left\{\frac{\pi}{2} - \arctan\left(\frac{l(\rho)}{m}\right)\right\}\right]\right]. \tag{A.4}$$

General constraints on b and Φ to produce a traversable wormhole, as enumerated in (C1)–(C5), are satisfied by the functions in (A3) and (A4). It may be verified that: (C1) throughout the spacetime, $1 - \frac{b(\rho)}{\rho} \geqslant 0$ and $\frac{b(\rho)}{\rho} \to 0$ as $\rho \to \pm\infty$. (C2) throat occurs at the minimum of ρ where $b(\rho_0) = \rho_0$. This minimum ρ_0 corresponds to $l_0 = m\beta$ so that, from (A1), we find

$$\rho_0 = m(1 + \beta^2)^{\frac{1}{2}}\exp\left[\beta\left\{\frac{\pi}{2} - \arctan\beta\right\}\right]. \tag{A.5}$$

(C3) The spacetime (A2) has no horizon, that is, Φ is everywhere finite. (C4) The coordinate time t measures proper time in asymptotically flat regions because $\Phi \to 0$ as $\rho \to \pm\infty$. (C5) The spacetime has no singularities, as already discussed in section 4.

Some additional constraints, as enumerated in (H1)–(H4), are necessary if the trip is to be undertaken by a human traveler. These are also satisfied by the functions in (A3) and (A4): (H1) trip begins and ends at stations located on either side of the throat where gravity field should be weak. This demands that (i) the geometry at the stations must be nearly flat, or, $\frac{b(\rho)}{\rho} \ll 1$, (ii) the gravitational redshift of signals sent from stations to infinity must be small, or, $|\Phi| \ll 1$ and (iii) the acceleration of gravity at the stations must be less than one Earth gravity $g_\oplus = 980\ \mathrm{cm\,s^{-2}}$, or, $\left|c^2\left(1 - \frac{b}{\rho}\right)^{\frac{1}{2}}\frac{\mathrm{d}\Phi}{\mathrm{d}\rho}\right| \lesssim g_\oplus$. While the first two constraints (i) and (ii) are easily met in virtue of the general constraints (C1) and (C4) respectively, (iii) gives

$$\left|\frac{m\beta}{l^2 + m^2}\,\mathrm{e}^{\Phi}\right| \lesssim \cdot\frac{g_\oplus}{c^2}. \tag{A.6}$$

For fixed finite values of m and β, $\mathrm{e}^{\Phi} \to 1$ for large l and hence this constraint can be easily satisfied at the stations. (H2) The tidal forces suffered by a human traveler should be tolerable, which means that the magnitude of the differential of four acceleration $|\Delta\overrightarrow{a}|$ should be less than $g_\oplus$ in the orthonormal frame $(e_{\widehat{0'}}, e_{\widehat{1'}}, e_{\widehat{2'}}e_{\widehat{3'}})$ of the traveler. This constraint translates, for a traveler of length (head to foot) $\sim$2 m, to the following bounds on the curvature tensor computed in his/her frame. The radial stretch along the human body has to be constrained by the inequality,

$$|R_{\hat{1}'\hat{0}'\hat{1}'\hat{0}'}| = \left|\left(1 - \frac{b}{\rho}\right)\left(-\frac{\mathrm{d}^2\Phi}{\mathrm{d}\rho^2} + \frac{\rho\frac{\mathrm{d}b}{\mathrm{d}\rho} - b}{2\rho(\rho - b)}\frac{\mathrm{d}\Phi}{\mathrm{d}\rho} - \left(\frac{\mathrm{d}\Phi}{\mathrm{d}\rho}\right)^2\right)\right|$$
$$\lesssim \frac{g_\oplus}{c^2 \times 2\,\mathrm{m}} \simeq \frac{1}{(10^{10}\ \mathrm{cm})^2}. \tag{A.7}$$

This bound is essentially meant to constrain the redshift function $\Phi(\rho)$. We see that, at the throat its value is $\Phi(\rho_0) = \beta\left[\arctan\beta - \frac{\pi}{2}\right]$, which tends to -1 as $\beta \to \infty$, and to 0 as $\beta \to 0$. In general, as $\rho \to \pm\infty$, Φ and all its derivatives vanish. Thus the function $\Phi(\rho)$ is well behaved everywhere and consequently the constraint (A7) is easily satisfied. This result is expected since all the curvature components fall off with distance and vanish at infinity [18]. Let us look at the lateral bounds given by

$$|R_{\hat{2}'\hat{0}'\hat{2}'\hat{0}'}| = |R_{\hat{3}'\hat{0}'\hat{3}'\hat{0}'}| = \left|\frac{\gamma^2}{2\rho^2}\left[\left(\frac{v}{c}\right)^2\left(\frac{\mathrm{d}b}{\mathrm{d}\rho} - \frac{b}{\rho}\right) + 2(\rho - b)\frac{\mathrm{d}\Phi}{\mathrm{d}\rho}\right]\right|$$
$$\lesssim \frac{g_\oplus}{c^2 \times 2\,\mathrm{m}} \simeq \frac{1}{(10^{10}\ \mathrm{cm})^2}. \tag{A.8}$$

These bounds constrain the speed v with which the traveler crosses the wormhole, $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$. For the metric (A2), the above work out to

$$|R_{\hat{2}'\hat{0}'\hat{2}'\hat{0}'}| = |R_{\hat{3}'\hat{0}'\hat{3}'\hat{0}'}| = \left|\frac{\gamma^2}{2\rho^2}\left[\left(\frac{v}{c}\right)^2\left(\frac{2m(m + l\beta)}{l^2 + m^2}\right) + \left(\frac{2m\beta(l - m\beta)}{l^2 + m^2}\right)\right]\right|$$
$$\lesssim \frac{g_\oplus}{c^2 \times 2\,\mathrm{m}} \simeq \frac{1}{(10^{10}\ \mathrm{cm})^2}. \tag{A.9}$$

The maximum tidal force experienced by the traveler should occur at the throat $l = m\beta$ or equivalently at $\rho = \rho_0$. Hence the velocity v at the throat is determined by the inequality (A9),

$$\frac{\gamma^2}{\rho_0^2}\left[\left(\frac{v}{c}\right)^2\right] \lesssim \frac{1}{(10^{10}\ \mathrm{cm})^2}. \tag{A.10}$$

It is possible to adjust m and β such that we have any throat radius, $\rho_0 \sim 10\,\mathrm{m}$ (say). Assuming that $v \ll c$ or $\gamma \sim 1$, we obtain $v \lesssim 30\ \mathrm{m\ s^{-1}}(\frac{\rho_0}{10\,\mathrm{m}})$. This shows that the speed v across the hole could be made reasonably small and easily achievable. (H3) The traveler should feel less than $g_\oplus$ acceleration throughout the trip, which requires

$$\left|\mathrm{e}^{-\Phi}\left(1 - \frac{b}{\rho}\right)^{\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}\rho}(\gamma\mathrm{e}^{\Phi})\right| \lesssim \frac{g_\oplus}{c^2}. \tag{A.11}$$

With $\gamma \sim 1$, this works out to the same constraint as in (A6), which is already satisfied. (H4) The total proper time interval $\Delta\tau$ measured by the traveler and the coordinate time interval Δt measured at the stations should not exceed a year (say) for the entire trip, that is,

$$\Delta\tau = \int_{-L_1}^{L_2}\frac{\mathrm{d}L}{v\gamma} \lesssim 1\ \text{year} \tag{A.12}$$

$$\Delta t = \int_{-L_1}^{L_2}\frac{\mathrm{d}L}{v\mathrm{e}^{\Phi}} \lesssim 1\ \text{year}, \tag{A.13}$$

where $L = -L_1$ and $L = +L_2$ are the proper radial distances of the stations measured from the throat at $L = 0$. The element dL of proper radial distance L is defined by

$$\mathrm{d}L = \frac{\mathrm{d}\rho}{\sqrt{1 - \frac{b(\rho)}{\rho}}}. \tag{A.14}$$

Let us locate the two stations at large enough radii ρ so that $1 - \frac{b(\rho)}{\rho} \sim 1$ and $\Phi \sim 0$. Accordingly, we take $\rho_1 = \rho_2 \sim 10^4 \rho_0$ corresponding to $L_1 = L_2 \sim 10^4 \rho_0$. Since $\gamma \sim 1$, we can write

$$\Delta\tau \approx \Delta t \approx \int_{-L_1}^{L_2} \frac{\mathrm{d}L}{v} \simeq 2 \times 10^4 \frac{\rho_0}{v} \sec . \tag{A.15}$$

Hence, with velocity not exceeding 30 m s^{-1} between stations, a traveler can complete the trip across a 10 m throat in comfortable 1.7 h.

The rotating wormhole is also expected to be traversable, the only difference being that the traveler might feel an additional centrifugal force determined by the adjustable parameter a. A detailed mathematical treatment of this case will be presented elsewhere.

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