

The magnetic field of the relativistic stars in the 5D approach

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Abstract

It is well-known that the 5D equations without sources may be reduced to the 4D ones with sources, provided an appropriate definition for the energy-momentum tensor of matter in terms of the extra part of the geometry. The advantage consists on the naturally appearance of gravitational and electromagnetic fields from this decomposition. With this ansatz an algorithm is presented, which permits to express the physical parameters in terms of gauge potentials and scalar field. An explicit form for the exterior magnetic field of neutron star in terms of the scalar field and the gauge potentials is deduced for a static, spherically-symmetric metric.

1 Introduction

Since a great amount of astrophysical objects in Cosmos, like pulsars, quasars or black holes, are gravitational bodies endowed with magnetic field, it is of interest to study the nature and behavior of magnetic field in interaction with gravitation. Classically, the Einstein-Maxwell theory should be sufficient to describe such cases and, indeed, there exist a set of exact solutions of those equations representing exterior fields of gravitational objects endowed with magnetic field [1]. Some of them are reasonably small, but they do not have the right behavior of the gravitational field far away from the sources; the other ones are acceptable in their behavior at infinity, but are rather cumbersome to be studied in analytic form. Furthermore, this theory actually is not an unification theory, but rather a superposition one: Einstein plus Maxwell, where the electromagnetic field appears like a model, there is no explanation for its existence. Recently [2] it has been shown that vacuum solutions in scalar-tensor theories are equivalent to solutions of general relativity with imperfect fluid as source, but the scalar fields do not arise from a natural framework of unification, instead they are put by hand, as in inflationary models and are therefore artificial fields.

On the other hand, it is well known that the 5D solutions of Einstein equations in vacuum when projected into four dimensional space-time correspond to solutions of Einstein's equations with an energy-momentum tensor of a gravitational field coupled with an electromagnetic field and also with a massless scalar field [3]. For theories like Kaluza-Klein (KK) and Low Energy Superstring (LESS), the electromagnetic field is a component of a more general field,

the existence of gravitation and electromagnetism following from its decomposition. These theories give rise to the hypothesis that the magnetic field of some astrophysical objects could be of fundamental origin and the magnetic field could be a consequence of a more general scalar-gravito-electromagnetic field. The problem that would arise is the existence of a scalar field whose interaction is so weak that it has not been measured till now in gravitational fields like the sun, even though it would possess a scalar field inherent. Nevertheless, that attractive or repulsive scalar force could have effects in stronger gravitational fields like pulsars and this gives rise to the hypothesis that the magnetic field of some astrophysical objects could be fundamental.

Wesson and Leon [4] demonstrated by algebraic means that the 5D KK equations without sources may be reduced to the Einstein equations with sources and proved that the extra part of the 5D geometry could be used appropriately to define an effective 4D energy-momentum tensor. Hongya and Wesson obtained that 5D black hole solutions of KK theory can be reduced to new exact solutions of 4D general relativity in which matter is a spherically symmetric anisotropic fluid of radiation. Patel and Naresh [5] obtained from the five-dimensional cylindrically symmetric space-time, by dimensional reduction, the radiation Friedman- Robertson -Walker flat model. Matos [6] presented a method for generating exact solutions of Einstein field equations as harmonic maps using the Chiral formalism in 5D. These solutions represent exterior fields of a gravitational body with arbitrary electromagnetic fields and whose gravitational potential possesses a Schwarzschild - like behaviour. He further elaborated a model [7], [8], [9] for the magnetic dipole of static bodies based on a 5D gauge theory that in 4D corresponds to a massive magnetic dipole coupled to a massless scalar field. Louis O Pimentel [10] and then Montesino and Matos [11] have obtained an effective energy-momentum tensor using the 4-dimensional part of the 5D Einstein equations with cosmological constant.

In this paper we start from the 5D metric on the principal fibre bundle, with scalar field and non-vanishing gauge potentials and study the correspondence between 5D Einstein field equations with cosmological constant and the 4D Einstein equations with sources. The source of the gravitation is considered to be a perfect magneto-fluid and from the 5D-4D reduction yields a natural expression for the magnetic field in terms of the gauge potentials and of scalar field.

2 The geometry

High-dimensional relativity seems to be an elegant way for unifying all interactions in physics and the first idea has been more and more transformed from the original suggestion of Kaluza and Klein. In this work the attention is focused upon the five-dimensional relativity, which unifies gravitation with electromagnetism.

It is not clear how to interpret physically any higher dimension. Interpretation of the fifth dimension has been done as a massless scalar field which can be or not associated with a fluid density [12], or, in other works, has been interpreted as a magnetic mass [13]. Other interpretation has been done as a fifth geometric property which shows up near horizon. In any case, it is worth to explore ways of interpreting the properties of the 5D solutions in a 4D word.

The version adopted here considers that the 5D Riemannian space P is a principal fiber bundle with typical fiber S^1 , the circle. This version is more convenient for three reasons: it is a natural generalization of $U(1)$ - gauge theory (Maxwell theory) to curved space-times, there is no necessity to recur to the so-called Kaluza-Klein or to the n - mode ansatz and it is no need to impose any restrictions to the functional dependence of the metric terms on P . The geometry used is shortly explain bellow.

The formalism of gauge theory [14] is described in the framework of fibre bundle, which is a collection of elements (P, π, M, G) , explicitly given by three differentiable manifolds: the principal bundle P , the base manifold M (usually taken to be the space-time manifold with metric g) and the typical fibre F which embodies the gauge freedom being a structure Lie group with transition functions that acts on F from the left and the projection $\pi : P \rightarrow M$ whose inverse image $\pi^{-1}(p) \equiv F_p$ is the fibre at p . A certain gauge correspond to a certain section of P and the gauge transformations are vertical automorphism $f_p : F_p \rightarrow F_p$ that change the section according to $s \rightarrow f_p \circ s$. The gauge field potentials are given by the connection forms ω on P that specifies the way in which a point of P is to be parallel transported along a curve lying in the base manifold M and yields a corresponding curvatures or field strengths Ω . In practice one uses their section-dependent counterparts $s^*\omega$ and $s^*\Omega$ defined on the base manifold. The matter fields are forms on the base manifold with values in a vector space U of an associated vector bundle and the elements of the corresponding frame bundle constitute the reference frames used to decompose the matter fields with respect to U .

Let $U \subset M$ be an open subset of M . If $\phi : F \rightarrow U \times F$ is a trivialisation of an open subset of F then the physical quantities can be mapped into the set $U \times F$ through the trivialisation ϕ , which means that the total space P looks locally like the direct product of M and F . Let $\tilde{g} = I^2 \hat{\omega} \otimes \hat{\omega}$ be a metric on F and $\hat{g} = \eta_{AB} \hat{\omega}^A \otimes \hat{\omega}^B = \eta_{AB} \hat{\omega}^A \otimes \hat{\omega}^B + I^2 \hat{\omega} \otimes \hat{\omega}$; $A, B = 1, \dots, 5$; $a, b = 1, \dots, 4$ be the metric on $P = H \otimes V$ where V is the vertical space and H its complement. Let $\{e_a\}$ be the projection of the complement base $\{\hat{e}_a\}$ such that $d\pi(e_a) = e_a$ and π_1 be the first projection, $\pi_1 : U \times F \rightarrow U$, $\pi_1(x, y) = x$. Thus, because of the identity $\pi = \pi_1 \circ \phi$, one finds that the corresponding base in the trivialisation set $U \times F$ is $d\phi(\{\hat{e}_a, e\}) = \{e_a - A_a(\partial/\partial y), (\partial/\partial y)\}$ whose dual base is $\{\omega^a, dy + A_a \omega^a\}$. Thus, the corresponding metric on $U \times F$ will be:

$$\bar{g} = \eta_{ab} \hat{\omega}^a \otimes \hat{\omega}^b + I^2 (A_a \omega^a + dy) \otimes (A_b \omega^b + dy) \quad (1)$$

where on observe that $s^*(\hat{\omega}) = A_a \omega^a$. This means that A_a are the pullback components of the one-form connection A through a cross section s . Since the group $U(1) \cong F$ is acting on P , there exist an isometry $Is : P \rightarrow P$, $(x^a, y) = (x^a, y + 2\pi)$ such that $Is^* \bar{g} = \bar{g}$. This implies the existence of a Killing-vector ξ and therefore we can be choose a coordinate system where the metric components of \bar{g} do not depend on $x^5 = y$. With the gauge-theory philosophy, the action of $U(1)$ on P means that there are electromagnetic interactions on P , which implies that there is a coordinate system where the metric components do not depend on x^5 .

3 Space-time field equations

In the present work the results of [13] are extended with the assumption of nonvanishing electromagnetic potential. The intention is to demonstrate that the Einstein equations on M without cosmological constant and with perfect charged fluid as source, are obtained from the field equations for vacuum with cosmological constant on the principal fibre bundle $P(1/JM, U(1))$, J being the scalar field which correspond to the radius of the internal space $U(1)$, whose units and magnitude depends on the particular cases of the cosmological or astrophysical model [6]. Louis O. Pimentel [10] used the following identification between the velocity 4-vector and the scalar field:

$$u_a = \frac{J_{;a}}{\sqrt{-J_{;c}J^{;c}}} \quad (2)$$

in order to exprime the energy-impuls tensor in terms of J in the framework of the 5D gravity. If the velocity 4-vector u_a corresponds to the comoving observers with the fundamental fluid-flow lines, then u_a is a timelike normalized vector field. In this case, a dependence on the t coordinate of the scalar field will be imposed: $J = J(t)$, which means that the scalar field includes the time evolution of the Universe.

The base space M is the space-time of general relativity and we can choose its metric from the verified metric solutions of the Einstein equations. The exterior of a spherical, rotating, charged astrophysical object can be modeled with a static, spherically symmetric metric. Therefore, the metric will admit two Killing vectors associated with the (t, φ) coordinates and the physical quantities will depend only on (r, θ) . To simplify the calculation, without losing generality, the normalized gravitational potentials $g_{ab} = (g_1, r^2, r^2 \sin^2 \theta, -g_4)$, the gauge potentials $A_a = (A_1, A_2, A_3, A_4)$ and the physical parameters will be chosen to depend only on r .

After all those considerations the 5D metric is rather complicated:

$$\bar{g}_{AB} = \begin{pmatrix} \frac{1}{J}g_1 + J^2A_1^2 & 0 & 0 & 0 & J^2A_1 \\ 0 & \frac{1}{J}r^2 + J^2A_2^2 & 0 & 0 & J^2A_2 \\ 0 & 0 & \frac{1}{J}r^2 \sin^2 \theta + J^2A_3^2 & 0 & J^2A_3 \\ 0 & 0 & 0 & -\frac{1}{J}g_4 + J^2A_4^2 & J^2A_4 \\ J^2A_1 & J^2A_2 & J^2A_3 & J^2A_4 & J^2 \end{pmatrix} \quad (3)$$

and the Einstein field equations on the 5D manifold in vacuum with cosmological constant Λ , given by: $G_{AB} = \Lambda \bar{g}_{AB}$ are cumbersome to calculate (all the calculations were performed within the GRTensor package [16])

Simplifications could be made to the chose of the Yang-Mills potentials. The simplest case is for vanishing electromagnetic potential: $A_a = 0$. The 4D form of the Einstein tensor is calculated for the static spherically symmetric metric:

$$ds^2 = g_1 dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) - g_4 dt^2 \quad (4)$$

which for $g_4 = 1/g_1 = g$ is a Schwarzschild-type and for $g_1 = a^2/(1 - kr^2)$, $g_4 = 1$, is a FRW-type metric.

The extra terms obtained from the identification of the 4D form with the corresponding part of the 5D Einstein field equation are interpreted to be the source of the gravity: $G_{ab} = 8\pi T_{ab}$. Thus, a nonvanishing energy-momentum tensor associated with the scalar field J can be defined (comma means derivation with t):

$$T_r^r = T_\theta^\theta = T_\varphi^\varphi = \frac{1}{8\pi} \left(\frac{3}{4} \left(\frac{J'}{J} \right)^2 \frac{1}{g_1} + \frac{\Lambda}{J} \right) \quad (5)$$

$$T_t^t = -\frac{1}{8\pi} \left(\frac{3}{4} \left(\frac{J'}{J} \right)^2 \frac{1}{g_1} - \frac{\Lambda}{J} \right) \quad (6)$$

In this case, it is possible the identification with the corresponding physical energy-momentum tensor of the perfect fluid:

$$T_{ab} = (p + \rho) u_a u_b + p g_{ab} \quad (7)$$

where u_a is the unity velocity vector tangent to the flow lines: $u_a u_a = -1$.

From this comparison, the pressure and the density of energy can be expressed in term of the scalar field:

$$\rho = \frac{3}{4} \left(\frac{J'}{J} \right)^2 \frac{1}{g_4} - \frac{\Lambda}{J} \quad (8)$$

$$p = \frac{3}{4} \left(\frac{J'}{J} \right)^2 \frac{1}{g_4} + \frac{\Lambda}{J} \quad (9)$$

where the gravitational constant is $\Lambda = G_{yy}$.

The case of nonvanishing Yang-Mills potentials will be considered further. In the rest frame of the spherical charge distribution, if the gauge has the form: $A_a = (0, 0, A_3, 0)$, the electromagnetic tensor: $F_{ab} = A_{b,a} - A_{a,b}$ will admit only one nonvanishing component corresponding to the magnetic field: $B_a = 1/2\epsilon^{abcd} u_b F_{cd} = (0, B_\theta, 0, 0)$.

The introduction of the magnetic potential in the 5D form of the metric has been made with the intention of finding an identification between the 4D part of Einstein tensor obtained from the reduction 5D-4D and the energy-momentum tensor which describes the external behavior of a magnetized astrophysical object. The existence of the azimuthal magnetic field is realistic for the neutron star, this being the only stable configuration for a stationary axisymmetric flow of the internal stellar plasma in the MHD approximation.

The energy-momentum tensor is constituted with the perfect fluid tensor and the magnetic part of the electro-magnetic tensor, yielding the complete form of the perfect magneto-fluid tensor:

$$T_{ab} = \left(p + \rho + \frac{B^2}{4\pi} \right) u_a u_b + \left(p + \frac{B^2}{8\pi} \right) g_{ab} - \frac{B_a B_b}{4\pi} \quad (10)$$

which for the chosen metric has the components:

$$T_b^a = \begin{pmatrix} p + \frac{B_\theta^2}{8\pi r^2} & 0 & 0 & 0 \\ 0 & p - \frac{B_\theta^2}{8\pi r^2} & 0 & 0 \\ 0 & 0 & p + \frac{B_\theta^2}{8\pi r^2} & 0 \\ 0 & 0 & 0 & -\rho - \frac{B_\theta^2}{8\pi r^2} \end{pmatrix} \quad (11)$$

In terms of the components introduced by the fifth dimension, the tensor (11) has the form (dot means derivation with r):

$$T_r^r = \frac{1}{8\pi} \left(\frac{3}{4} \left(\frac{J'}{J} \right)^2 \frac{1}{g_4} + \frac{\Lambda}{J} + \frac{J^3 (\dot{A})^2}{4r^2 g_1 \sin^2 \theta} \right) \quad (12)$$

$$T_\varphi^\varphi = \frac{1}{8\pi} \left(\frac{3}{4} \left(\frac{J'}{J} \right)^2 \frac{1}{g_4} + \frac{\Lambda}{J} - \frac{J^3 (\dot{A})^2}{4r^2 g_1 \sin^2 \theta} \right) \quad (13)$$

$$T_\varphi^\varphi = \frac{1}{8\pi} \left(\frac{3}{4} \left(\frac{J'}{J} \right)^2 \frac{1}{g_4} + \frac{\Lambda}{J} + \frac{J^3 (\dot{A})^2}{4r^2 \sin^2 \theta} \right) \quad (14)$$

$$T_t^t = \frac{1}{8\pi} \left(-\frac{3}{4} \left(\frac{J'}{J} \right)^2 \frac{1}{g_4} + \frac{\Lambda}{J} - \frac{J^3 (\dot{A})^2}{4r^2 g_1 \sin^2 \theta} \right) \quad (15)$$

From the comparison of (11), with (12 – 14) the expression for the magnetic field is naturally obtained:

$$B_\theta = \frac{J^3 (\dot{A})^2}{4g_1 \sin^2 \theta} \quad (16)$$

The equation (16) clearly describes the universal constitution of the magnetic field and its essential contribution to metric geometry of the space-time.

The expression for the cosmological constant, derived from the fifth component of the Einstein tensor, is rather complicated, but the nondiagonal component $G_{\varphi y}$ provides with an interesting relation:

$$\frac{\dot{A}}{A} \left(\frac{\dot{g}_4}{g_4} - \frac{\dot{g}_1}{g_1} \right) = 2 \frac{\ddot{A}}{A} \quad (17)$$

which can be easily integrated and a formal expression for the metric potential in terms of the scalar and gauge field is deduced:

$$\frac{g_1}{g_4} = \dot{A}^2 \quad (18)$$

The expression (18) is then substituted in (16) and yields an interesting formulae that gives the time-dependence of the magnetic field:

$$B_\theta = \frac{J^3}{4g_4 \sin^2 \theta} \quad (19)$$

For $g_4 = 1/g_1$ (Schwarzschild case) we obtain: $g_1 = \dot{A}$, and for $g_4 = 1$ (FRW case), we have: $g_1 = \dot{A}^2$. Considering those metric functions, the magnetic field becomes in the first case:

$$B_\theta = \frac{J^3 \dot{A}}{4 \sin^2 \theta} \quad (20)$$

and in the second:

$$B_\theta = \frac{J^3}{4 \sin^2 \theta} \quad (21)$$

4 Conclusions

In this paper it has been shown that the field equations with cosmological constant Λ on the principal fibre bundle $P(1/JM, U(1))$ with nonvanishing magnetic potential, can lead to the energy-momentum tensor of a perfect magnetofluid endowed with azimuthal magnetic field, which is the case of a neutron star [17]. The magnetic field yields from a natural decomposition in direct connection with the fifth dimension and this is of evidence for the fundamental role of the magnetic field to the constitution of matter and of the Universe.

The study of the above problems involve a lot of interesting aspects besides the ones analyzed here. It can be pointed out, for instance, the study of the 4D energy momentum tensor which corresponds to this solution; the study of the behavior of test particles and of future interest is the generalization of this solution to the stationary case in order to modelate in a more realistic way a rotating astrophysical object. It is also convenient to make a deeper investigation on the relationship between this model and the string theory for low energies.

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