ASYMPTOTICALLY FLAT FIVE-DIMENSIONAL AXISYMMETRICAL KALUZA-KLEIN SPACETIME SOLUTION

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In this paper we show a asymptotially flat five-dimensional axisymmetric Kaluza-Klein spacetime solution with electrostatic field, which depends on the two parameters w and q.

1 Introduction

The Five-dimensional Kaluza-Klein theory is a united theory of gravity and electromagnetism[1,2], in which the fifth dimension is a compact circle with the radius R of the order of the Planch length. Consider the standard five-dimensional Einstein-Hilbert action

$$S = -\frac{1}{16\pi G^{(5)}} \int d^5x \sqrt{-g^{(5)}} R^{(5)} \tag{1}$$

where $G^{(5)}$ is the five-dimensional gravitational constant and $g^{(5)}$ the determinant of five-dimensional of metric g_{AB} with the signature $(+,+,+,-,+)$. Similarly to the four-dimensional situations the five-dimensional Riemann tensor, Ricci tensor and the curvature scalar are defined, respectively, as follows

$$\begin{gathered} R^A_{BCD} = \partial_C \Gamma^A_{BD} - \partial_D \Gamma^A_{BC} + \Gamma^A_{KC}\Gamma^K_{BD} - \Gamma^A_{KD}\Gamma^K_{BC} \\ R_{BC} = R^A_{BCA} \\ R^{(5)} = R^C_C \end{gathered} \tag{2}$$

For the lower-energy case, the metric function is independent of the fifth dimension, and can take the form

$$g_{AB} = \begin{pmatrix} e^{-\alpha\phi} g_{\mu\nu} + e^{2\alpha\phi} A_\mu A_\nu & e^{2\alpha\phi} A_\mu \\ e^{2\alpha\phi} A_\nu & e^{2\alpha\phi} \end{pmatrix} \tag{3}$$

where the constant $\alpha = \frac{2}{\sqrt{3}}$, A_μ is the electromagnetic potentials, and $g_{\mu\nu}$ is the four-dimensional spacetime metric. From the action(1) and the metric (3) one can obtain the four-dimensional effective low-energy action

$$S_{effective} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}[R - 2\partial_\mu\phi\partial^\mu\phi - e^{2\sqrt{3}\phi}F_{\mu\nu}F^{\mu\nu}] \tag{4}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $G = \frac{G^{(5)}}{2\pi R}$ is the four-dimensional gravitational constant, g is the determinant of the metric $g_{\mu\nu}$ and R is the curvature of the four-dimensional metric $g_{\mu\nu}$. For the low-eneger case, the advantage of the Kaluza-Klein theory is that solving the vacuum Kaluza-Klein field equation is equivalent to solving the four-dimensional field equition with non-zero energy-monentum tensor depending on the electromagnitism and scalar field. For the five-dimensional vacuum case, some axisymmetric solutions have been worked out[3−14]. For the Maxwell-Einstein equation, some axisymmetric solutions including the black hole solutions are also obtained[15−18]. Recently, we give a so-called five-dimensional Kaluza-Klein TS-like solution which describes the axisymmetric spacetime with eletrostatic field[19]. In this paper, on the base of the previous paper[19] we will give the new Kaluza-Klein spacetime solutions with eletrostatic field, which depends on the two independent parameters w and q taking arbitrary real values; the asymptotical behavior of the solution is discussed in detail, as will be seen the electric charge, the electric dipole moment and scalar charge of the gravitational source can be taken finite values under the certain contraint condition. In section 3 the spherically symmetric limit is given, which is consistent with the Gibbons' result.

2 The Axsymmetric Five-Dimensional Kaluza-Klein Solution

For the five-dimensional Kaluza-Klein axisymmetric stationary spacetime, the correponding metric reads

$$ds^2 = e^{\chi}(d\rho^2 + dz^2) + G_{AB}dx^A dx^B \tag{5}$$

from the action (1) one obtain the five-dimensional vacuum field equation

$$(\rho G_{,\rho}G^{-1})_{,\rho} + (\rho G_{,z}G^{-1})_{,z} = 0 \tag{6}$$

$$\chi_{,\rho} = \rho^{-1} + (4\rho)^{-1}tr(U^2 - V^2) \tag{7}$$

$$\chi_{,z} = (2\rho)^{-1}(trUV) \tag{8}$$

where $U = \rho G_{,\rho}G^{-1}$ and $V = \rho G_{,z}G^{-1}$. For the metric (5) we obtain the following solution

$$ds^2 = \frac{p^2x^2 - q^2y^2 - 1}{p^2}k^2 e^{\gamma(\Psi)}(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}) + k^2(x^2-1)(1-y^2)e^{-2\Psi}d\varphi^2$$

$$+e^{\Psi}[\frac{4(wpx+uqy)^2-(p^2x^2-q^2y^2-1)^2}{(p^2x^2-q^2y^2+2upx+2wqy+1)(px^2-q^2y^2-1)}dt^2$$
$$+\frac{2wpx+2uqy}{p^2x^2-q^2y^2-1}dx^5dt+\frac{p^2x^2-q^2y^2+2upx+2wqy+1}{p^2x^2-q^2y^2-1}(dx^5)^2] \quad (9)$$

with

$$[(x^2-1)\Psi_{,x}]_{,x}+[(1-y^2)\Psi_{,y}]_{,y}=0 \quad (10)$$

$$\gamma_{,x}=\frac{3}{2}\frac{1-y^2}{x^2-y^2}[x(x^2-1)(\Psi_{,x})^2-x(1-y^2)(\Psi_{,y})^2-2y(x^2-1)\Psi_{,x}\Psi_{,y}]-2\Psi_{,x} \quad (11)$$

$$\gamma_{,y}=\frac{3}{2}\frac{x^2-1}{x^2-y^2}[y(x^2-1)(\Psi_{,x})^2-y(1-y^2)(\Psi_{,y})^2+2x(x^2-1)\Psi_{,x}\Psi_{,y}]-2\Psi_{,y} \quad (12)$$

where

$$x=\frac{1}{2k}\{[\rho^2+(z+k)^2]^{\frac{1}{2}}+[\rho^2+(z-k)^2]^{\frac{1}{2}}\} \quad (13)$$

$$y=\frac{1}{2k}\{[\rho^2+(z+k)^2]^{\frac{1}{2}}-[\rho^2+(z-k)^2]^{\frac{1}{2}}\} \quad (14)$$

and k is a nonzero real constant, the constants u, w, p and q satisfy $p^2-q^2=1$ and $u^2-w^2=1$; when the parameter $w=0$ or $u=\pm1$ (9) will reduce to the soluton given in Ref.[19]. For the equation (10), one can choose the following solution with the asymptotic behavior $\Psi\to0$ as $(x^2+y^2)\to\infty$

$$\Psi=c\ln\frac{x-1}{x+1}+\sum_{n=0}^{\infty}\frac{a_n}{(x^2+y^2-1)^{\frac{n+1}{2}}}P_n(\frac{xy}{\sqrt{x^2+y^2-1}}) \quad (15)$$

According to the metric (3) the corresponding four-dimensional axisymmetric metric is given

$$d\tau^2=k^2[(p^2x^2-q^2y^2+2upx+2wqy+1)(p^2x^2-q^2y^2-1)]^{\frac{1}{2}}$$
$$\times e^{\gamma+\frac{\Psi}{2}}\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right)$$
$$+k^2(x^2-1)(1-y^2)^2\left(\frac{p^2x^2-q^2y^2+2upx+2wqy+1}{p^2x^2-q^2y^2-1}\right)^{\frac{1}{2}}e^{\frac{-3\Psi}{2}}d\varphi^2$$
$$+\left(\frac{p^2x^2-q^2y^2+2upx+2wqy+1}{p^2x^2-q^2y^2-1}\right)^{-\frac{1}{2}}e^{\frac{3\Psi}{2}}dt^2 \quad (16)$$

with the electricsatic potential

$$A_4=\frac{2wpx+2uqy}{p^2x^2-q^2y^2+2upx+2wqy+1} \quad (17)$$

the gravitational potential

$$f = -g_{44} = (\frac{2wpx + 2uqy}{p^2x^2 - q^2y^2 + 2upx + 2wqy + 1})^{\frac{1}{2}} e^{\frac{3\Psi}{2}} \tag{18}$$

and the scalar potential

$$\phi = \frac{\sqrt{3}}{4} \ln \sigma = \frac{\sqrt{3}}{4} (\ln \frac{p^2x^2 - q^2y^2 + 2upx + 2wqy + 1}{p^2x^2 - q^2y^2 - 1} + \Psi) \tag{19}$$

Choosing the Boyer-Linquist coordinates

$$x = \frac{r - m}{k}, \qquad y = \cos\theta \tag{20}$$

at large r, we have

$$A_4 \to \frac{2wk}{pr} + \frac{2uqk^2 \cos\theta + 2wpkm - 4wuk^2}{p^2r^2} + o(r^{-3}) \tag{21}$$

$$f \to [1 - \frac{uk}{pr} + o(r^{-2})] e^{\frac{3\Psi}{2}} \tag{22}$$

$$\phi \to \frac{\sqrt{3}}{4} [\frac{2uk}{pr} + \Psi] + o(r^{-2}) \to \frac{\Sigma}{r} + o(r^{-2}) \tag{23}$$

From the asymptotic behaviors of the electric potential A_4, the gravitational potential f and the scalar field ϕ we get the total electric charge, the total mass and the total scalar charge of the source

$$Q = \frac{2wk}{p} \tag{24}$$

$$M = \frac{uk}{2p} + M(\Psi) \tag{25}$$

$$\Sigma = \frac{\sqrt{3}uk}{2p} + \Sigma(\Psi) \tag{26}$$

where $M(\Psi)$ and $\Sigma(\Psi)$ are the mass contribution and the scalar charge contribution to the total mass M and the total scalar charge Σ depending on the function Ψ, respectively. If the function $\Psi \to \frac{\overline{m}}{r} + o(r^{-2})$ at large r, then $M(\Psi) = -\frac{3\overline{m}}{4}$ and $\Sigma(\Psi) = \frac{\sqrt{3}\overline{m}}{4}$, where the real parameter $\overline{m}$ take finite value in order that the metric is regular. Noting that the parameters w, u and p take values in the ranges $[0, \pm\infty)$, $[\pm 1, \pm\infty)$ and $[\pm 1, \pm\infty)$, respectively, one can find the electric charge Q, the mass M and the scalar charge Σ take values in the ranges $[0, \pm\infty)$, $[M(\Psi), \infty)$ and $[\Sigma(\Psi), \infty)$, where we have made use of the assumption that both the parameter k and $\frac{u}{p}$ are positive, and $M(\Psi)$ is also positve in view of the mass M being always positive in the case $M(\Psi) = 0$ or $\frac{u}{p} \to 0$. From the above one can see that Q, M and Σ are infinite. Now, let us consider such an constraint condition

$$w(pm - 2uk) = 0 \tag{27}$$

namely, $w = 0$ and $pm = 2uk$. For the former $u = 1$, $p \geq 1$ or $u = -1$, $p \leq -1$ we have the electric charge , the mass and the scalar charge

$$Q = 0 \tag{28}$$

$$M = \pm\frac{k}{2p} + M(\Psi) \tag{29}$$

$$\Sigma = \pm\frac{\sqrt{3}k}{2p} + \Sigma(\Psi) \tag{30}$$

and the second term of (21) denotes the electric dipole term, the electric dipole moment is

$$P = \frac{2qk^2}{p^2} \tag{31}$$

for the latter we have

$$Q = \pm m\sqrt{1 - \frac{4k^2}{p^2m^2}} \tag{32}$$

$$M = \frac{m}{4} + M(\Psi) \tag{33}$$

$$\Sigma = \frac{\sqrt{3}m}{4} + \Sigma(\Psi) \tag{34}$$

and the electric dipole moment

$$P = \frac{qkm}{p} \tag{35}$$

Noting that the parameters p and q are real and $p^2 - q^2 = 1$, for both case $w = 0$ and $pm = 2uk$ all quantities Q, M, Σ and P are finite. For the former there are the limits $M = M(\Psi)$, $\Sigma = \Sigma(\Psi)$ and $P = 0$; for the latter, there are the limits $Q = \pm m$ and $P = 0$, where the mass M and scalar charge Σ do not formally independ of p. In fact, the equation (28)-(31) are included in (32)-(35), i.e. the constraint condition $w = 0$ corresponds to the case $u = \pm 1$ of condition $pm = 2uk$, so we only need consider the latter condition in the following discussion.

3 The Spherically Symmetric Metric

Letting the parameter $q = 0$ and $\Psi = 0$, then metric (16) reduces to the spherically symmetric metric

$$d\tau^2 = \left[(r-m)^2 + 2ku\,(r-m) + k^2\right]^{\frac{1}{2}} [(r-m)^2 - k^2]^{\frac{1}{2}} [d\theta^2 + \sin^2\theta d\varphi^2] + \left[\frac{(r-m)^2+2ku(r-m)+k^2}{(r-m)^2-k^2}\right]^{\frac{1}{2}} dr^2 - \left[\frac{(r-m)^2+2ku(r-m)+k^2}{(r-m)^2-k^2}\right]^{-\frac{1}{2}} dt^2 \tag{36}$$

and the electric charge, the mass and the scalar charge are

$$Q = 2wk \tag{37}$$

$$M = \frac{uk}{2} \tag{38}$$

$$\Sigma = \frac{\sqrt{3}uk}{2} \tag{39}$$

letting

$$2uk = \beta, 2k = a \tag{40}$$

then

$$Q^2 = \beta^2 - a^2 \tag{41}$$

$$M = \frac{\beta}{4} \tag{42}$$

$$\Sigma = \frac{\sqrt{3}}{4}\beta \tag{43}$$

noting that the scalar charge takes the form [20,21] $\Sigma = \frac{\sqrt{3}}{4}(\beta + \alpha)$ instead of $\Sigma = \sqrt{\frac{3}{4}}(\beta + \alpha)$ and the electric potential differs by the facror 2, one can see this result is consistent with the result of Ref.[20,21] with $\alpha = 0$. the problem is that electric charge Q, mass M and scalar charge Σ are still infinite. Similarly, for the case $q = 0$ we can use the constraint condition $m = \pm 2uk$, then

$$Q = \pm m\sqrt{1 - \frac{2k}{m}} \tag{44}$$

$$M = \frac{m}{4} \tag{45}$$

$$\Sigma = \frac{\sqrt{3}m}{2} \tag{46}$$

where the parameters m and k are nonzero real. So the mass and scalar charge are nonzero.

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