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RESEARCH ARTICLE

Kerr-like phantom wormhole

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Abstract In this work we study a Kerr-like wormhole with an scalar field with opposite sign as source (Phantom). It has three parameters: mass, angular momentum and scalar field charge. This space-time has a naked ring singularity, otherwise it is regular everywhere. The main feature of this wormhole is that the mouth of the throat lies on a sphere of the same radius as the ring singularity and apparently does not allow any observer to reach the singularity, it behaves like an anti-horizon. After analyzing the geodesics of the wormhole we find that an observer can go through the wormhole without troubles, but the equator presents an infinite potential barrier which does not allow any geodesic from reaching the throat. From an analysis of the Riemann tensor we obtain that the tidal forces are small and could allow the wormhole to be traversable, from the north pole, for an observer like a human being.

Keywords Kerr-like wormhole · Phantom field · Ring singularity

1 Introduction

The research on wormholes started with Einstein trying to give a field representation of particles [1]. The idea was further developed by Ellis [2], who tries to model particles

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as bridges between two regions of the space-time. Later on, in the seminal Morris and Thorne's work [3], the idea of considering such solutions as connections between two separated regions of the Universe attracted a lot of attention.

In the 80's, the astronomer Carl Sagan asked for help to physicist and turned out that the solution proposed by Ellis [2], actually could be interpreted as the identification, the union, of two different regions of the space-time. The idea of Ellis is the following, no matter how far the two regions of the space-time are -or even if those two regions were in different space-times- by means of this union one could just identify them, and obtain a solution which allows to go from one region of the space-time to another. Morris and Thorne [3] showed that these wormhole solutions need to violate the energy conditions. Such type of matter is called exotic (see [4] for a detailed review on this subject). The solutions can exist but they need to be generated by matter which apparently does not exists. This is the reason why these solutions remained in the realm of fiction.

However, more and more evidence was building towards the presence of unknown types of matter and energy in our Universe, which do not necessarily satisfy energy conditions. Now we know that the Universe is formed by 68% of dark energy. This new type of matter composes the majority in the Universe, and happens to be everywhere [5]. There is now an agreement among the scientific community that matter which violates some of the energy conditions is very plausible to exist. Thus, the issue that wormholes can be rejected due to the fact that they violate energy conditions is, at least, diminished. The existence of wormholes in the universe is very interesting; they could be highways to stars and galaxies, otherwise impossible to visit. Of course, this is very speculative to this stage, but no observation can discard the existence of phantom energy. Nowadays, we know that a combination of phantom and quintessence fields could be the dominant component in the Universe [6]. The fact that phantom energy can be the source of wormholes [1–5] is more than exciting, and brings up new questions to investigate about the existence of stars made of this kind of matter.

Maybe, the main problem faced by wormholes is their stability. The stability problem of the bridges has been studied since the 60's by Penrose [7], in connection to the stability of the Cauchy horizons. The stability of the throat of a wormhole was studied numerically by Shinkay and Hayward [8], they show that the wormhole proposed by Thorne [3] collapses, possibly towards a black hole and its throat closes, when it is perturbed by a scalar field with stress-energy tensor defined with the usual sign. They also found that the throat of the wormhole grows exponentially, when the perturbation is due to a scalar field of the same type as the one making the wormhole; thus showing that the solution is highly unstable (see also [9]). In [10] it was conjectured that rotation of the magnetic field could stabilize a phantom star. The idea is that a rotating solution would have more possibilities to be stable than the one proposed by Thorne, as well as more possibilities to be stable than the general static spherically symmetric solutions of wormholes.

We look for wormhole solutions which are traversable, that is, a test particle can go from one side of the throat to the other in a finite time without facing large tidal forces and rotating, following the conjecture that the rotation can stabilize the wormhole. Some rotating solutions were studied in the past, as an approximation [11,12] or as an exact solution of the Einstein equations [10]. Nevertheless, this last solution has a main problem, it is not asymptotically flat and must be matched with a static one.

In this work we study the solution obtained in [13]. It has three parameters: mass, angular momentum and scalar charge. It has a naked ring singularity which is the same as in the Kerr solution. We will show that it is possible to go through the throat from the north pole.

This work is organized as follows: In Sect. 2 we introduce and analyse the metric, including its invariants and parameters. We give some of the limiting cases of the space-time, analyse the functions that comprise the metric and study the geometry of the wormhole and study the geometry of the wormhole. In Sect. 3 we study null geodesics and find that are repelled by the ring singularity. In Sect. 4 it is shown that null and weak energy conditions are violated. In Sect. 5 we analyse the tidal forces and find that the easiest way to go through the throat is by the polar geodesics which can go from one part of the throat to the other side, while the tidal forces are infinite on the ring singularity, it is not possible for a traveller to go through the wormhole approaching from the equatorial plane. Finally we give some discussion in Sect. 6.

2 The line element

We consider a stationary and axially symmetric space-time possessing a time-like Killing vector field $(\xi)^\alpha = (\partial/\partial_t)^\alpha$, generating invariant time translations and a space-like Killing vector field $\mu^\alpha = (\partial/\partial\varphi)^\alpha$ generating invariant rotations with respect to φ. The corresponding line element can be expressed in Boyer-Lindquist coordinates,

$$ds^2 = -f\,(dt + \omega\, d\varphi)^2 + \frac{1}{f}\left(\Delta\left(\frac{dl^2}{\Delta_1} + d\theta^2\right) + \Delta_1 \sin^2\theta\, d\varphi^2\right), \tag{1}$$

where

$$\Delta = (l - l_1)^2 + \left(l_0^2 - l_1^2\right)\cos^2\theta, \tag{2}$$

$$\Delta_1 = (l - l_1)^2 + \left(l_0^2 - l_1^2\right), \tag{3}$$

The line element components ω and f read

$$\omega = a\,\frac{(l - l_1)}{\Delta}\,\sin^2\theta, \tag{4}$$

$$f = \frac{\left(a^2 + k_1^2\right)\, e^{\lambda}}{a^2 + k_1^2\, e^{2\lambda}}, \tag{5}$$

where

$$\lambda = \frac{a^2 + {k_1}^2}{2\,k_1\,\Delta}\,\cos\theta, \tag{6}$$

l_1, l_0 being parameters with units of distance, such that $g_{ll} > 0$, that is

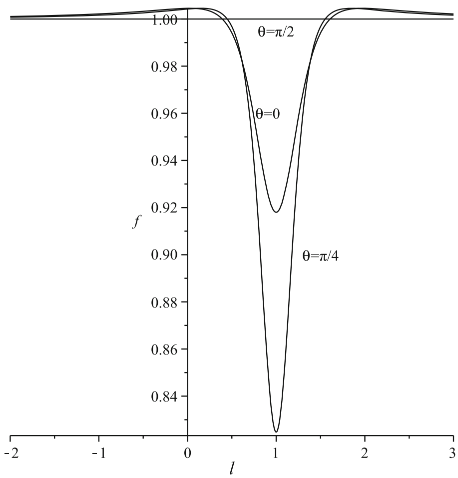


Fig. 1 The function f with $l_1 = 1.0, l_0 = 1.1, a = 0.11, k_1 = 0.10$ for different values of θ. Observe that there is a minimum in 0.918 when $\theta = 0$

$$l_0^2 > l_1^2 > 0, \tag{7}$$

these parameters are related with the size of the throat, while a and k_1 are parameters with units of angular momentum. Observe that the function λ is well behaved everywhere, except in $\Delta = 0$ just in the ring singularity. In Fig. 1 we plot the function f.

Line element (1) is a solution of the Einstein's equations $R_{\mu\nu} = -\kappa \Phi_\mu \Phi_\nu$ for an opposite sign scalar field given by

$$\Phi = \frac{1}{\sqrt{2\kappa}} \lambda, \tag{8}$$

where $\kappa = \frac{8\pi G}{c^4}$. It is due to this opposite sign that we call this solution a Phantom Wormhole.

The asymptotic behaviour of metric (1) can be seen as follows. For large positive or negative values of $l \to \pm\infty$, we have that $\omega \to 0$ and $\lambda \to 0$ thus

$$f \to 1$$

and $\Delta, \Delta_1 \to l^2$, so that the line element

$$ds^2 \to -dt^2 + dl^2 + l^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right), \tag{9}$$

is a flat space in spherical coordinates, *i.e.*, metric (1) is asymptotically flat.

The scalar curvature (and the other invariants of the metric) are of the form

$$Invariants = \frac{F}{8\, k_1^2\, \Delta^{r_1}\, \Delta_1^{r_2}\, \left(a^2 + k_1^2 e^{2\lambda}\right)^{r_3}}, \tag{10}$$

where F is a complicated function free of singularities that takes different forms for each invariant of the metric, and r_1, r_2, and r_3 are positive coefficients whose exact

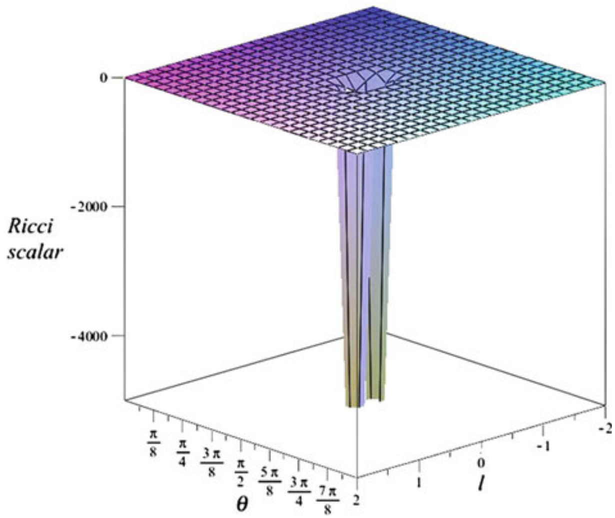


Fig. 2 The Ricci scalar for the metric (1), with $l_1 = 0$, $l_0 = 1.0$, $a = 0.5$ and $k_1 = 1.0$

value depend upon the chosen invariant. For the scalar curvature ($r_1 = 4, r_2 = r_3 = 1$), it is clear that the condition $\Delta = 0$ makes it diverge at the location of the ring singularity (see Fig. 2).

The ADM mass and angular momentum [14] are given by

$$M = -\frac{1}{8\pi}\lim_{S_t\to\infty}\oint_{S_t}(\mathrm{k}-\mathrm{k}_0)\sqrt{\sigma}\mathrm{d}^2\theta, \tag{11}$$

$$J = -\frac{1}{8\pi}\lim_{S_t\to\infty}\oint_{S_t}\left(\mathrm{K_{ij}}-\mathrm{K}\,\gamma_{ij}\right)\phi^{\mathrm{i}}\,\mathrm{r^j}\sqrt{\sigma}\mathrm{d}^2\theta. \tag{12}$$

In our case we obtain

$$M = -l_1, \tag{13}$$

$$J = a. \tag{14}$$

Thus the wormhole has a negative mass, which is responsible for the bounce of geodesics from the singularity, and an angular momentum given by the parameter a.

If we set $a = 0$, from (4) we get $\omega = 0$; the metric (1) transforms into

$$ds^2 = -f\,dt^2 + \frac{1}{f}\left(\Delta\left(\frac{dl^2}{\Delta_1} + d\theta^2\right) + \Delta_1\,\sin^2\theta\,d\varphi^2\right), \tag{15}$$

where the line element component f now reads

$$f = e^{-\lambda}, \tag{16}$$

and the parameter λ is give by

$$\lambda = \frac{k_1}{2\Delta}\cos\theta, \tag{17}$$

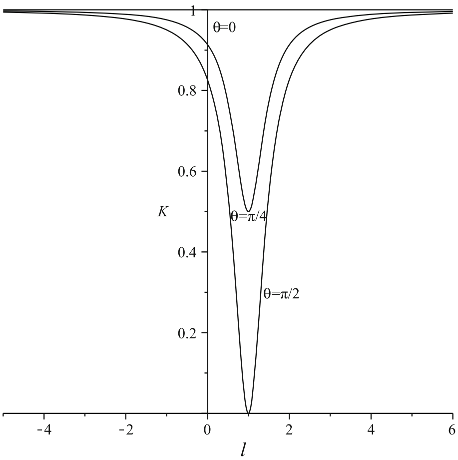


Fig. 3 The function K with $l_1 = 1, l_0 = 1.1, a = 0.11, k_1 = 0.10$ for different values of θ. Observe that $K = 1$ everywhere except close to the ring singularity

To see the geometry of the wormhole, it is convenient to write metric (1) as

$$ds^2 = -f\,(dt + \omega\, d\varphi)^2 + \frac{K}{f}\, dl^2 + \frac{\Delta_1}{f}\left[K\, d\theta^2 + \sin^2\theta\, d\varphi^2\right], \tag{18}$$

where $K = \Delta/\Delta_1$. Again, the quantity between brackets [ ] can be interpreted as the solid angle element modified by the function K multiplying $d\theta^2$. The modification of the solid angle is due the fact that the space-time is now axially symmetric. Nevertheless, this modification is small everywhere (except near a sphere of radius l_1, the radius of the ring singularity). At large distances of the phantom star $|l| \to \infty$ it takes on the usual form $K = 1$ (see Fig. 3).

The wormhole throat might be described by an embedding diagram in three dimensional Euclidean space at fixed time and for a fixed value of θ. Under these conditions, metric (18) reduces to

$$ds^2 = f^{-1}K\left(\frac{dl}{d\mathcal{R}}\right)^2 d\mathcal{R}^2 + \mathcal{R}^2 d\varphi^2, \tag{19}$$

with $\mathcal{R}^2 = R^2\sin^2\theta_0$, where $R^2 = f^{-1}\Delta_1$. The embedding surface is given by

$$z(l, \theta_0) = \int\limits_{l_1} \sqrt{\frac{K}{f} - \left(\frac{d\mathcal{R}}{dl}\right)^2}\, dl, \tag{20}$$

Since $g_{\mathcal{R}\mathcal{R}} = f^{-1}K(dl/d\mathcal{R})^2 < 1$ near $\theta = \pi/2$, the term inside the square root is negative and the function z is not well-behaved, the throat can not be visualized near the equatorial plane. In Fig. 4 we show the embedding diagram seen from different angles.

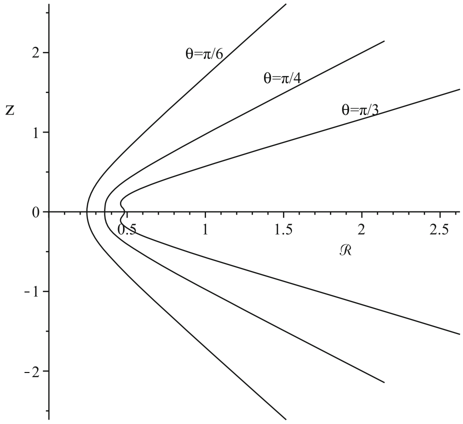


Fig. 4 The throat of the wormhole for different values of θ. Here we set $l_1 = 1.0$, $l_0 = 1.1$, $a = 0.11$ and $k_1 = 0.1$. This *plot* shows the embedding diagram for a wormhole with metric given by (1). Since our metric is axially symmetric, the *shape* of the throat change for different values of θ

3 Geodesics

We start with the study of the geodesics using the Boyer–Lindquist coordinates. We first are interested in radial geodesics to see whether an observer can penetrate the wormhole or not. Of course the ring singularity apparently does not allow any observer to penetrate the wormhole, at least going by the equator. Nevertheless, the easiest possibility of a traveller for going trough the throat is that the observer travels by the polar geodesic, that means, the line that joints the north pole with the south pole. In the surface of the sphere of radius $l = l_1$ the traveller meets the mouth of the throat, where the tidal forces reduce its magnitude due to the symmetry of the singularity as we saw in the former section. In order to see that, we obtain the polar geodesics in different coordinate systems.

For doing so, let τ be an affine parameter and $u^\mu = (\dot{t}, \dot{r}, \dot{\theta}, \dot{\varphi})$, with $\dot{t} = \frac{dt}{d\tau}$, etc., the vector velocity of an observer, such that the equation $u^\mu u_\mu = -1$ holds. In Boyer–Lindquist coordinates this expression reduces to

$$-1 = -f\left(\dot{t} + \omega\,\dot{\varphi}\right)^2 + \frac{1}{f}\left(\Delta\left(\frac{\dot{l}^2}{\Delta_1} + \dot{\theta}^2\right) + \Delta_1\,\sin^2\theta\,\dot{\varphi}^2\right), \tag{21}$$

For the polar timelike geodesics $\theta = 0$, this implies that $\Delta_1 = \Delta$ and $\omega = 0$, that means that from this geodesic, the observer does not feel the rotation of the wormhole. The Eq. (21) reduces to

$$-1 = -f\dot{t}^2 + \frac{1}{f}\dot{l}^2, \tag{22}$$

In this metric $\frac{\partial}{\partial t}$ is a Killing vector, thus we obtain that

$$f\dot{t} = \sqrt{2}E, \tag{23}$$

E being a constant. Then the geodesic Eq. (22) reduces to

$$\frac{1}{2}\dot{l}^2 + \frac{1}{2}f = E^2 \tag{24}$$

This last expression is a dynamical equation where we can know the motion of the test particle if we know the "potential", in this case given by $1/2\, f$, in terms of the prolate coordinates ($\sigma x = l - l_1$ and $y = \cos\theta$, where $\sigma = l_0^2 - l_1^2$) is

$$\frac{1}{2}\dot{x}^2 + \frac{f}{2\sigma^2} = E^2. \tag{25}$$

In order to have a better interpretation of Eq. (25) we can transform it into

$$\frac{1}{2}\dot{x}^2 + \frac{f}{2\sigma^2} + \frac{1}{2\sigma^2} = E^2 + \frac{1}{2\sigma^2} \tag{26}$$

where now the potential and the total energy are given by

$$V \to \frac{f}{2\sigma^2} + \frac{1}{2\sigma^2} \qquad E^2 \to E^2 + \frac{1}{2\sigma^2} \tag{27}$$

Interpreting the dynamical equation $\frac{\dot{x}^2}{2} + V = E^2$, we see that the potential V is a potential well, the particle can fall into the well and remain there or, if the total energy of the test particle is bigger than E^2, the test particle continues through the throat. It is important to mention that this analysis is only valid for geodesics with $\theta = 0$, since other values of $\theta = const.$ lead us here to non-physical geodesics.

We can integrate Eq. (25) in terms of the variables τ and x. This equation transform into

$$\frac{d\tau}{dx} = \pm\left(2E^2 - \frac{f}{\sigma^2}\right)^{-1/2} \tag{28}$$

which can be reduced into quadratures as

$$\tau = \pm\int\left(2\,E^2 - \frac{\left(a^2 + {k_1}^2\right)e^{\frac{a^2+{k_1}^2}{2k_1\left(\sigma^2x^2+{l_0}^2-{l_1}^2\right)}}}{\sigma^2\left(a^2 + {k_1}^2 e^{\frac{a^2+{k_1}^2}{k_1\left(\sigma^2x^2+{l_0}^2-{l_1}^2\right)}}\right)}\right)^{-1/2} dx \tag{29}$$

This integration can be performed numerically, the results are shown in Fig. 5, where we see that an observer can travel through the throat in finite proper time τ. The plot show different possibilities for the journey. As usual, positive values for x means that the observer is on one region of the space-time and negative values for x means that the observer is on the other one.

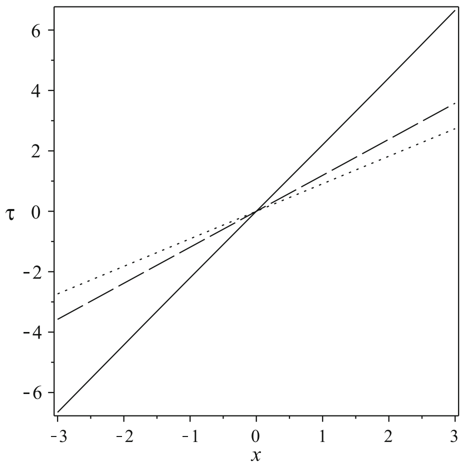


Fig. 5 Time-like polar geodesics in the proper time for the values $E^2 = 0.5$ (*solid line*), $E^2 = 0.75$ (*dashed line*) and $E^2 = 1$ (*doted line*) of the total energy. Here we set $l_1 = 1$, $l_0 = 1.5$, $a = 0.1$, $k_1 = 0.11$ and $\sigma = 1.25$

Now for the general case, with $\theta \neq const.$, in terms of the momenta in Boyer-Lindquist coordinates

$$p_l = \frac{\Delta}{\Delta_1 f}\dot{l}, \qquad p_\theta = \frac{\Delta}{f}\dot{\theta},$$

and the conserved quantities

$$E = f(\dot{t} + \omega\dot{\varphi}), \qquad L + \omega E = \frac{\Delta_1 \sin^2\theta}{f}\dot{\varphi}$$

the Euler-Lagrange equations for the geodesics of (1) are,

$$\begin{aligned}
\dot{p}_l =& \frac{(a^2 + k_1^2)(l - l_1)\cos\theta}{2k_1 f_- \Delta^2}E^2 + \frac{a(l - l_1) f \Delta_-}{\Delta_1 \Delta^2}E(L + \omega E) \\
&+ \frac{(l - l_1) f^2 \Delta_1}{\Delta^2}\left(\frac{1}{f} + \frac{(a^2 - k_1^2)\cos\theta}{2k_1 \Delta f_-}\right)\left(p_l^2 + \frac{p_\theta^2}{\Delta_1}\right) \\
&+ \frac{f^2}{\Delta_1^2 \sin^2\theta}\left(\frac{1}{f} + \frac{(a^2 - k_1^2)(l - l_1)\Delta_1 \cos\theta}{2k_1 \Delta^2 f_-}\right)(L + \omega E)^2 \\
&- \frac{(l - l_1) f}{\Delta}p_l^2, \qquad (30) \\
\dot{p}_\theta =& \frac{(a^2 - k_1^2)\sin\theta \Delta_-}{4k_1 \Delta^2 f_-}E^2 - \frac{a(l - l_1)\sin 2\theta f}{\Delta^2 \sin^2\theta}E(L + \omega E) \\
&+ \frac{f^2}{\Delta_1 \sin\theta}\left(\frac{\sin 2\theta}{2f\sin^3\theta} + \frac{(a^2 - k_1^2)\Delta_-}{4k_1 \Delta^2 f_-}\right)(L + \omega E)^2
\end{aligned}$$

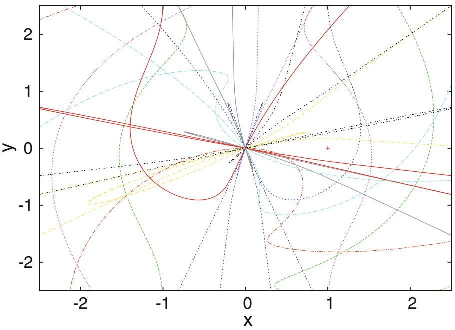


Fig. 6 Paths of geodesics falling into the wormhole for different θ, with the values $l_1 = 1$, $l_0 = 1.1$, $a = 0.1$, $k_1 = 0.11$, $E = 1.0$ and $L = 0.5$. With the singularity (*cross*) at $l = l_1$ and $\theta = \pi/2$

$$+\frac{f^2}{K\Delta}\left(\frac{(a^2-k_1^2)\Delta_-\sin\theta}{4k_1\Delta f_-}-\frac{(l_0^2-l_1^2)\sin 2\theta}{2f}\right)$$
$$\times\left(p_l^2+\frac{p_\theta^2}{\Delta_1}\right). \tag{31}$$

where

$$\Delta_- = (l-l_1)^2-(l_0^2-l_1^2)$$

and

$$f_- = \frac{(a^2-k_1^2)e^{\lambda}}{a^2-k_1^2e^{2\lambda}}.$$

For the general case for a non constant θ the geodesic equations can be solved numerically. In Fig. 6 we can see the paths of null geodesics projected on the xy-plane, ($x = l\sin\theta$, $y = l\cos\theta$). In general, null-geodesics are repelled from the singularity. This follows from the potential $\log g_{tt}$ being repulsive near the singularity (if $a \geq 0.1$ and $k_1 > 0$) and from the fact that the mass of the wormhole is negative and is located at the ring singularity.

4 WEC and NEC Violation

In this section we analyse the Null Energy Condition (NEC) and Weak Energy Condition (WEC) violation of the phantom matter in this space-time. NEC violation stipulates, $T_{\hat{\alpha}\hat{\beta}}\mu^{\hat{\alpha}}\mu^{\hat{\beta}} < 0$, for all null vectors μ, while WEC violation establishes that for all timelike vector u, $T_{\hat{\alpha}\hat{\beta}}u^{\hat{\alpha}}u^{\hat{\beta}} < 0$

To make the analysis tractable, we choose an orthonormal basis $e_{\hat{t}}, e_{\hat{r}}, e_{\hat{\theta}}, e_{\hat{\phi}}$,

$$e_{\hat{t}} = f^{-\frac{1}{2}}e_t,$$
$$e_{\hat{r}} = (f^{-1}K)^{\frac{1}{2}}e_r,$$

$$e_{\hat{\theta}} = f^{-1} K \Delta_1^{-\frac{1}{2}} e_\theta,$$
$$e_{\hat{\varphi}} = \frac{\omega e_t + e_\varphi}{\sqrt{f^{-1} K \Delta_1} \sin(\theta)}. \tag{32}$$

For simplicity we use a radial outgoing null vector, $\mu = e_{\hat{t}} + e_{\hat{r}}$, thus

$$T_{\hat{\alpha}\hat{\beta}} \mu^{\hat{\alpha}} \mu^{\hat{\beta}} = T_{\hat{t}\hat{t}} + T_{\hat{r}\hat{r}} = \frac{1}{2\kappa}(G_{\hat{t}\hat{t}} + G_{\hat{r}\hat{r}}). \tag{33}$$

$$\rho - \tau = T_{\hat{t}\hat{t}} + T_{\hat{r}\hat{r}} = -\frac{(a^2 + k_1^2)^2}{4 k_1^2} \frac{(l - l_1)^2 \Delta_1 \cos^2\theta f}{\Delta^5} \leq 0, \tag{34}$$

Near the singularity ($l = l_1$ and $\theta = \pi/2$) the inequality in (34) holds, while at the throat $l = l_1$ and far from the wormhole we have $\rho - \tau = 0$. Taking $\mu = e_{\hat{t}}$, the energy density is given by

$$\rho = -\frac{(a^2 + k_1^2)^2}{16 k_1^2} \frac{\Delta \sin^2\theta + 4(l - l1)^2 \cos^2\theta}{\Delta^5}, \tag{35}$$

From (35), we get $\rho < 0$ near the wormhole, WEC is violated. From (34) and (35), near the wormhole NEC and WEC are violated.

5 The gravitational tidal forces

From the physical point of view it is important to know whether the wormhole is traversable or not. In this section we study the tidal forces constraints [3,11,12] of metric (1). For simplicity we consider a traveller going radially through the wormhole, beginning at rest in space station in the lower universe and ending at rest in space station in the upper universe. We introduce the orthonormal basis of its own reference frame,

$$e_{\hat{0}} = \gamma e_{\hat{t}} \mp \gamma (v/c) e_{\hat{r}},$$
$$e_{\hat{1}} = \mp \gamma e_{\hat{r}} + \gamma (v/c) e_{\hat{t}}, \tag{36}$$
$$e_{\hat{2}} = e_{\hat{\theta}}, \quad e_{\hat{3}} = e_{\hat{\varphi}}.$$

where $\gamma = [1 - (\frac{v}{c})^2]^{-\frac{1}{2}}$. For simplicity we assume here that the traveller do not feel any acceleration larger than about 1 Earth's gravity. Following the same idea as in the work of Morris and Thorne [3] and [11,12], the radial tidal constraint is given by

$$|R_{\hat{1}\hat{0}\hat{1}\hat{0}}| \leq \frac{g_\oplus}{c^2 \times 2m} \approx (10^8 m)^{-2}, \tag{37}$$

where the height of our traveller is $2m$ (in SI units). We have $|R_{\hat{1}\hat{0}\hat{1}\hat{0}}| = |R_{\hat{t}\hat{l}\hat{t}\hat{l}}|$. Then the first tidal constraint reduces to

$$|R_{\hat{l}\hat{t}\hat{l}\hat{t}}| = \frac{\Delta_1}{4f\sin^2\theta}|2\sin^2\theta\Delta f\frac{\partial^2 f}{\partial l^2} - \Delta_1 f\sin^2\theta\frac{\partial f}{\partial l}\frac{\partial K}{\partial l} - K\sin^2\theta\left(\frac{\partial f}{\partial\theta}\right)^2 + f\sin^2\theta\frac{\partial f}{\partial\theta}\frac{\partial K}{\partial\theta} + f^4K\left(\frac{\partial\omega}{\partial l}\right)^2| \le (10^8m)^{-2}. \tag{38}$$

While the lateral constraints are reduced to the study of $|R_{\hat{2}\hat{0}\hat{2}\hat{0}}| \le (10^8m)^{-2}$, and $|R_{\hat{3}\hat{0}\hat{3}\hat{0}}| \le (10^8m)^{-2}$. Since our metric is axially symmetric, we have that

$$|R_{\hat{2}\hat{0}\hat{2}\hat{0}}| = \gamma^2|R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}}| + \gamma^2(v^2/c)|R_{\hat{\theta}\hat{r}\hat{\theta}\hat{r}}|. \tag{39}$$

Now we assume that, for example, the traveller is at rest at the throat [3], this implies $v \to 0$ and $\gamma \to 1$. Then $|R_{\hat{2}\hat{0}\hat{2}\hat{0}}| = |R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}}|$. Thus, the second tidal constrain is given by

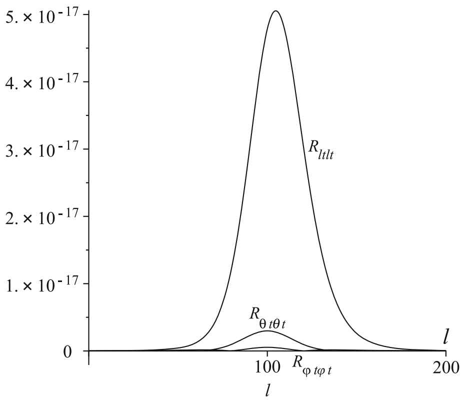
$$|R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}}| = \frac{1}{4\Delta^2 f\sin^2\theta}| - \Delta\Delta_1\sin^2\theta\left(\frac{\partial f}{\partial l}\right)^2 + f\Delta_1\sin^2\theta\frac{\partial f}{\partial l}\frac{\partial\Delta}{\partial l} + 2f\Delta\sin^2\theta\frac{\partial^2 f}{\partial\theta^2} - f\sin^2\theta\frac{\partial f}{\partial\theta}\frac{\partial\Delta}{\partial\theta} + f^4K\left(\frac{\partial\omega}{\partial\theta}\right)^2| \le (10^8m)^{-2} \tag{40}$$

Finally our last constraint can be written as

$$|R_{\hat{\phi}\hat{t}\hat{\phi}\hat{t}}| = \frac{1}{4f\Delta\sin^2\theta}| - \Delta_1\sin^2\theta\left(\frac{\partial f}{\partial l}\right)^2 + f\sin^2\theta\frac{\partial f}{\partial l}\frac{d\Delta_1}{dl} + f^4\left(\frac{\partial\omega}{\partial l}\right)^2 + 2f\sin\theta\cos\theta\frac{\partial f}{\partial\theta} - \sin^2\theta\left(\frac{\partial f}{\partial\theta}\right)^2 + \frac{f^4}{\Delta_1}\left(\frac{\partial\omega}{\partial\theta}\right)^2| \le (10^8m)^{-2}. \tag{41}$$

We can also force the traveller to approach the throat using the geodesics with $\theta = 0$, in that case we have that $\partial w/\partial\theta = 0$ and $\partial f/\partial\theta = 0$. We can see that through this geodesic the Riemann tensors simplifies. From Fig. 7 we can see that the tidal

Fig. 7 The tidal forces for $\theta = 0$ using $l_1 = 100$, $l_0 = 110$, $a = k_1 = 10^{-4}$. All components reach their peak value at the throat of the wormhole ($l = l_1$)



forces, for a wormhole with $l_1 \approx 5 \times 10^{-3} M_\odot$, can be smaller than $(10^8 m)^{-2}$ for certain values of a and k_1, therefore the wormhole could be traversable if the traveller approaches the throat form the north pole. Let us now analyses the tidal forces at the equator. Setting $\theta = \pi/2$, we get

$$|R_{\hat{l}\hat{t}\hat{l}\hat{t}}| = 2|R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}}| = |R_{\hat{\phi}\hat{t}\hat{\phi}\hat{t}}| = \frac{1}{16}\left|\frac{a^4 - 6a^2k_1^2 + k_1^4}{k_1^2(l-l_1)^6}\right|. \tag{42}$$

from (42) we see that near the throat the tidal forces go to infinity, as expected (the singularity lies at $l = l_1$ and $\theta = \pi/2$), hence the wormhole is not traversable if a traveller approaches the throat from the equator. The wormhole is traversable for $\theta = 0$ but is non-traversable for $\theta = \pi/2$. The wormhole is traversable since there is at least one θ which makes the tidal forces small enough to allow a traveller to go through the wormholes throat.

6 Conclusions

We have analysed metric (1) and shown that this metric represents a wormhole, which throat is shown in Fig. 4. This metric contains a ring singularity very similar to the Kerr solution, all the invariants of the metric are regular everywhere except on this ring. The throat is found outside of the ring singularity, the mouth of the throat is on a sphere of radius $l = l_1$, around the wormhole. We have shown that polar geodesics, that means, geodesics going through the polar line are regular, an observer can go through the throat if the observer trajectory remains on the polar geodesics and contains an energy bigger than $1/2f$, for any values of the free parameters. On this trajectory the tidal forces are very small, therefore this wormhole could be traversable. For any other angle the mouth of the wormhole lies on the sphere, but close to the equator, the effect of the wormhole is to repel the test particles. On the equator, the repulsion is infinity and everything bounces from the singularity, even the light is repelled by the wormhole. Thus, the sphere $l = l_1$ has an effect contrary to the horizon of a black hole, namely, an observer can reach the sphere, goes thought the throat, but this sphere does not allow the traveller to reach the singularity. Of course, the traveller can come back to its original world without much troubles.

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