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Space-time curvature signatures in Bose-Einstein condensates

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Abstract. We derive a generalized Gross-Pitaevski (GP) equation for a Bose Einstein Condensate (BEC) immersed in a weak gravitational field starting from the covariant Complex Klein-Gordon field in a curved space-time. We compare it with the traditional GP equation where the gravitational field is added by hand as an external potential. We show that there is a small difference of order gz/c^2 between them that could be measured in the future using Bose-Einstein Condensates. This represents the next order correction to the Newtonian gravity in a curved space-time.

Among the four forces of nature, the gravitational force is the hardest one to study. It is orders of magnitude weaker than the other three forces and requires astronomical masses to see corrections beyond the basic Newtonian formula. The lab scale masses that we can manipulate introduce gravitational forces that are very hard to measure accurately. This is reflected in the poor precision that we currently have on the determination of the Newtonian constant of gravitation ($G = 6.67384(80) \times 10^{-11}$ kg^{-1} m^3 s^{-2}) [1].

Most of the experiments that measure the gravitational force use basically the same method pioneered by Cavendish [1]. Atomic interferometers have been recently introduced as an alternative way to implement sensitive gravimeters [2]. Here, cold atoms are launched vertically and subjected to a sequence of laser pulses that split, recombine and interfere the atoms. Using a Bose-Einstein Condensate (BEC) reduces the ballistic expansion of the atomic cloud and allows for large fall times, thus increasing the precision of the gravimeters.

Sensitive atomic gravimeters are useful in the detection of gravitational waves [3], tests to Einstein's general relativity [4], study of short range gravitational forces [5] and coherent evolution of delocalized quantum objects [6]. Measurements of g (the gravitational acceleration of the earth) typically reach 9 digits of precision. This has been recently improved by two orders of magnitude by launching a BEC in free fall in a 10 m tower [7]. Planned improvements to this include using a 146 m tower [8], sending the atoms in parabolic flights [9] or putting the atoms in space [10].

BECs are usually described by a Gross-Pitaevskii (GP) equation, where an external field is added in order to confine the system or to include the Earth's gravitational acceleration ([11]). Nevertheless, the GP equation is non-relativistic and non-covariant; it is not invariant under Lorenz transformations unlike the electromagnetic field. A GP equation obtained this way could only show Newtonian effects of gravity. A more complete description is obtained by treating the gravitational field not as a force, but as the curvature of space-time as in Einstein's theory of relativity. In reference [12] it was shown that the Klein-Gordon equation in a curved space takes the GP form after a simple transformation and includes extra terms that are interpreted as the gravitational field and a finite temperature contribution. The gravitational components include terms beyond the Newtonian gravity.

In this work we highlight these differences and propose that these terms could become the focus of future experimental measurements. The derivation gives a generalized Gross-Pitaevsky equation for the description of a BEC in a gravitational field. The first order corrections beyond Newtonian gravity is given by gz/c^2 and we discuss the potential for observation of such effect in atomic interferometry.

1 Klein-Gordon in a weak gravitational field

References [13,14] derive the relativistic theory of a BEC in a quantum field theory description (see [15,16] for similar approaches). Starting from the Klein-Gordon equation in a flat space-time, a generalized GP equation is obtained for relativistic and finite temperature fields. The generalization for an expanding universe is given in reference [12]. Here we apply the same approach on a curved space for a BEC living in a weak gravitational field. The Klein

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Gordon equation for a complex scalar field is given by

$$\Box\Phi - \frac{dV}{d\Phi^*} = \left(\nabla^\mu + i\frac{e}{\hbar c}A^\mu\right)\left(\nabla_\mu + i\frac{e}{\hbar c}A_\mu\right)\Phi - \frac{dV}{d\Phi^*} = \frac{2m^2}{\hbar^2}U_{ext}\Phi, \tag{1}$$

where Φ is the complex scalar field and A_μ is the corresponding electromagnetic four vector. The derivation is similar to that in reference [17] except that here we consider minimal coupling only. Mathematical consistency requires the coupling to an electromagnetic type field, but we set $e = 0$ from now on to apply the calculations for a BEC of neutral atoms. In this work we use the Mexican hat scalar field potential given by

$$V = \frac{m^2c^2}{\hbar^2}\Phi\Phi^* + \frac{\lambda}{4\hbar^2}|\Phi|^4. \tag{2}$$

We include an external interaction U_{ext} that can represent the potential that confines the condensate or the gravitational field if we add it by hand. The metric of a weak gravitational field is [18]

$$ds^2 = -(1+2\psi)dt^2 + (1-2\phi)g_{ij}dx^i dx^j, \tag{3}$$

where ψ, ϕ are the gravitational potentials and g_{ij} is the 3-dimensional flat-space metric. The gravitational potentials are the dimensionless expansion parameters, and we neglect higher order terms since they are too small to be observed in terrestrial experiments (we show below that $\phi \approx 10^{-16}$). Here we consider scalar fluctuations ϕ and ψ only, the vector and tensor fluctuations are one and two orders of magnitude smaller than the scalar one [18]. The Klein-Gordon equation with this metric reads

$$\Box_{NEW}\Phi - \frac{dV}{d\Phi^*} = \frac{2m^2}{\hbar^2}U_{ext}\Phi. \tag{4}$$

The metric modifies the D'Alambertian as

$$\Box_{NEW} = -(1-2\psi)\frac{\partial^2}{c^2\partial t^2} + (1+2\phi)\nabla^2 + [\dot\psi + 3\dot\phi]\frac{\partial}{c^2\partial t} + \nabla(\psi - \phi)\nabla. \tag{5}$$

The gravitational potential appears naturally from the metric but there is also a correction to the kinetic energy term, something that does not happen in the traditional treatment that adds the gravitational potential by hand.

To write down the Klein-Gordon equation (Eq. (4)) in its GP form we apply the transformation $\Phi = \Psi e^{-imc^2t/\hbar}$ (see also [19])

$$i(1-2\phi)\hbar\dot\Psi = -\frac{\hbar^2}{2m}\Box_{NEW}\Psi + \frac{\lambda}{2m}|\Psi|^2\Psi + mc^2(\phi + U_{ext})\Psi. \tag{6}$$

This last equation is again the Klein-Gordon equation (Eq. (1)), but written in terms of the function Ψ.

If we remove the gravitational fields ϕ and ψ in equation (6), we recover the Gross Pitaevskii equation in the non-relativistic limit. Therefore we interpret equation (6) as the generalization of the GP equation for a relativistic Bose particle in a weak gravitational field.

In the general relativity theory of Einstein with a weak gravitational field we have $\phi = \psi$. In some alternative theories of gravity this identity does not follow [20]. For simplicity, we assume a static gravitational field ($\partial_t\phi = \dot\phi = 0$). This approximates the gravitational field of the earth ignoring variations like those introduced by tides for example. We do not consider here contributions from the earth's rotation that can be compensated in asymmetric atomic interferometers by tilting the retro-reflection mirror [7]. The result after taking the Newtonian limit is

$$i(1-2\phi)\hbar\dot\Psi = -(1+2\phi)\frac{\hbar^2}{2m}\nabla^2\Psi + \frac{\lambda}{2m}|\Psi|^2\Psi + mc^2\phi\Psi + mc^2U_{ext}\Psi. \tag{7}$$

We interpret equation (7) as the generalization of the GP equation in a gravitational field.

We compare here the differences between the traditional approach to Newtonian gravity ($\phi = 0$) to the curved space-time derivation used in this work ($U_{ext} = 0$). Indeed, taking $\phi = 0$, we recover the flat space result that corresponds to the usual GP equation

$$i\hbar\dot\Psi = -\frac{\hbar^2}{2m}\nabla^2\Psi + \frac{\lambda}{2m}|\Psi|^2\Psi + mc^2U_{ext}\Psi. \tag{8}$$

Taking $U_{ext} = gz/c^2$ we obtain the gravitational potential of the Earth (mgz). Here the gravitational interaction is added as an external potential and represents the Newtonian version of a BEC in a gravitational field.

In the curved space description there is no need to add the gravitational interaction externally ($U_{ext} = 0$). Setting $\phi = gz/c^2$ (working locally at a position far away from the center of the source) gives the correct gravitational potential, but there is a correction since ϕ appears also in the kinetic and $\dot\Psi$ term of equation (7). This last correction corresponds to the gravitational redshift that was first demonstrated experimentally by Pound and Rebka [21] and more recently by directly lifting an atomic clock [22]. We determine the magnitude of the correction in atomic interferometry. Here an atom (or the BEC wave function) is divided in two, and the two components evolve in free fall for some time before recombining them. The separation is usually smaller than 1 cm [7], but it is conceivable to achieve separations up to 1 m in the near future by combining long free fall times with large momentum transfer. A z =1 m separation gives a $\phi \simeq 10^{-16}$, that is small compared to the 1 in the $\dot\Psi$ term of equation (7).

The combination gz/c^2 is the perturbative parameter. The observation of the correction requires doing atomic interferometry with 16 digits of precision. The current record is at 11 digits [7], but the projected sensitivity of the space project is at 15 digits of precision [9]. Having the atoms in space is not the best strategy to see the correction since that eliminates g. Instead, the present work suggests moving towards atomic interferometry in strong gravitational fields.

It is possible to reach higher values of g by taking advantage of the equivalence principle. Instead of the gravitational field of the Earth one could accelerate the complete experimental setup. This is clearly complicated, but in the case of acceleration introduced by Bloch oscillations things might be simpler. Here it is only necessary to sweep the phase of the two counter propagating lasers beams to produce an accelerated lattice [23]. There is a price to pay in the precision of the measurement due the reduced measurement time since, for example, a linear acceleration of $100g$ gives already a displacement of 5 m in 0.1 s.

We apply a Madelung transformation $\Psi = \sqrt{n}e^{iS}$ to equation (7), with n representing the density number of particles and S the velocity super potential

$$\mathbf{v} = \frac{\hbar}{m}\nabla S. \tag{9}$$

The real and imaginary parts of equation (7) in terms of the n and S variables are

$$(1-2\phi)\dot{n} + (1+2\phi)\nabla\cdot(n\mathbf{v}) - (1-2\phi)\dot{j} = 0 \tag{10}$$

$$(1-2\phi)\frac{v}{c} + \frac{1}{2}(1+2\phi)\frac{\mathbf{v}^2}{c^2} + \phi + \frac{\lambda}{2m^2}n - \frac{\hbar^2}{2m^2c^2}\frac{\Box_{NEW}\sqrt{n}}{\sqrt{n}} + \frac{1}{2}(1-2\phi)\frac{v^2}{c^2} = 0 \tag{11}$$

respectively, with the flux $\mathbf{j} = n\mathbf{v}$ and

$$j = n\frac{\hbar}{mc^2}\dot{S} = n\frac{v}{c}. \tag{12}$$

We interpret equation (10) as the generalized continuity equation in a gravitational field. The equation differs from the one derived using pure fluid mechanics in the factors in front of the density and velocity terms. Equations (10) and (11) are the Klein-Gordon equations in a weak gravitational field, in other words, they are the Einstein-Klein-Gordon equations written in the variables n and S. We calculate the gradient of equation (11) to obtain the corresponding momentum equation

$$\begin{aligned}(1-2\phi)n\dot{\mathbf{v}} + (1+2\phi)n(\mathbf{v}\cdot\nabla)\mathbf{v} = n\mathbf{F}_{ext} \\ + n\left(1 + 2\frac{v}{c} + 2\frac{v^2}{c^2}\right)\mathbf{F}_\phi \\ - \nabla p + n\mathbf{F}_Q + \nabla\sigma,\end{aligned} \tag{13}$$

where p is the pressure with a state equation $p = (\lambda/m^2)n^2$, $\mathbf{F}_Q = -\nabla((\hbar^2/2m^2)\nabla^2\sqrt{n}/\sqrt{n})$ is the quantum force and σ is the viscosity (see [24] for details of the non-gravitational case). As expected, the external force (by unit of mass) is defined by $\mathbf{F}_{ext} = -\nabla U_{ext}$ and the gravitational force by $\mathbf{F}_\phi = -c^2\nabla\phi$. Equations (10) and (13) correspond to the Klein-Gordon equation (Eq. (1)) written in terms of n and $\mathbf{v}$.

The difference between the two procedures to include the gravitational potential becomes evident from equation (13). If we consider the gravitational field as an external force (non-covariant case), we go back to a flat space-time with $\phi = 0$ and $\mathbf{F}_\phi = 0$. In this case the hydrodynamic equation corresponds to the traditional one. If instead we use the covariant form of the equations we can set $\mathbf{F}_{ext} = 0$ and include the gravitational force in the $\mathbf{F}_\phi$ term. The coefficient in front of both terms differs by $2v/c + 2v^2/c^2 \sim 2v/c$. The modification is very small and has its origin in the gravitational redshift. To connect with the previous treatment, the wave function phase S evolves at a different rate at different heights z because of the gravitational potential, so that $S = mgzt/\hbar$. Using equation (12) we have $2v/c = 2gz/c^2$ which is the same parameter we obtained before.

In summary, we present a field theoretical approach to describe a BEC in a curved space. Rather than adding the gravitational interaction as an external potential, we obtain it directly as a consequence of the curved space. The derivation results in a generalization of the Gross-Pitaevski equation in a gravitational field. From the equation we identify an expansion parameter gz/c^2 and we focus on the first order corrections beyond Newtonian gravity. We quantify the magnitude of the corrections to atomic interferometry signals that sets the precision of the measurements required to see them. This makes BECs in strong gravitational (or accelerated) fields a very interesting candidate to study corrections beyond Newtonian gravity.

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