Exact rotating magnetic traversable wormhole satisfying the energy conditions

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In this work we wonder if there is a way to generate a wormhole (WH) in nature using "normal" matter. In order to give a first answer to this question, we study a massless scalar field coupled to an electromagnetic one (dilatonic field) with an arbitrary coupling constant as source of curvature. Using this source, we obtain an exact solution of the Einstein equations, which represents a magnetized rotating WH. The space-time is everywhere regular except for a naked ring singularity, which we show to be causally disconnected from the rest of the Universe in the case of a slowly rotating WH. The throat of the WH lies on the disc bounded by the ring singularity and, surprisingly enough, it can be kept open without requiring exotic matter, which means, satisfying all the energy conditions. After analyzing the geodesic motion and the tidal forces we find that a test particle can go through the WH without trouble.

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I. INTRODUCTION

In 1916, Flamm suggested that our Universe might not be simply connected [1]. This idea opened the possibility of the existence of tunnels connecting different regions in the Universe, or even completely different universes. In 1935, Einstein and Rosen rediscovered this solution trying to give a field representation of particles [2]; this concept was furthered by Ellis [3], who modeled particles as bridges between two regions of space-time. Many years later, Morris and Thorne considered such solutions as means of interstellar travel [4]. Unfortunately, these wormholes (WH) need to violate the energy conditions. Today this type of matter is called exotic (see [5] for a detailed review on this subject).

In [6,7] it was shown that the violation of the null energy condition (NEC) was a generic feature for regular traversable WHs. This was done assuming that the throat is a compact two-dimensional surface with minimum area. However, if the throat is no longer a compact object (i.e., a starlike structure) one might deal with cylindrical WHs, which from afar appear as cosmic strings and thus, would avoid the presence of exotic matter [8–10].

On the other side, numerical simulations seem to reveal that static WHs are unstable [11]. To overcome this problem, it was conjectured in [12] that the rotation of the WH could stabilize a ghost star. The idea is that a rotating WH would have more possibilities to be stable than the general static spherically symmetric WHs. Some rotating solutions were studied in the past, as an approximation [13,14] or as an exact solution of the Einstein equations [12,15,16]. However, all of them violate some energy condition.

Nevertheless, we wonder if it is possible to generate WHs where the source is some kind of matter that can be found in nature. In this work we look for WHs made of normal matter that are traversable, that is, WHs made of matter that satisfy the energy conditions and where a test particle can go from one side of the throat to the other in a finite time without facing large tidal forces. Also, following the conjecture that rotation can stabilize the WH, we search for rotating WHs.

In order to answer this question, at least partially, we look for scalar fields that could be formed by particles coupled with the electromagnetic field. Certainly, these kinds of particles exist and are common in nature, but all of them are massive. The problem is that so far it has not been possible to get exact solutions of the Einstein equation with massive scalar fields. However, if the scalar field is massless, standard techniques can be used to find exact space-times from these sources. Doing so loses some important features of the properties of matter, but gains precision to determine the form of the space-time itself, which is the most important aspect for this kind of analysis. Thus, we start from the Lagrangian,

$$\hat{\mathcal{L}} = -R + 2\epsilon\nabla_\mu\Phi\nabla^\mu\Phi + e^{-2\alpha\Phi}F_{\mu\nu}F^{\mu\nu} + V(\Phi), \quad (1)$$

where R is the Ricci scalar, $F_{\mu\nu}$ is the electromagnetic field tensor, Φ is the scalar field of a spin zero (composed)

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particle and $V(\Phi)$ the scalar field potential. We separate the dilatonic field from the ghost field using $\varepsilon = +1$ for the dilatonic and $\varepsilon = -1$ for the ghost field. As a first approximation we set $V = 0$. As we see, we can solve the Einstein equations exactly, giving us a space-time that we can study with all precision. The Einstein-Maxwell-Dilaton field equations from the Lagrangian (1) are

$$R_{\mu\nu} = 2\varepsilon\nabla_\mu\Phi\nabla_\nu\Phi + 2e^{-2\alpha\Phi}\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\delta\gamma}F^{\delta\gamma}\right), \tag{2}$$

with coupling constant α. In this work we report an asymptotically flat rotating magnetized solution of the Einstein equations, which is obtained by using the ansatz proposed in [17]. The paper is organized as follows: in Sec. II we describe the generation process for the metric of the WH and present its parameters along with the general form of its scalar invariants. It is also shown that it is asymptotically flat. The local and global geometry of the space-time is analyzed in Sec. III. In Sec. IV the energy conditions of the matter source of the WH are studied. We prove here that null and weak energy conditions are satisfied for the case of the WH with dilatonic field. The tidal forces that a traveler could experience crossing through the throat of the WH are studied in Sec. V, where we show that it is possible to do so while moving in the polar plane. We then examine, in Sec. VI, geodesic motion in the equatorial plane of the space-time. Finally, in Sec. VII, we focus on the case of a slowly rotating WH and find that the ring singularity is inaccessible for an observer traveling in geodesics.

II. THE LINE ELEMENT

Exact solutions for the Einstein-Maxwell phantom field equations (2) can be generated by using the method described in [17]. This method is based on a generalized harmonic map ansatz in the potential space defined by the so-called superpotentials f, ϵ, χ, ψ and κ. These quantities are related to physical potentials such as gravitational, electromagnetic and rotational. We focus on the second class of solutions presented in [17], for which

$$\begin{aligned} f &= f_0, \\ \kappa &= \kappa_0 e^{c\lambda}, \\ \psi &= -e^{-c\lambda}\sqrt{f_0}/\kappa_0 + \psi_0, \\ \chi &= -\sqrt{f_0}\kappa_0 e^{c\lambda} + \chi_0, \\ \epsilon &= -\sqrt{f_0}e^{c\lambda}\psi_0\kappa_0 + \epsilon_0, \end{aligned} \tag{3}$$

where c, f_0, ϵ_0, χ_0, ψ_0 and κ_0 are integration constants. Also, λ is the harmonic map used to generate the solution and for this particular class is proportional to the scalar field Φ; for this paper we use $\Phi = -c\lambda/\alpha$. The harmonic map needs to satisfy the Laplace equation given in Boyer-Lindquist coordinates by

$$((l^2 - 2l_1 l + l_0^2)\lambda_{,l})_{,l} + \frac{1}{\sin(\theta)}(\sin(\theta)\lambda_{,\theta})_{,\theta} = 0; \tag{4}$$

here l is the radial coordinate, θ the polar angle, and l_0 as well as l_1 are constants whose units are those of length. A comma is being used to denote a partial derivative. We have assumed that λ is a function of l and θ only. The electromagnetic vector potential has the form $A_\mu = [A_0, 0, 0, A_3]$, while the stationary and axially symmetric space-time metric reads

$$\begin{aligned} ds^2 &= -f(dt + \Omega d\varphi)^2 \\ &\quad + \frac{1}{f}\left[\Delta e^K\left(\frac{1}{\Delta_1}dl^2 + d\theta^2\right) + \Delta_1 \sin^2(\theta)d\varphi^2\right], \end{aligned} \tag{5}$$

with

$$\begin{aligned} \Delta &= (l - l_1)^2 + (l_0 - l_1^2)\cos^2\theta, \\ \Delta_1 &= l^2 - 2l_1 l + l_0^2. \end{aligned} \tag{6}$$

The superpotentials expressed in (3) define the components of the electromagnetic four-potential $A_0 = \psi/2$ and A_3, as well as the parameters of the metric Ω and K, through the following set of partial differential equations:

$$\begin{aligned} \Omega_{,l} &= \frac{1}{f^2}(\epsilon_{,\lambda} - \psi\chi_{,\lambda})\sin\theta\lambda_{,\theta}, \\ \Omega_{,\theta} &= -\frac{1}{f^2}(\epsilon_{,\lambda} - \psi\chi_{,\lambda})\Delta_1\sin\theta\lambda_{,l}, \\ 2A_{3,l} &= -\frac{1}{f\kappa^2}\chi_{,\lambda}\lambda_{,\theta}\sin\theta + \Omega\psi_{,\lambda}\lambda_{,l}, \\ 2A_{3,\theta} &= \frac{1}{f\kappa^2}\chi_{,\lambda}\lambda_{,l}\Delta_1\sin\theta + \Omega\psi_{,\lambda}\lambda_{,\theta}, \end{aligned} \tag{7}$$

along with

$$\begin{aligned} K_{,l} &= \frac{k_0\sin\theta}{\Delta}(2\Delta_1\lambda_{,\theta}\lambda_{,l}\cos\theta \\ &\quad + (l - l_1)(\Delta_1\lambda_{,l}^2 - \lambda_{,\theta}^2)\sin\theta), \\ K_{,\theta} &= \frac{k_0\Delta_1\sin\theta}{\Delta}(-(\Delta_1\lambda_{,l}^2 - \lambda_{,\theta}^2)\cos\theta \\ &\quad + 2(l - l_1)\lambda_{,\theta}\lambda_{,r}\sin\theta). \end{aligned} \tag{8}$$

Among the various options for the harmonic map λ (see [17] for details), we choose to use $\lambda = \lambda_0\cos\theta/\Delta$, which represents a magnetic dipole. With this choice, the solutions for Eqs. (7) are $\Omega = -c\lambda_0(l - l_1)\sin^2\theta/f_0\Delta$ and $A_3 = -\sqrt{f_0}\Omega/2\kappa$ up to an integration constant that we set to 0. At this point, for simplicity, it is useful to define $\lambda_0 = -a$ as the parameter of scalar charge with units of angular momentum and to set the values of the rest of the integration constants to unity.

On the other hand, Eqs. (8) can be more easily solved introducing oblate spheroidal coordinates $Lx = l - l_1$ and $y = \cos\theta$, where $L^2 = l_0^2 - l_1^2$ is constant. In this coordinate system $x \in \mathbb{R}$ and $|y| \leq 1$. With this coordinate change (8) transforms into

$$K_{,x} = \frac{k_0\sqrt{1-y^2}}{\Delta}\Big[-2y\sqrt{1-y^2}\Delta_1\lambda_{,x}\lambda_{,y} + x\sqrt{1-y^2}(\Delta_1\lambda_{,x}^2 - (1-y^2)\lambda_{,y}^2)\Big]$$
$$K_{,y} = \frac{k_0\Delta_1}{\Delta}[y(\Delta_1\lambda_{,x}^2/L^2 - (1-y^2)\lambda_{,y}^2) + 2x(1-y^2)\lambda_{,x}\lambda_{,y}]. \quad (9)$$

Notice that now $\Delta = L^2(x^2+y^2)$ and $\Delta_1 = L^2(x^2+1)$. After integrating Eqs. (9) one finds that

$$K = \frac{k}{L^4}\frac{(1-y^2)(8x^2y^2(x^2+1) - (1-y^2)(x^2+y^2)^2)}{(x^2+y^2)^4}, \quad (10)$$

where we have absorbed all integration constants into a newly defined one k. With this, we have found all of the components for metric (5).

The space-time that we have generated represents a magnetized rotating WH without gravitational potential (recall that $f = 1$). Instead, curvature arises from the presence of a scalar field and an electromagnetic potential coupled with a dilation field. The magnetic field associated with the vector potential A_μ represents a magnetic dipole.

For the rest of this paper we choose to adopt oblate spheroidal coordinates since they enable us to express our results in the most compact manner (with the disadvantage that these coordinates are not physically intuitive). So, we can finally present explicitly the line element of this rotating magnetized WH as

$$ds^2 = L^2\left[(x^2+y^2)e^K\left(\frac{dx^2}{x^2+1} + \frac{dy^2}{1-y^2}\right) + (x^2+1)(1-y^2)d\varphi^2\right] - \left(dt + \frac{a}{L}\frac{x(1-y^2)}{x^2+y^2}d\varphi\right)^2. \quad (11)$$

The characteristic parameters of this space-time are the length $L = \sqrt{l_0^2 - l_1^2}$ and the scalar charge a with units of angular momentum. We consider $l_0 > l_1$ such that $L \in \mathbb{R}$ is related to the size of the WH's throat. Additionally, the scalar field Φ and the electromagnetic vector potential A_μ are respectively given by

$$\Phi = \frac{ay}{\alpha L^2(x^2+y^2)}, \quad (12)$$

$$A_\mu = -\frac{e^{\alpha\Phi}}{2}\left[1 - e^{-\alpha\Phi}, 0, 0, \frac{ax(1-y^2)}{L(x^2+y^2)}\right]. \quad (13)$$

A constraint for α, the scalar charge a and the free constant k can be found by first computing the components of the Ricci tensor $R_{\mu\nu}$ with its definition in terms of the connection coefficients $\Gamma^\rho_{\mu\nu}$ of metric (11); then, by calculating the right-hand side of the Einstein equations (2) using the expressions for the scalar field (12) and electromagnetic potential (13). Comparing both results one realizes that these fields, with metric (11), constitute a solution of the Einstein equations only if the following constraint holds:

$$\alpha^2(a^2 - 8k) - 4\varepsilon a^2 = 0. \quad (14)$$

Interesting special cases for the coupling constant are $\alpha^2 = 1$, which represents a low-energy string theory, and $\alpha^2 = 3$, in which the Lagrangian (1) reduces to that of a five-dimensional Kaluza-Klein theory. In Table I we show the values of the constant k in the previous cases for the dilatonic and ghost fields. It also contains the value of α^2 for which $k = 0$ (only the dilatonic field is possible).

All of the invariant quantities of the line element (11), e.g., the Ricci scalar, the quadratic Riemann tensor $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$, etc. are of the form

$$\text{Invariants} = \frac{F(x,y)}{(x^2+y^2)^\beta}e^{-K}, \quad (15)$$

where β is a positive integer and $F(x,y)$ is a polynomial of degree less than the degree of $(x^2+y^2)^\beta$ and of less order than e^{-K}. For instance, for the Ricci scalar $\beta = 4$ and $F(x,y) = (a^2 - 8k)(y^2(1-y^2) + x^2(1+3y^2))$. From (15) we see that the space-time (11) has an anisotropic naked ring singularity of radius L at $x = y = 0$. Another important aspect to notice from the general form of the invariants is that $y = 1$ is nothing but a coordinate singularity in the line element.

TABLE I. Real values of k for some cases of α^2 for both dilatonic and ghost scalar fields.

	k	
α^2	Dilatonic field ($\varepsilon = 1$)	Ghost field ($\varepsilon = -1$)
1	$-3a^2/8$	$5a^2/8$
3	$-a^2/24$	$7a^2/24$
4	0	…

Moreover, (11) is asymptotically flat since

$$\lim_{x\to\infty} e^K \to 1 \quad \text{and} \quad \lim_{x\to\infty} \frac{a}{L}\frac{x(1-y^2)}{x^2+y^2} \to 0.$$

The throat of the WH lies on the disc $x = 0$ bounded in the equatorial plane ($y = 0$) by the ring singularity. The throat connects two three-dimensional spaces, one with $x > 0$ and another with $x < 0$. Since there are no discontinuities in the extrinsic curvature on the disc, it is possible to cross the surface $x = 0$, i.e., to travel through the WH. Crossing this bounded surface represents leaving one universe and entering another. WHs such as this are often referred to as ring WHs [18].

We finalize this section by mentioning that in the static case, that is, $a \to 0$, the line element (11) reduces to that of a flat space-time.

III. GEOMETRY OF THE WORMHOLE

In this section we show the geometrical features of this space-time. We focus in the local geometry of the WH's throat and the global properties of the metric.

To grasp the local geometry of the throat it is commonly used to embed hypersurfaces with $y = y_0$ and t constant in three-dimensional Euclidean space. With this, the line element (11) reduces to

$$ds^2 = \frac{L^2(x^2+y_0^2)e^{K_0}}{x^2+1}dx^2 + \left[L^2(x^2+1)(1-y_0^2) - \frac{a^2}{L^2}\frac{x^2(1-y_0^2)^2}{(x^2+y^2)^2}\right]d\varphi^2, \quad (16)$$

where $K_0 = K(x, y_0)$. We embed the resulting two-manifold by considering now the Euclidean metric in cylindrical coordinates, $ds^2 = d\rho^2 + dz^2 + \rho^2 d\varphi^2$, and assuming $\rho = \rho(x)$ and $z = z(x)$, which yields

$$ds^2 = \left[\left(\frac{d\rho}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2\right]dx^2 + \rho^2(x)d\varphi^2. \quad (17)$$

Comparing (16) and (17) we obtain

$$\rho^2(x) = L^2(x^2+1)(1-y_0^2) - \frac{a^2}{L^2}\frac{x^2(1-y_0^2)^2}{(x^2+y^2)^2}, \quad (18)$$

$$\left(\frac{d\rho}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2 = \frac{L^2(x^2+y_0^2)e^{K_0}}{x^2+1}. \quad (19)$$

Equation (19) can be solved numerically once the derivative of the known function $\rho(x)$ is inserted. In Fig. 1 we show solutions of this equation for different values of y_0 with the initial condition $z(0) = 0$, such that the throat is located at $z = 0$.

It can be observed from Fig. 1 that, as expected, the embedding profile has a minimum radius at the throat of the WH ($x = z = 0$). This minimum radius corresponds also to

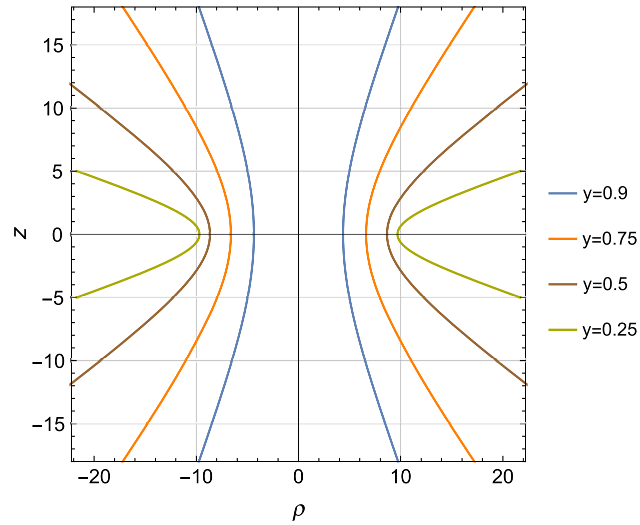

FIG. 1. Geometry of the throat for different values of y_0 with $L = 10$, $a = 0.1$ and $k = -3a^2/8$. Here, z and ρ are parametrized by the coordinate x. The two-dimensional manifold embedded in three-dimensional Euclidean space is obtained by rotating the curves $z(\rho)$ about the z axis.

a surface of minimum area due to the throat being a disc; i.e., there is axial symmetry. Note also that the throat gets wider, reaching a maximum value of $L = 10$, as y decreases. It should be mentioned that, although we only present a set of curves for the WH with dilatonic field and $\alpha^2 = 1$ ($k = -3a^2/8$), the choice of k does not alter significantly the geometry of the throat as long as $a < L^2$. The reason for this is readily explained. In Sec. VII we show that a slowly rotating WH is characterized by $a/L^2 \ll 1$. Applying this condition to Eqs. (18) and (19) leads to an analytical solution for $z(x)$, namely, $z(x) = \pm L y_0 x$, while $\rho^2(x) = L^2(x^2+1)(1-y_0^2)$ to first order in a/L^2, both of which are independent of k. These expressions yield very similar curves as the ones contained in Fig. 1.

As for the global geometrical properties of this space-time, we employ a Carter-Penrose diagram to describe them [19]. For this purpose, we consider a slice of metric (11) with fixed y and φ. Thus, we have

$$ds^2 = -dt^2 + \frac{L^2(x^2+y_0^2)e^{K_0}}{x^2+1}dx^2, \quad (20)$$

which can be expressed as $ds^2 = -dt^2 + du^2$ through the coordinate change

$$u = L\int\left[\frac{(x^2+y_0^2)e^{K_0}}{x^2+1}\right]^{1/2}dx. \quad (21)$$

We map the future and past null infinity of each universe $\mathcal{J}^{\pm}_{1,2}$ to a finite region by making use of the conformal transformation: $\psi = \arctan(t+u) + \arctan(t-u)$ and $\xi = \arctan(t+u) - \arctan(t-u)$. This finite region corresponds to straight lines with unity slope in the

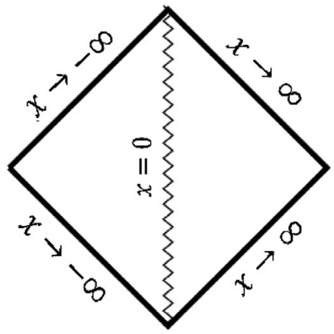

FIG. 2. Carter-Penrose diagram for a slice of the space-time with y and φ constant. For $y \neq 0$ an observer meets the throat of the WH when reaching $x = 0$. Observe that causal curves can get from one universe to the other passing through the throat.

$\psi - \xi$ plane. Now, the sliced metric considered is conformally flat: $ds^2 = F(\psi, \xi)(-d\psi^2 + d\xi^2)$, where $F(\psi, \xi) = (1/4)\sec^2((\psi + \xi)/2)\sec^2((\psi - \xi)/2)$ is the conformal factor. Note that we can cover this entire slice of space-time with a single patch (whose boundary will be $\mathcal{J}^{\pm}_{1,2}$) in a Carter-Penrose diagram because the integrand in (21) is everywhere regular, and so u is everywhere regular too. The throat will be located in the region described implicitly by $u(0) = \tan(\psi + \xi) - \tan(\psi - \xi)$. See Fig. 2.

Despite the somewhat complicated expression for the line element, the diagram shown in Fig. 2 reveals the simplicity of the structure of this WH. The main feature here is that the asymptotically flat regions $x > 0$ and $x < 0$ are communicated through the hypersurface defined by $x = 0$, i.e., the throat. It is clear that traveling between these two regions is possible, as far as causality is concerned, if and only if the throat is crossed.

IV. ENERGY CONDITIONS

Hochberg and Visser have previously shown that for WHs whose throat is a regular compact two-dimensional surface with a finite minimum area, the violation of the NEC near or at the throat is required [6,7]. No assumptions are made about the symmetry of the metric or the existence of any asymptotically flat region in the references mentioned; however, regularity is required.

We now inspect whether this WH violates or satisfies the NEC. In order to analyze the energy conditions, we choose an orthonormal basis [20],

$$e_{\hat{t}} = e_t, \qquad e_{\hat{x}} = \sqrt{\frac{x^2+1}{L^2(x^2+y^2)e^K}}e_x,$$
$$e_{\hat{y}} = \sqrt{\frac{1-y^2}{L^2(x^2+y^2)e^K}}e_y, \qquad e_{\hat{\varphi}} = \frac{e_\varphi - \Omega e_t}{\sqrt{L^2(x^2+1)(1-y^2)}}, \tag{22}$$

where $\Omega = ax(1-y^2)/L(x^2+y^2)$, and $e_\alpha = \partial/\partial x^\alpha$ is the canonical vectors basis. We use an outgoing null vector in the x direction, $\mu = e_{\hat{t}} \pm e_{\hat{x}}$; thus in the orthonormal basis, $T_{\hat{\alpha}\hat{\beta}}\mu^{\hat{\alpha}}\mu^{\hat{\beta}} = T_{\hat{t}\hat{t}} + T_{\hat{x}\hat{x}} = \frac{1}{2}(R_{\hat{t}\hat{t}} + R_{\hat{x}\hat{x}})$. Thereby, we have

$$\begin{aligned}\rho - \tau &= T_{\hat{t}\hat{t}} + T_{\hat{x}\hat{x}} \\ &= \frac{a^2 e^{-K}}{2L^6(x^2+y^2)^5}\Big[x^4 + x^4y^2 + 2x^2y^4 + y^4(1-y^2) \\ &\quad -2\left(\frac{\alpha^2 - 4\varepsilon}{\alpha^2}\right)x^2y^2(x^2+1)\Big],\end{aligned} \tag{23}$$

where $\rho = T_{\hat{t}\hat{t}}$ is the total energy density of mass energy[1] and $-\tau = T_{\hat{x}\hat{x}}$ is the tension per unit area measured by the static observer in the x direction.

For the energy density we get

$$\rho = \frac{a^2}{4L^6}\frac{x^2(1+3y^2) + y^2(1-y^2)}{(x^2+y^2)^4} > 0. \tag{24}$$

From (23) and (24) it can be seen that if $\alpha^2 \leq 4\varepsilon$ then $\rho > \tau$ everywhere, and NEC is satisfied if a traveler moves along the x direction in an outgoing null vector. There is no need for exotic matter to keep the throat open. For a ghost field ($\varepsilon = -1$), the condition $\rho > \tau$ does not always hold; therefore there will be regions where NEC is violated. On the other hand, for a dilatonic field ($\varepsilon = 1$) NEC is always fulfilled as long as $\alpha^2 \leq 4$. Finally, the weak energy condition (WEC), i.e., $\rho > 0$, is satisfied everywhere for any case.

The fact that NEC is satisfied for some cases does not contradict the previous results from Hochberg and Visser since we are dealing with a space-time with a singular ring in it [8]. One may regard the ring singularity as responsible for the fulfillment of the energy conditions in this WH.

V. TIDAL FORCES

Due to the presence of the ring singularity ($x = y = 0$) it is possible that a traveler crossing the throat experiences strong gravitational forces. To make sure the throat is traversable we analyze the tidal forces following [4,13,20]. We take the reference frame of a traveler moving in the x direction, that is,

$$e_{\hat{0}} = \gamma e_{\hat{t}} \mp \gamma(v/c)e_{\hat{x}}, \qquad e_{\hat{1}} = \mp\gamma e_{\hat{x}} + \gamma(v/c)e_{\hat{t}},$$
$$e_{\hat{2}} = e_{\hat{y}}, \qquad e_{\hat{3}} = e_{\hat{\varphi}}, \tag{25}$$

$\gamma = [1-(v/c)^2]^{-\frac{1}{2}}$ being the Lorentz factor. In the x direction the tidal constraint is given by

[1]We have dropped the use of the symbol ρ of the previous section, which referred to the radial cylindrical coordinate.

$$|R_{\hat{1}\hat{0}\hat{1}\hat{0}}| \leq g_{\oplus}/(c^2 \times 2\ \mathrm{m}) \approx 1/(10^5\ \mathrm{km})^2, \qquad (26)$$

where we have used 2 m as the height of our traveler. For the lateral constraints (polar and azimuthal directions) the above condition is the same: $|R_{\hat{2}\hat{0}\hat{2}\hat{0}}| \leq (10^5\ \mathrm{km})^{-2}$, and $|R_{\hat{3}\hat{0}\hat{3}\hat{0}}| \leq (10^5\ \mathrm{km})^{-2}$.

So, in the reference frame of the traveler and because the metric is axially symmetric, we have explicitly that

$$\begin{aligned} |R_{\hat{1}\hat{0}\hat{1}\hat{0}}| &= |R_{\hat{x}\hat{t}\hat{x}\hat{t}}|, \\ |R_{\hat{2}\hat{0}\hat{2}\hat{0}}| &= \gamma^2 |R_{\hat{y}\hat{t}\hat{y}\hat{t}}| + \gamma^2 (v^2/c^2) |R_{\hat{y}\hat{x}\hat{y}\hat{x}}|, \\ |R_{\hat{3}\hat{0}\hat{3}\hat{0}}| &= \gamma^2 |R_{\hat{\varphi}\hat{t}\hat{\varphi}\hat{t}}| + \gamma^2 (v^2/c^2) |R_{\hat{\varphi}\hat{x}\hat{\varphi}\hat{x}}|. \end{aligned} \qquad (27)$$

We then assume that the traveler is at rest at the throat [4]; this implies $v \to 0$ and $\gamma \to 1$. Thus, $|R_{\hat{2}\hat{0}\hat{2}\hat{0}}| = |R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}}|$ and $|R_{\hat{3}\hat{0}\hat{3}\hat{0}}| = |R_{\hat{\varphi}\hat{t}\hat{\varphi}\hat{t}}|$.

The components of interest of the Riemann tensor, for both dilatonic and ghost scalar field, are given by

$$|R_{\hat{x}\hat{t}\hat{x}\hat{t}}| = \frac{e^{-K}}{4} \frac{a^2(1-y^2)(x^2-y^2)^2}{L^6(x^2+y^2)^5}, \qquad (28)$$

$$|R_{\hat{y}\hat{t}\hat{y}\hat{t}}| = e^{-K} \frac{a^2 x^2 y^2 (x^2+1)}{L^6(x^2+y^2)^5}, \qquad (29)$$

$$|R_{\hat{\varphi}\hat{t}\hat{\varphi}\hat{t}}| = \frac{e^{-K}}{4} \frac{a^2(x^2+3x^2y^2+y^2(1-y^2))}{L^6(x^2+y^2)^4}. \qquad (30)$$

Forcing our traveler to approach the throat with $y = 1$, the tidal force in the x direction is 0 everywhere, while the remaining tidal forces go to 0 as the traveler approaches the throat at $x = 0$. It is possible to traverse the throat without feeling the presence of the ring singularity traveling on the plane $y = 1$.

VI. GEODESIC MOTION

In what follows we study the geodesics of a freely falling particle in the space-time. We are interested in radial geodesics to see whether an observer can penetrate the WH or not. Of course, the ring singularity apparently does not allow any observer to penetrate the WH, at least going by the equator.

For this purpose, let λ be an affine parameter and $u^\mu = (\dot{t}, \dot{x}, \dot{y}, \dot{\phi})$, with $\dot{t} = \frac{dt}{d\lambda}$, etc., the vector velocity of an observer, such that the equation $u^\mu u_\mu = \kappa$ holds, with $\kappa = 0$ for lightlike geodesics and $\kappa = -1$ for timelike geodesics. It follows that

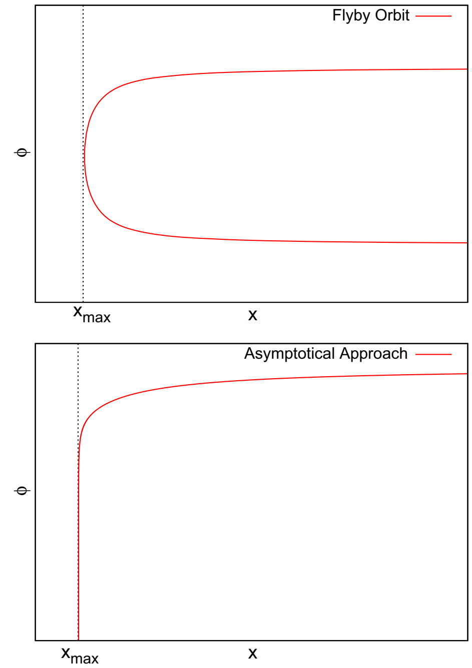

FIG. 3. Geodesics on the $y = 0$ plane. The coordinate φ is plotted as a function of x. In the top panel a geodesic is deflected after reaching its closest approach x_{max}. In the bottom panel, a geodesic approaches x_{max} asymptotically.

$$\begin{aligned} \kappa = &-\left(\dot{t} + \frac{ax(1-y^2)}{L(x^2+y^2)}\dot{\varphi}\right)^2 \\ &+ L^2\left[(x^2+y^2)e^K\left(\frac{\dot{x}^2}{x^2+1} + \frac{\dot{y}^2}{(1-y^2)}\right)\right. \\ &\left. + (x^2+1)(1-y^2)\dot{\varphi}^2\right]. \end{aligned} \qquad (31)$$

We analyze the geodesics constrained to the plane $y = 0$. Hence, (31) reduces to

$$\begin{aligned} L^6 x^4 e^{K_x} \dot{x}^2 &= L^4 x^2 (x^2+1)(\mathcal{E}^2 + \kappa) - (\mathcal{L} L x + a\mathcal{E})^2 \\ &= \hat{X}(x), \end{aligned} \qquad (32)$$

with $K_x = K(x, 0) = -k/L^4 x^4$ and constants of motion $\mathcal{E} = \dot{t} + \Omega\dot{\varphi}$ and $\mathcal{L} + \Omega\mathcal{E} = L^2(x^2+1)(1-y^2)\dot{\varphi}$. The condition $\hat{X}(x) \geq 0$ dominates the geodesics on $y = 0$ [21]. Since Eq. (32) always admits at least a real root ($x_+ \in \mathbb{R}$) two types of motion are possible (the orbits are illustrated in Fig. 3) [22].

(1) Flyby orbit. If the right-hand side polynomial has roots such that there is a maximum one with $x_{\mathrm{max}} > 0$ and $(\partial\hat{X}/\partial x)(x_{\mathrm{max}}) > 0$, the particle departs to infinity after approaching x_{max}.

(2) Critical orbit. If $\hat{X}(x)$ has a root x_{max} and $\hat{X}(x_{\mathrm{max}}) = (\partial\hat{X}/\partial x)(x_{\mathrm{max}}) = 0$, then the particle takes an infinite proper time to approach x_{max}.

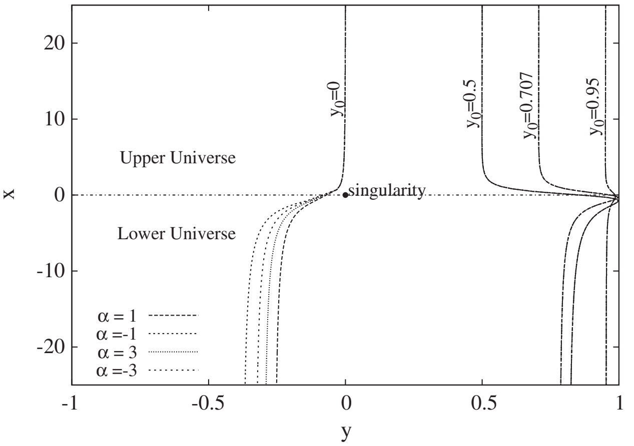

FIG. 4. Geodesics crossing the WH through the throat at $x = 0$. The throat size is $L = 10$ and the traveler initially at $(y_0, x_0 = 25)$. With $\mathcal{E} = 10$, $\mathcal{L} = 5$ and $a = 0.1$, with two different values of α.

Observe that $\hat{X}(x = 0) = -a^2\mathcal{E}^2 < 0$, so it is not possible for any geodesic to go through the WH throat or to reach the singular ring from the plane. If a test particle starts its motion on one side of the throat with $y = 0$, this particle is going to remain on the same side of the throat all the time. Thus, the disc cannot be crossed on the plane $y = 0$ due to the presence of the ring singularity.

It is worth mentioning that motion in the equatorial plane is geodesic. A simple computation reveals that the term $\Gamma^y_{\mu\nu}u^\mu u^\nu$ vanishes for $y = 0$, which implies that $\dot{y} = \ddot{y} = 0$. This makes the y component of the geodesic equation consistent. The t and φ components of said equation will hold as long as $\mathcal{E}$ and $\mathcal{L}$ are constants of motion. We are left then only with the radial component to solve, namely, $\ddot{x} + \Gamma^x_{\mu\nu}u^\mu u^\nu = 0$. This equation is later used in Sec. VII to find geodesics with constant x in the equatorial plane.

We finalize this section presenting some numerical solutions to the geodesic equation, $\dot{u^\sigma} + \Gamma^\sigma_{\mu\nu}u^\mu u^\nu = 0$, for the coordinate accelerations $\ddot{x}$ and $\ddot{y}$. The curves that represent these solutions are shown in Fig. 4 parametrized by λ. It can be observed that there exist trajectories that can travel from one side of the WH to the other universe without touching the ring singularity.

VII. SLOWLY ROTATING LIMIT

Now we study a slowly rotating WH following [23]. Similarly for this metric, a slowly rotating limit can be introduced by considering the angular velocity of a zero angular momentum particle falling through the WH; this is described by

$$\omega = \frac{\dot{\varphi}}{\dot{t}} = \frac{g^{\varphi\varphi}p_\varphi + g^{\varphi t}p_t}{g^{t\varphi}p_\varphi + g^{tt}p_t}, \tag{33}$$

with $p_\phi = 0$. This is no different from the concept of locally nonrotating frames described in [24] where a set of local observers are said to “rotate with the geometry.” Thereby, the angular velocity of the particle is the angular velocity of the WH ω_{WH}. Here, p_μ are the conjugate momenta and $\dot{t}$, $\dot{\varphi}$ the coordinate velocities of the falling particle. So, for ω_{WH} we get

$$\omega_{\mathrm{WH}} = \frac{a}{L^3}\frac{x(x^2+y^2)}{\frac{a^2}{L^4}x^2(y^2-1) + (x^2+1)(x^2+y^2)^2}. \tag{34}$$

We now compare this angular velocity to that of null rays rotating in the φ direction with x and y fixed. For compactness we do not express explicitly the components of the metric tensor (except for $g_{tt} = -1$). From (31) we have

$$0 = -1 + 2g_{t\varphi}\omega_{\mathrm{ray}} + g_{\varphi\varphi}\omega^2_{\mathrm{ray}}, \tag{35}$$

where $\omega_{\mathrm{ray}} = d\varphi/dt$ is the angular velocity of the null rays. Solving this equation for ω_{ray} one obtains $\omega_{\mathrm{ray}} = (-g_{t\varphi} \pm \sqrt{g^2_{t\varphi} + g_{\varphi\varphi}})/g_{\varphi\varphi}$. Now that ω_{ray} has been computed, we consider that the WH is slowly rotating if $|\omega_{\mathrm{WH}}/\omega_{\mathrm{ray}}| \ll 1$, that is,

$$\left|\frac{-g_{t\varphi}}{-g_{t\varphi} \pm \sqrt{g^2_{t\varphi} + g_{\varphi\varphi}}}\right| \ll 1, \tag{36}$$

where we have used $\omega_{\mathrm{WH}} = g^{\varphi t}/g^{tt} = -g_{t\varphi}/g_{\varphi\varphi}$. Equation (36) implies that $1 \ll |(g^2_{t\varphi} + g_{\varphi\varphi})^{1/2}/g_{t\varphi}|$, which can be explicitly expressed as

$$\frac{ax}{L^2(x^2+y^2)}\sqrt{\frac{1-y^2}{x^2+1}} \ll 1. \tag{37}$$

We have already shown that geodesic motion is possibly constrained to the equatorial plane. Consequently, in order for the null curves being discussed here (with x and y fixed) to be geodesics, and thus trajectories that light would follow, we set $y = 0$. Furthermore, from the null condition $p^\mu p_\mu = 0$ we have that

$$-\mathcal{E}^2 + \frac{(\mathcal{L} + \Omega\mathcal{E})^2}{L^2(x^2+1)} = 0, \tag{38}$$

where $p_t = -\mathcal{E}$ and $p_\varphi = \mathcal{L}$. We can rewrite (38) as $\mathcal{L} = (\pm L^2 x\sqrt{x^2+1} - a)\mathcal{E}/Lx$. Notice that $\mathcal{L}$ continues to be a constant of motion even though it depends on x; this is due to the fact that $x = x_0$ is constant and so $\dot{x} = 0$. With $y = 0$ and $x = x_0$, the radial component of the geodesic equation, that is, $\ddot{x} + \Gamma^x_{\mu\nu}u^\mu u^\nu = 0$, becomes

$$(a\mathcal{E} + L\mathcal{L}x_0)(L\mathcal{L}x_0^3 + a\mathcal{E}(1 + 2x_0^2)) = 0. \tag{39}$$

Substituting $\mathcal{L}$ into (39) and reducing it, we obtain the following equation, $L^4x_0^6 - a^2(x_0^2 + 1) = 0$, for x_0. For the values $a = 0.1$, $L = 10$, we have that $x_0 \approx \pm 0.1$ with the rest of the roots being complex. Thus, null curves with $x = x_0$ and $y = 0$ fixed are geodesics and can be light trajectories.

The previous analysis reduces (37) to $a/L^2 x_0\sqrt{x_0^2+1} \ll 1$. For this to hold in general we take the unitless parameter $a/L^2 \ll 1$ and therefore consider it as the slowly rotating limit.

Returning to Eq. (34), the cases $a \sim L^2$ and $L^2 \ll a$ lead to infinite and highly discontinuous angular velocities at the roots $x = \hat{x_0}$ and $y = \hat{y_0}$ of its denominator. On the other hand, in the case $a \ll L^2$ the angular velocity is well behaved everywhere but near the ring singularity. We show the behavior of ω_{WH} for these three different cases in Fig. 5.

The line element (11) can be simplified applying the slowly rotating limit to the parameter $e^K \approx 1$ to first order in a/L^2. However, as $K = K(x, y)$, there are regions in the space-time where this limit is no longer valid; i.e., $e^K \approx 1$ does not hold. Note that for a dilatonic field with $\alpha^2 = 4$, $e^K = 1$ even without using the slowly rotating limit.

To determine the region of validity of the slowly rotating approximation we introduce the criteria $|K| < 0.1$. This follows from the fact that $e^K \approx 1 + K$ to second order in a/L^2. The objective of this criteria is to indicate for which values of x and y the zeroth order term is the leading term and thus, the zeroth order approximation suffices. See Fig. 6.

In a slowly rotating WH, the tidal forces (28)–(30) and the energy conditions (23) and (24) approach 0 except at the ring singularity. In this limit it is possible to find a fourth conserved quantity $\mathcal{K}$. Using the Hamilton-Jacobi formalism, and keeping only terms to first order in a/L^2, we get that Eq. (31) can be separated for the variables x and y. This results in

$$\begin{aligned}\mathcal{K} &= -L^2(\mathcal{E}^2 + \kappa)x^2 + (x^2 + 1)p_x^2 - \frac{\mathcal{L}^2}{x^2 + 1} \\ &\quad + \frac{2a\mathcal{L}\mathcal{E}x}{L(x^2 + 1)} + \mathcal{O}(a^2/L^4) \\ &= L^2(\mathcal{E}^2 + \kappa)y^2 - (1 - y^2)p_y^2 - \frac{\mathcal{L}^2}{1 - y^2} + \mathcal{O}(a^2/L^4),\end{aligned} \tag{40}$$

in which we have used the relations for the conjugate momenta[2] $p_x = L^2(x^2 + y^2)\dot{x}/(x^2 + 1)$ and $p_y = L^2(x^2 + y^2)\dot{y}/(1 - y^2)$. Rewriting and separating (40) we obtain the following equations,

$$\Delta^2\dot{x}^2 = X(x), \qquad \Delta^2\dot{y}^2 = Y(y),$$

where we define

$$X(x) \equiv \Delta_1((\kappa + \mathcal{E}^2)x^2 + \mathcal{K}/L^2) - 2a\mathcal{E}\mathcal{L}x/L + \mathcal{L}^2, \tag{41}$$

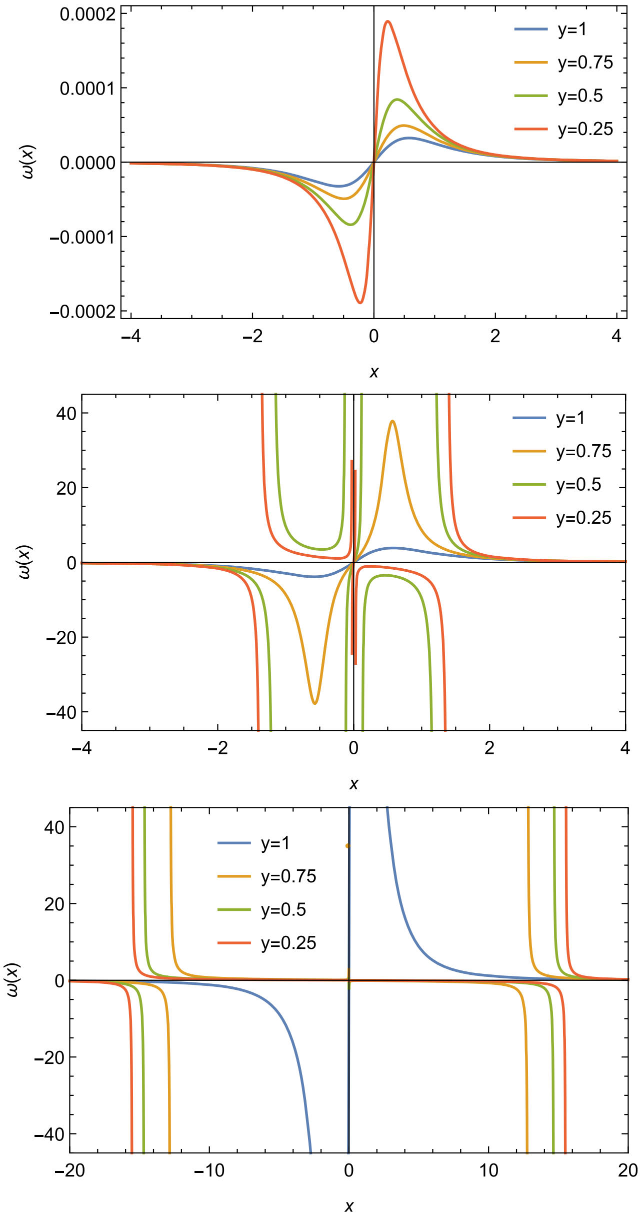

FIG. 5. Angular velocity of the WH as a function of x for different values of y. In the top panel a case where $a \ll L^2$ with $a = 0.1$ and $L = 10$. In the middle panel the $a \sim L^2$ case with $a = 0.11$ and $L = 0.21$. In the bottom panel $L^2 \ll a$ with $a = 11$ and $L = 0.21$. The function grows to infinity as it approaches the ring singularity ($x = 0$, $y = 0$) for any case.

[2]From this point forward every equality has an implicit $\mathcal{O}(a^2/L^4)$ term at the end of the expression, which we omit for compactness.

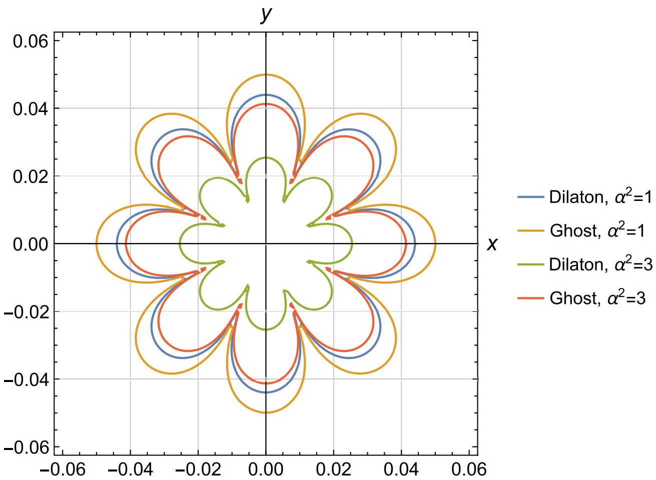

FIG. 6. Region of validity in the x-y plane of the slowly rotating approximation for the dilatonic and ghost fields with different coupling constants. Inside the area of the closed curves the slowly rotating approximation is no longer valid.

$$Y(y) \equiv (1-y^2)(L^2(\kappa+\mathcal{E}^2)y^2-\mathcal{K})-\mathcal{L}^2. \tag{42}$$

A second rank Killing tensor $\mathcal{K}^{\mu\nu}$ can be found for the slowly rotating metric $g_{\mu\nu}^{SR}$ ($g_{\mu\nu} = g_{\mu\nu}^{SR} + \mathcal{O}(a^2/L^4)$). This tensor is given by (see Appendix A for details)

$$\mathcal{K}^{\mu\nu} = -h_1 g_{SR}^{\mu\nu} + \mathcal{X}^{\mu\nu} = h_2 g_{SR}^{\mu\nu} - \mathcal{Y}^{\mu\nu}, \tag{43}$$

with $h_1(x) = L^2x^2$ and $h_2(y) = L^2y^2$. Also

$$\mathcal{X}^{\mu\nu} = \begin{bmatrix} -L^2x^2 & 0 & 0 & -aLx/\Delta_1 \\ 0 & \Delta_1/L^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -aLx/\Delta_1 & 0 & 0 & -L^2/\Delta_1 \end{bmatrix}$$

$$\mathcal{Y}^{\mu\nu} = diag\left[-L^2y^2, 0, 1-y^2, \frac{1}{1-y^2}\right]. \tag{44}$$

Contracting twice the Killing tensor (43) with the conjugated momenta p_μ yields the previously introduced fourth conserved quantity of motion $\mathcal{K}$, i.e., $\mathcal{K}^{\mu\nu}p_\mu p_\nu = \mathcal{K}$. One can easily verify that $\nabla_{(\sigma}\mathcal{K}_{\mu\nu)}$ vanishes for terms below second order in a/L^2.

We proceed to study the properties of the fourth degree polynomials (41) and (42), in particular, the nature of their roots, which gives us information about the accessible regions in the WH to our traveler.

In order to go through the WH, we need that $X(x) > 0$ at $x = 0$. From (41), it follows that $\mathcal{K} + \mathcal{L}^2 > 0$. Furthermore, if the traveler were to freely move through the upper universe ($x > 0$) and the lower universe ($x < 0$), it would be necessary that $X(x) > 0 \ \forall \ x \in \mathbb{R}$. This is accomplished by demanding that $\mathcal{K} + \mathcal{L}^2 > 0$ is satisfied and that (41) has four complex conjugate roots.

Careful study of $X(x) = 0$ reveals that if (41) has four complex conjugate roots then, as can be seen in Appendix B, the following conditions should be met:

(1) $\kappa + \mathcal{E}^2 > 0$ (this is trivially satisfied for null geodesics).
(2) Either $L^2(\kappa + \mathcal{E}^2) + \mathcal{K} > 0$ or

$$4L^2(\kappa+\mathcal{E}^2)(\mathcal{K}+\mathcal{L}^2) > (L^2(\kappa+\mathcal{E}^2)+\mathcal{K})^2. \tag{45}$$

On the other hand, $Y(y)$ has the following properties:

(1) $Y(0) = -\mathcal{K} - \mathcal{L}^2 = -X(0)$.
(2) If inequality (45) holds, then $Y(y) = 0$ has four complex conjugate roots. The case of four real roots corresponds when (45) fails to be satisfied and both $\mathcal{K} + \mathcal{L}^2 > 0$ and $L^2(\kappa + \mathcal{E}^2) + \mathcal{K} > 0$.

From the first property it can be readily seen that, if the condition for the traveler to cross the throat of the WH is fulfilled ($\mathcal{K} + \mathcal{L}^2 > 0$), then $Y(0) < 0$. Unfortunately, from this fact we cannot yet conclude that the traveler would never be able to touch the ring singularity (recall that the slowly rotating limit is not valid in $x = y = 0$). However, this hints towards the existence of inaccessible regions to the traveler where the slowly rotating limit may hold. This follows from the possibility that $Y(y)$ could still be negative for values of y other than $y = 0$.

Of course, the case where (42) has four real roots is the one we are interested in, since this implies there are regions in which $Y(y) > 0$ and a traveler could freely move on the plane $y = y_0$ with $Y(y_0) > 0$ as long as $X(x) > 0$.

Given the conditions for the nature of the roots of (41) and (42), it can be established that if $X(x) > 0 \ \forall \ x \in \mathbb{R}$ then $\mathcal{K} + \mathcal{L}^2 > 0$ and $\kappa + \mathcal{E}^2 > 0$. Additionally, if (45) does not hold and $L^2(\kappa + \mathcal{E}^2) + \mathcal{K} > 0$, then $Y(y) = 0$ has four real roots. See Fig. 7.

The physical significance of the above statement is that, fulfilling the derived inequalities on the constants of motion, a traveler can cross back and forth both universes constrained to a plane $y = y_0$, where $Y(y_0) > 0$, without ever touching the ring singularity. In fact, as a result of the negative behavior of the polynomial $Y(y)$ close to the ring singularity, an observer traveling in geodesic motion would be repelled from its nearby region. In Fig. 8 the regions of interest near the ring singularity are shown in the x-y plane.

It is important to clarify that, even though the slowly rotating limit fails very near the ring singularity, there are always regions of the space-time close enough to it where this limit is valid and the repulsion of geodesics is observed. In this sense, the ring singularity can be considered to be surrounded by a repulsive potential which prevents any observer from reaching it. This includes light itself, as the analysis done applies for null geodesics too, so we conclude that in this slowly rotating WH the ring

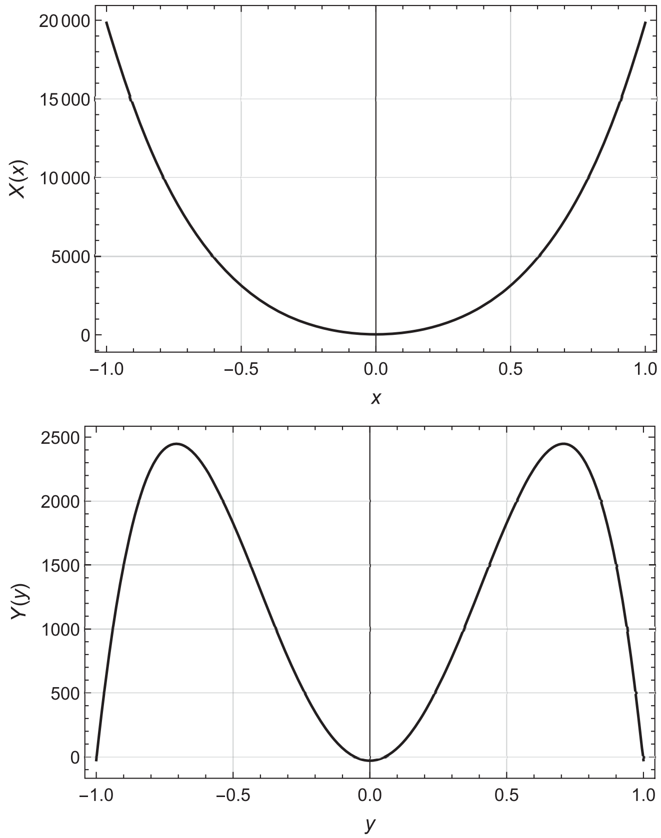

FIG. 7. The polynomials $X(x)$ and $Y(y)$ with $\mathcal{E} = 10$, $\mathcal{L} = 5$, $a = 0.1$, $L = 10$, $\mathcal{K} = 5$ and $\kappa = -1$. Note that the roots of $Y(y)$ are at $y_{1,2} = \pm 0.055$ and $y_{3,4} = \pm 0.998$.

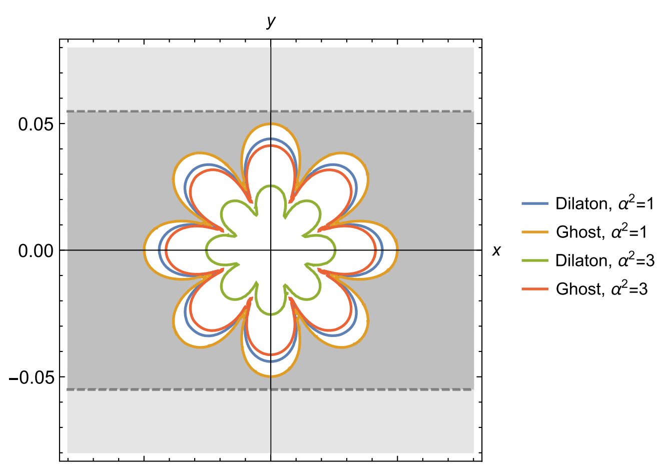

FIG. 8. Main regions of interest near the ring singularity in the x-y plane with $\mathcal{E} = 10$, $\mathcal{L} = 5$, $a = 0.1$, $L = 10$, $\mathcal{K} = 5$ and $\kappa = -1$. In the light gray area an observer is allowed to freely move; in the dark gray area repulsive effects emerge and in the white area the slowly rotating limit is no longer valid for each separate case. The first pair of roots of $Y(y)$ is at $y_{1,2} = \pm 0.055$ (dashed lines).

singularity is causally disconnected from the rest of the space-time.

VIII. CONCLUSIONS

Metric (11) describes the space-time of an asymptotically flat rotating magnetized WH. It is an exact solution of the Einstein-Maxwell-Dilaton field equations and contains a ring singularity at $x = y = 0$ of radius L. The disc $x = 0$ can be identified with a throat surrounded by the ring singularity. Due to the presence of this singularity we showed that NEC and WEC are satisfied for the dilatonic field ($\varepsilon = 1$) with coupling constant $\alpha^2 \leq 4$.

Despite the apparent flaw that the ring singularity may represent, in the slowly rotating limit a traveler moving on a plane $y = y_0$ (where y_0 is a nonzero constant) can cross back and forth the throat reaching another asymptotically flat space-time without facing extreme tidal forces and without touching the singularity itself. Furthermore, we observe that test particles following timelike or lightlike curves are repelled from the naked singularity and thus it is causally disconnected from the rest of the Universe.

The effect of the mass parameter in this space-time remains open.

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APPENDIX A

In this brief appendix we show that if the Hamiltonian $\mathcal{H}$ of a freely falling test particle in a space-time with inverse metric $g^{\mu\nu}$ is separable, then there exists a Killing tensor of second order $\mathcal{K}^{\mu\nu}$, which, when contracted with the conjugate momenta p_μ, yields an additional constant of motion $\mathcal{K} = \mathcal{K}^{\mu\nu} p_\mu p_\nu$.

Let $\{u^0, u^1, u^2, u^3\}$ be the coordinate system. We assume that the metric tensor only depends on two arbitrary coordinates $u^1 = x$ and $u^2 = y$, which makes $\partial/\partial u^a$ Killing vectors and therefore, p_a conserved quantities, for $a \neq 1$, 2. Then, the Hamiltonian $2\mathcal{H} = \kappa = g^{\mu\nu} p_\mu p_\nu$ is separable if and only if $g^{\mu\nu}$ can be written in the form

$$g^{\mu\nu} = \frac{\mathcal{X}^{\mu\nu}(x) + \mathcal{Y}^{\mu\nu}(y)}{f(x) + h(y)}, \tag{A1}$$

with $\mathcal{X}^{y\mu} = \mathcal{X}^{\mu y} = \mathcal{Y}^{x\mu} = \mathcal{Y}^{\mu x} = 0$. Because $g^{\mu\nu}$ is a symmetrical tensor, the sum $\mathcal{X}^{\mu\nu}(x) + \mathcal{Y}^{\mu\nu}(y)$ is also symmetrical and, for any given metric of the form (A1), $\mathcal{X}^{\mu\nu}(x)$ and $\mathcal{Y}^{\mu\nu}(y)$ can always be chosen such that they are both symmetrical separately.

Inserting $g^{\mu\nu}$ in the Hamiltonian we get

$$(f(x) + h(y))\kappa = (\mathcal{X}^{\mu\nu}(x) + \mathcal{Y}^{\mu\nu}(y))p_\mu p_\nu. \qquad \text{(A2)}$$

Applying the Hamilton-Jacobi theory, we choose Hamilton's principal function as $S(u^\mu, p_\mu, \lambda) = p_a u^a + W_1(x) + W_2(y) - \lambda\kappa/2$, with $a \neq 1$, 2, $W_1(x)$ and $W_2(y)$ being auxiliary functions and λ the affine parameter. Now, the Hamilton-Jacobi equation $\mathcal{H} + \partial S/\partial\lambda = 0$ yields the Hamiltonian of the geodesics. From Hamilton-Jacobi theory we also have that $p_\mu = \partial S/\partial u^\mu$, so $p_x = dW_1(x)/dx$ and $p_y = dW_2(y)/dy$. With this conditions the variables x and y in (A2) can be easily separated, implying the existence of a new constant $\mathcal{K}$,

$$\kappa f(x) - \mathcal{X}^{\mu\nu}(x)p_\mu p_\nu = \mathcal{Y}^{\mu\nu}(y)p_\mu p_\nu - \kappa h(y) = \mathcal{K}. \qquad \text{(A3)}$$

Note that separability could not have been accomplished if we had not previously demanded that $\mathcal{X}^{y\mu} = \mathcal{Y}^{x\mu} = 0$. Assuming there exists a second rank tensor such that $\mathcal{K} = \mathcal{K}^{\mu\nu}p_\mu p_\nu$, by comparing it to (A3), it has to be that

$$\mathcal{K}^{\mu\nu} = f(x)g^{\mu\nu} - \mathcal{X}^{\mu\nu}(x) = \mathcal{Y}^{\mu\nu}(y) - h(y)g^{\mu\nu}. \qquad \text{(A4)}$$

Alternatively, this tensor can also be expressed as

$$\mathcal{K}^{\mu\nu} = \frac{f(x)\mathcal{Y}^{\mu\nu}(y) - h(y)\mathcal{X}^{\mu\nu}(x)}{f(x) + h(y)}. \qquad \text{(A5)}$$

A straightforward (but extremely tedious) computation reveals that $\nabla_{(\sigma}\mathcal{K}_{\mu\nu)} = 0$, i.e., $\mathcal{K}_{\mu\nu}$ is a second rank Killing tensor. The conserved quantity $\mathcal{K}$ is associated with this hidden symmetry of the metric.

APPENDIX B

The roots of a fourth degree (or quartic) polynomial can be found by radicals, in fact, fourth is the highest degree for which solutions can be found analytically for any polynomial of this class. Therefore, the nature of the roots of these polynomials can be determined by a set of discriminants which, naturally, depend on their coefficients [25]. For the general case these discriminants are rather long expressions. In Sec. VII, a fourth degree polynomial that does not possess a cubic term is found. Thus, in this appendix the following particular case is discussed,

$$Ax^4 + Bx^2 + Cx + D = 0, \qquad \text{(B1)}$$

with $A, B, C, D \in \mathbb{R}$. For this polynomial, the discriminants are given by

$$\begin{aligned} \Delta &= 16A(16A^2D^3 - 8AB^2D^2 \\ &\quad + 36ABDC^2 - 27AC^4 - B^3C^2), \\ P_1 &= 8AB, \qquad P_2 = 16A^2(4AD - B^2). \end{aligned} \qquad \text{(B2)}$$

Rather than their multiplicity, we focus on whether the roots of (B1) are real or complex conjugate. The conditions on the discriminants (B2) are as follows:

(i) For two real and two complex conjugate roots $\Delta < 0$.
(ii) For four real different roots $\Delta > 0$ and $P_{1,2} < 0$.
(iii) For two pairs of complex conjugate roots $\Delta > 0$ and either $P_1 > 0$ or $P_2 > 0$.

Multiple roots, which can be either real or complex, are obtained only when $\Delta = 0$. For instance, if $\Delta = 0$ and $P_{1,2} < 0$ there is a double real root and two different simple real roots.

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