



Energy balance of a Bose gas in a curved space-time

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Abstract

Classical solutions of the Klein–Gordon equation are used in astrophysics to model galactic halos of scalar field dark matter and compact objects such as cores of neutron stars. These bound solutions are interpreted as Bose–Einstein condensates whose particle number density is governed by the Gross–Pitaevskii (GP) equation. It is well known that the Gross–Pitaevskii–Poisson (GPP) system arises as the non-relativistic limit of the Klein–Gordon–Einstein (KGE) equations and, conversely, the KGE system may be interpreted as a generalization of the GPP equations in a curved space-time. In the present work, we consider a 3+1 ADM foliation of the space-time in order to construct a general-relativistic version of the GP equation. Besides, we derive a general energy balance equation for the boson gas in the hydrodynamic variables, where different energy potentials are identified as kinetic, quantum, electromagnetic and gravitational. In addition, we find a correspondence between the energy potentials in the balance equation and actual components of the scalar energy–momentum tensor. We also study the Newtonian limit of the hydrodynamic formulation and the balance equation. As an illustrative case, we study the effects in the energy potentials of a relativistic correction in the GP equation.

Keywords Scalar field theory · Klein–Gordon equation · Gross–Pitaevskii equation · Bose–Einstein condensates · Bohm’s interpretation · Boson stars · Dark matter · General relativity · Quantum mechanics

1 Introduction

Scalar fields are ubiquitous in modern physics, from the inflaton responsible for the primordial acceleration of the Universe and the Higgs scalar that gives mass to matter particles at fundamental scales, up to huge astronomical and cosmological scales

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where they are used to model dark matter and dark energy. In most physical systems of interest, the dynamics of these scalars is well described by the original Klein–Gordon–Maxwell (KGM) equations, which are Lorentz and $U(1)$ invariant. It is well known that the KG modes can be interpreted as a set of independent bosonic particles living in a Minkowski space-time. These bosons can be endowed with charge if a complex field is considered, whose electromagnetic interaction is mediated by a $U(1)$ gauge field. This is mathematically achieved by promoting the global $U(1)$ symmetry to a local one. Self-interactions between bosons are encoded in a potential $V(\Phi)$.

The first attempt to describe astronomical objects as macroscopic bosonic states was made by Wheeler, who aimed at constructing stable particle-like solutions from classical electromagnetic fields coupled to general relativity that he called *geons* (an abbreviation for “gravitational-electromagnetic entity”) [1]. Later, Kaup [2] and Ruffini and Bonazzola [3] introduced the notion of boson stars that could be useful to model compact stars having certain advantages over fluid neutron stars models.¹ More recently, scalar fields as dark matter were suggested as a set of bosonic modes all laying in the ground state making up a macroscopic wave function which corresponds to a galactic halo (see e.g. [5–16], among others). These studies suggest that these objects are gigantic Bose–Einstein condensates (BECs). In a general context, these configurations are classical solutions of the Klein–Gordon–Einstein (KGE) system and their evolution and stability have been widely studied in the last decades.

Due to the Bose–Einstein statistics, all individual particles lay in a common quantum state and, therefore, the macroscopic wave function scales as its occupation number. Consequently, the corresponding mean-field non-linear Schrödinger equation, called the Gross–Pitaevskii (GP) equation, encodes the evolution of the number density of the condensate. It has been shown in a host of works that the Schrödinger–Poisson (SP) and Gross–Pitaevskii–Poisson (GPP) equations arise as the non-relativistic limit of the Klein–Gordon–Einstein (KGE) equations (see [17] for a review). Therefore, it is common to interpret the KGE solutions as densities of number of particles of bosonic systems laying in a curved space-time background. Usually, such non-linear system is solved numerically (see e.g. [4,18]) and the existence of bounded stable soliton-like solutions of finite mass, dubbed as mini-boson stars, has been mathematically demonstrated [19].

The stability theory of these configurations can be understood from the elliptical nature of the equations. A simple and intuitive idea is the following: because these solutions are self-gravitating and dispersive by nature, the balance between these competing features determines the stability of these objects [20–22]. In those works, it has been shown from numerical full-relativity calculations that the critical maximum mass of a scalar configuration depends on the inverse of the mass of the boson. This implies that for heavy bosons, bound solutions cannot be of astrophysical size. Besides, they found that the stability of an initial configuration is determined by its initial total mass and the initial central value of the scalar field ϕ_0^i . For small initial central values of the density, stable bound solutions along the whole range of masses are possible. After passing a critical value of the ϕ_0^i , a small branch of solutions collapse into a

¹ They both have similar solutions of mass as a function of radius but, for boson stars, dynamical equations avoid developing discontinuities (e.g. there is no sharp stellar surface), there is no concern about solving turbulence issues, and one avoids uncertainties in the equation of state [4].

black hole and finally, when the total rest mass of the bosons exceeds the gravitational binding energy for larger values of ϕ_0^j , dispersive solutions cannot be held together and hence they fade away or migrate to a stable branch solution (see e.g. [4,23]).

In the equilibrium regime, these boson stars solutions have been used to model dark matter halos of galaxies for a long time. From the last 3 decades the implications of these sorts of models of dark matter have been analyzed in a systematic manner in different regimes (see e.g. [24–39], among others). A particularly interesting feature of this model is that it naturally predicts the non-existence of small substructures in contrast to the cold dark matter (CDM) model and therefore this model overcomes some potential issues of CDM such as the missing satellite problem, for example.

Another interesting application of this model is the formation of black holes from the collapse of unstable boson stars. This mechanism has been used to explain the formation of black holes of different kinds in the Universe, such as supermassive black holes [40–42]. Despite the fact that the no-hair theorems condemn stable scalar field configurations to exist around black holes, it has been shown that within this scenario the existence of scalar quasi-resonant solutions that decay very slowly is possible, and they can be used to model actual galactic dark matter halos surrounding supermassive black holes [41,43–45]. This idea provides a consistent mechanism of formation of supermassive black holes even at very early stages in the history of the Universe [45]. It is also important to study the scalar fields in the strong field regime, specially after the discovery of gravitational waves which opens a new quest about strong gravity. In the context of boson stars, the previous quest leads us to face an outstanding and unsolved problem in science, that is, finding exact general solutions of the KGE equations. Although such a task has been unreachable so far, approximations have always allowed us to turn around impossible tasks in physics, especially if one aims to make applications in astrophysics. A commonly used approach in order to turn over these difficulties is to consider equilibrium solutions either of the metric and the fields, that is, equilibrium solutions of the KGE equations with harmonic time dependence in a static background metric. These static solutions suffice to correctly describe several systems.

One of the most intriguing problems in general relativity is to describe the dynamics of matter in strongly relativistic regimes such as gases, fluids and fields near compact objects like neutron stars or black holes. The standard hydrodynamical theory does not allow us to identify the different energy contributions in these systems, since the standard laws of thermodynamics are not applicable in fully relativistic environments. This is due to the fact that it does not exist a fully consistent quantum field theory in curved space-times that would enable us to compute the accessible quantum states of the system in order to obtain fundamental thermodynamic potentials such as the entropy.

In this article, as a first step, we address this problem for a charged boson gas at zero temperature described by the KGM Madelung transformation of the scalar field. In this new field variables, the KG equation for the complex scalar field is transformed into a couple of hydrodynamic equations governing the dynamics of the boson gas. In this representation, it is easy to identify the different contributors to the total energy of the system through a balance equation that is an antecedent of the first law of thermodynamics in a curved space-time for a quantum gas made of

bosons. In addition, we find some relations between the energy potentials arising in the balance equation and the actual expected values of the energy and momentum for the system, quantified by the components of the energy–momentum tensor of the scalar field defined into the 3+1 foliation.

Although all this KG machinery has been the basis of great achievements as, e.g. the inflaton, the π –mesons, the Higgs boson, axions, etc., further developments of this framework are needed to model phenomena beyond the Minkowskian threshold. In this work, we are interested in solutions of the KG equation as models of objects at large scales like compact stars and dark matter halos.

In the non-relativistic limit, the KG equation endowed with a self-interaction term reduces to the GP equation. Since it is based on Newtonian gravity, the GP equation cannot model compact objects. However, it is possible to construct a more suitable framework by extending the GP equation to curved space-times, which is an important goal of this work. This article is organized as follows. In Sect. 2 we present the field equations describing our system of bosons. In Sect. 3 we generalize the GP equation, which provides a model for a macroscopic system of charged bosons laying in a curved space-time. In Sect. 4, we derive the hydrodynamic representation for the complex scalar field equations by a field redefinition using Madelung variables. In Sect. 5 we derive the generalized Euler and continuity equations from the complex Klein–Gordon equation. In Sect. 6 we derive a conservation equation that can be interpreted as the energy balance of the different components involved. In Sect. 7 we compute some ADM invariants formally interpreted as the energy and momentum defined along a generic 3 + 1 foliation. As usual, these quantities are written in terms of the components of the energy–momentum tensor and some other geometrical entities. By performing the Madelung field-redefinition, these quantities are written in terms of the energy potentials. Such relation provide valuable information helpful about the physical interpretation of the hydrodynamic energy potentials. In Sect. 8 we derive the Newtonian limit of the balance equation and its energy components. In Sect. 9 we present our simple case of study of a scalar field in flat space-time with a first-order relativistic correction. Finally, in Sect. 10 we present our conclusions.

2 Field equations

In what follows we model the boson gas as a set of excitations of a self-interacting, charged, complex scalar field which is minimally coupled to a gauge vector field mediating the electromagnetic interaction. Gravity is interpreted as a geometrical phenomenon, i.e. the surrounding space-time of a massive body acquires curvature as described by General Relativity. We shall not deal explicitly with the Einstein field equations but we consider an arbitrary space-time geometry. We aim to extend the KGM equations, with local $U(1)$ symmetry, by using coordinates for a 4-dimensional manifold playing the role of the physical curved space-time whose geometry is encoded by a metric g . From hereafter we use the units $c = \hbar = \epsilon_0 = \mu_0 = 1$, for c the speed of light, \hbar the reduced Planck constant, ϵ_0 and μ_0 the permittivity and permeability of free space, respectively. We define the electromagnetic d’Alembert operator as $\square_E \equiv (\nabla^\mu + ieA^\mu)(\nabla_\mu + ieA_\mu)$, where e is the charge unit and A_μ is the $U(1)$

gauge vector field corresponding to the Maxwell 4-potential, such that the KGM equations are given by

$$\square_E \Phi - \frac{dV}{d\Phi^*} = 0, \quad (1)$$

$$\nabla_\nu F^{\nu\mu} = J^{E\mu}, \quad (2)$$

for the complex scalar field $\Phi(t, \mathbf{x})$ and its complex conjugate $\Phi^*(t, \mathbf{x})$. The Faraday tensor is given by

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu, \quad (3)$$

and the conserved 4-current is defined as

$$J_\mu^E \equiv i \frac{e}{2m^2} [\Phi (\nabla_\mu - ieA_\mu) \Phi^* - \Phi^* (\nabla_\mu + ieA_\mu) \Phi]. \quad (4)$$

We introduce scalar self-interactions by using the “ Φ^4 ” self-interacting potential

$$V(\Phi) = m^2 |\Phi|^2 + \frac{\lambda}{2} |\Phi|^4, \quad (5)$$

describing a system of bosonic excitations that condenses into a single macroscopic ground state.

We use the 3 + 1 ADM foliation of the space-time in such a way that each 3-dimensional slice is space-like, corresponding to the level sets (hypersurfaces) of a parameter t that can be considered as a *universal time function*. For a 3-dimensional metric γ_{ij} measuring proper distances within hypersurfaces, the lapse of proper time $d\tau = N(t, x^i) dt$ between hypersurfaces (as measured by the *normal* or *Eulerian observers* moving along the world-line normal to the hypersurface), and the relative velocity N^i between Eulerian observers and the lines of constant spatial coordinates, the 3+1 metric of the space-time reads [46,47]

$$ds^2 = -N^2 dt^2 + \gamma_{ij} (dx^i + N^i dt) (dx^j + N^j dt), \quad (6)$$

where $N(t, x^i)$ is known as the *lapse function* and $N^i(t, x^j)$ as the *shift vector*. Notice that both the lapse function N and the shift vector N^i are free functions carrying on information of the choice of coordinates; they are known as *gauge functions*.

3 Generalized Gross–Pitaevskii equation

Here we show that the KG equation with the hat potential (5) can be transformed into a relativistic GP equation. The non-relativistic GP equation, which is a non-linear version of Schrödinger’s equation, has been of great interest in quantum and statistical mechanics since it accounts for correlations between quantum particles. An important application of this framework is the study of superfluidity and phase transitions. Bogolyubov [48,49] first tried to model superfluidity as an imperfect BEC

due to the weak repulsion between the bosons. It is worth noting that, in contrast to the standard GP equation for neutral particles in a spatially flat space-time, charged bosons in a curved space-time find interesting applications in the context of dark matter, boson stars and neutron stars with superfluid cores.

An important mathematical feature of the KG equation is that, in some circumstances, it admits non-dispersive solutions, as those relevant in scattering processes. The Derrick theorem states that non-regular, static, non-topological localized scalar field solutions are stable in a spatially flat space [50]. This constraint is avoided by adopting a harmonic decomposition for the complex scalar field,

$$\Phi(t, \mathbf{x}) = \Psi(t, \mathbf{x}) \exp(-i\omega_0 t), \quad (7)$$

where ω_0 is a constant that can be either the mass or the frequency of massless particles. Although the field is non-static, the space-time remains static and thus the KG equation admits soliton-like solutions [50,51]. By plugging such field-redefinition into the KG equation (1) we obtain

$$\begin{aligned} i\nabla^0\Psi - \frac{1}{2\omega_0}\square_E\Psi + \frac{1}{2\omega_0}(m^2 + \lambda n)\Psi \\ + \frac{1}{2}\left(-\frac{\omega_0}{N^2} - 2eA^0 + i\square t\right)\Psi = 0, \end{aligned} \quad (8)$$

where $n(t, \mathbf{x}) \equiv |\Phi|^2 = |\Psi|^2$ is defined as the scalar field density and $\square t = \nabla^\mu\nabla_\mu t$. Equation (8) is the KG equation in terms of Ψ and we name it the *generalized GP equation* in curved space-times for the potential (5). As mentioned before, in the $3+1$ formalism the choice of the coordinate system is given in terms of the *gauge variables*, the lapse function N and the shift vector N^i . Once a gauge condition is chosen, the specific form of the GP equation would be different from other GP equations in other gauges. A common approach uses harmonic coordinates defined by asking for the coordinates to satisfy $\square x^\alpha = 0$. In particular, for $\square t = 0$, the *harmonic slicing condition* on the lapse function is obtained [47]. Here we do not assume any specific foliation condition until Sect. 8, where we fix the longitudinal Newtonian gauge and obtain the traditional GP equation [52].

4 Hydrodynamic representation

The hydrodynamic approach for the Schrödinger equation was introduced by Madelung [53] (see also [54,55]), who showed that it is equivalent to Euler's equations for an irrotational fluid with an additional quantum potential. The boson gas that we study can be interpreted as a real fluid described by the quantum Euler equations. This hydrodynamic representation has been widely used in the literature for BEC dark matter [13,14,56,57], including electromagnetic interactions [17], and for relativistic BEC stars or neutron stars with a superfluid core [58,59]. In this section, we derive

the hydrodynamic representation for the generalized GP equation (8). We carry out the following Madelung transformation:

$$\Phi(t, \mathbf{x}) = \sqrt{n} \exp(i\theta) = \sqrt{n} \exp[i(S - \omega_0 t)], \tag{9}$$

where the amplitude Ψ is decomposed into a density $n(t, \mathbf{x})$ and a phase $S(t, \mathbf{x})$ encoding the geometry and evolution of the front-wave solution. In this way, the KG/GP equation splits into its imaginary and real parts, respectively:

$$\nabla_\mu \sqrt{n} (2\nabla^\mu \theta + eA^\mu) + e\nabla_\mu (A^\mu \sqrt{n}) + \sqrt{n} \square \theta = 0, \tag{10}$$

$$\square \sqrt{n} - \sqrt{n} [\nabla_\mu \theta (\nabla^\mu \theta + 2eA^\mu) + e^2 A^2 + m^2 + \lambda n] = 0, \tag{11}$$

where $A^2 = A^\mu A_\mu$. After applying the Madelung transformation, the current (4) turns into

$$J^E_\mu = \frac{ne}{m^2} (\nabla_\mu \theta + eA_\mu). \tag{12}$$

Interestingly, a relativistic quantum particle in a flat space-time with electromagnetic field has a mechanical momentum $m\mathbf{u} = \mathbf{p} - e\mathbf{A}$, with \mathbf{u} the 4-velocity, \mathbf{p} the canonical momentum and \mathbf{A} the magnetic vector potential. By writing its wave function in hydrodynamic variables, it results that $\mathbf{p} = \nabla S$. In a similar way for our boson gas, the electromagnetic 4-momentum corresponds to the sum of individual mechanical momenta, namely $J^E_\mu = (e/m)nu_\mu$. In terms of J^E_μ , Eq. (10) and (11) read

$$\nabla^\mu J^E_\mu = 0, \tag{13}$$

$$J^E_\mu J^{E\mu} + \frac{n^2 e^2}{m^4} \left(m^2 + \lambda n - \frac{\square \sqrt{n}}{\sqrt{n}} \right) = 0. \tag{14}$$

Then, by interpreting the KG equation as a general-relativistic GP equation, through the Madelung transformation it splits into the continuity (13) and quantum Hamilton-Jacobi (14) equations above. Such quantum version of the Hamilton-Jacobi equation differs from the classical one only by the last term on the left-hand side of Eq. (14), that corresponds to the de Broglie relativistic quantum potential [60].

Now, let us take the continuity equation (13) and notice that

$$\int_{\mathbb{V}} \nabla_\mu J^{E\mu} dV = \int_{\mathbb{V}} \nabla_0 J^{E0} dV + \int_{\mathbb{S}} k_j J^{Ej} dS = 0, \tag{15}$$

where \mathbb{S} is an arbitrary surface enclosing the volume \mathbb{V} containing the system and k^j is the vector orthogonal to \mathbb{S} . We assume that far away from the sources J^E_j is negligible, so we are free to choose a volume \mathbb{V} such that the surface integral of Eq. (15) vanishes. Thus, the quantity $Q = \int_{\mathbb{V}} J^{E0} dV$ is a conserved charge,

$$\frac{dQ}{dt} = \int_{\mathbb{V}} \nabla_0 J^{E0} dV = 0. \tag{16}$$

In conclusion, the continuity equation (13) expresses the conservation of the charge of the scalar field.

5 Continuity and Euler equations

Let us define the velocity v_μ of an individual particle as

$$mv_\mu \equiv \nabla_\mu S + eA_\mu. \tag{17}$$

It is important to mention that v^μ is not the unit normal 4-vector n^μ to the spatial hypersurfaces in the 3+1 formalism; in our case $v_\mu v^\mu \neq 1$. We also stress that we are working in a frame in which the contribution of the rest-mass energy has been subtracted from the 4-velocity. In terms of v_μ , the continuity and quantum Hamilton-Jacobi equations (13) and (14) become

$$\nabla^\mu (nv_\mu) - \frac{\omega_0}{m} (\nabla^0 n + n \square t) = 0, \tag{18}$$

$$v_\mu v^\mu - \frac{2\omega_0}{m} v^0 - \frac{\omega_0^2}{m^2 N^2} + 1 + \frac{\lambda}{m^2} n - \frac{1}{m^2} \frac{\square \sqrt{n}}{\sqrt{n}} = 0. \tag{19}$$

Equation (18) governs the evolution of the density of the boson gas whilst (19) governs the evolution of its phase.

After applying the covariant derivative ∇_α to Eq. (19), using the Leibniz rule to the first two terms and using Maxwell’s equations (2), we obtain the Euler equation

$$v_\mu \nabla^\mu v_\alpha - \frac{\omega_0}{m} \nabla^0 v_\alpha - \frac{\omega_0^2}{2m^2} \nabla_\alpha \left(\frac{1}{N^2} \right) \frac{\lambda}{2m^2} \nabla_\alpha n - \frac{1}{2m^2} \nabla_\alpha \left(\frac{\square \sqrt{n}}{\sqrt{n}} \right) + \frac{e}{m} \left[v_\mu (\nabla_\alpha A^\mu - \nabla^\mu A_\alpha) - \frac{\omega_0}{m} (\nabla_\alpha A^0 - \nabla^0 A_\alpha) \right] = 0. \tag{20}$$

At this point, we can identify different analogues to physical quantities. In first place, we spot the covariant definition of the Lorentz force in a curved space-time, given by

$$F^E_\alpha \equiv -\frac{e}{m} \left(v_\mu F_\alpha{}^\mu - \frac{\omega_0}{m} F_\alpha{}^0 \right). \tag{21}$$

In second place, we identify the gravitational “force” which, in spite of the geometrical nature of gravity assumed here, is an actual measurement of the curvature associated with the gravitational strength quantified by the time-time component of the metric

$$F^G_\alpha \equiv -2\nabla_\alpha U^G; \quad U^G \equiv -\frac{\omega_0^2}{4N^2 m^2}, \tag{22}$$

where U^G is the gravitational “potential” contribution due to the metric time-component related to the gravitational strength. In third place, the quantum force is given by

$$F^Q_\alpha \equiv -\nabla_\alpha U^Q; \quad U^Q \equiv -\frac{1}{2m^2} \frac{\square\sqrt{n}}{\sqrt{n}}, \tag{23}$$

where U^Q is the quantum potential. Finally, a measure of the temporal and spatial variations of the density of the scalar field is characterized by

$$F^n_\alpha \equiv -\nabla_\alpha h; \quad h \equiv \frac{\lambda n}{2m^2}, \tag{24}$$

where h is the enthalpy. Notice that F^n_α is only present when the self-interactions are turned-on ($\lambda \neq 0$).

For future purposes, we introduce the definition for the pressure $p \equiv \lambda n^2/4m^2$, which satisfies the Gibbs-Duhem relation $dh = dp/n$, and the internal energy $U^n = \lambda n/4m^2$, which satisfies the local law $dU^n = -p d(1/n)$.

In summary, if all the previous quantities (21-24) are plugged into Euler’s equation (20), we obtain

$$-\frac{\omega_0}{m}\nabla^0 v_\alpha + v_\mu \nabla^\mu v_\alpha = F^E_\alpha + F^G_\alpha + F^Q_\alpha + F^n_\alpha. \tag{25}$$

Equations (2), (18), (19) and (25) are dynamically equivalent to the KGM equations. However, written in terms of the n and v_μ variables, they give rise to a different physical interpretation. They may be viewed as the generalized continuity, Hamilton-Jacobi and Euler hydrodynamic equations.

6 Balance equation

We now derive the different energy contributions for the system of charged bosons and the total energy balance equation. Notice that contracting equation (25) with nv_α and using the Leibniz rule we get

$$\nabla^\mu (v_\mu n K) - \frac{\omega_0}{m}\nabla^0 (nK) = nv^\mu (F^E_\mu + F^G_\mu + F^Q_\mu + F^n_\mu), \tag{26}$$

where $K \equiv (1/2)v_\alpha v^\alpha$ is defined as the kinetic energy per unit mass.

In order to express the electromagnetic contribution in terms of the symmetric energy–momentum tensor, it is convenient to introduce the current of charge

$$J^{E\mu} = \frac{e}{m}n \left(v^\mu - \frac{\omega_0}{m}\nabla^\mu t \right). \tag{27}$$

With this current, the Lorentz force (21) takes the form

$$F^E_\alpha = -\frac{1}{n}J^{E\mu} F_{\mu\alpha}. \tag{28}$$

Using the covariant Maxwell equations (2), after some tensor algebra and using the Jacobi identity of the Riemann tensor, we obtain

$$F^{E\alpha} = \frac{1}{n} \nabla_{\beta} \Theta^{\alpha\beta}, \quad (29)$$

where the symmetric electromagnetic energy–momentum tensor is defined as

$$\Theta^{\alpha\beta} \equiv g^{\alpha\mu} F_{\mu\nu} F^{\nu\beta} + \frac{1}{4} g^{\alpha\beta} F^{\mu\nu} F_{\mu\nu}. \quad (30)$$

Its components $\{\Theta^{00}, \Theta^{0i}\} = \{U^E, P_i^E\}$ are the generalized electromagnetic energy density and the Poynting vector, respectively.

On the other hand, using Eq. (27) the first term on the r.h.s. of Eq. (26) reads

$$nv^{\alpha} F^E_{\alpha} = \left(\frac{m}{e} J^{E\alpha} + \frac{\omega_0}{m} n \nabla^{\alpha} t \right) F^E_{\alpha}. \quad (31)$$

The antisymmetry of the Faraday tensor implies that $J^{E\alpha} F^E_{\alpha} \propto J^{E\alpha} J^{E\mu} F_{\mu\alpha} = 0$, which leads us to obtain from Eqs. (29) and (31) the following:

$$nv^{\alpha} F^E_{\alpha} = -\frac{\omega_0}{m} n F^{E0} = -\frac{\omega_0}{m} \nabla_{\alpha} \Theta^{\alpha 0}. \quad (32)$$

Since Eq. (32) is a gauge invariant expression, the tensor structure of our results below will not be affected by the gauge fixing procedure used on the electromagnetic sector.

Now, using the continuity equation (18) and the Leibniz rule, it is easy to get the following relation for an arbitrary field U :

$$-\frac{\omega_0}{m} \nabla^0 (nU) + \nabla^{\mu} (v_{\mu} nU) - nv_{\mu} \nabla^{\mu} U + \frac{\omega_0}{m} n \nabla^0 U + \frac{\omega_0}{m} n U \square t = 0, \quad (33)$$

which can be applied to the internal energy U^n , the quantum potential U^Q and the gravitational contribution U^G .

A further simplification is made by noting that

$$n \nabla^0 U^Q = -\frac{1}{4m^2} \nabla_{\mu} \left[n \nabla^0 (\nabla^{\mu} \ln n) \right]. \quad (34)$$

This relation, along with Eq. (33), leads to

$$-\frac{\omega_0}{m} \nabla^0 (nU^Q) + \nabla^{\mu} (nv_{\mu} U^Q + J^Q_{\mu}) - nv_{\mu} \nabla^{\mu} U^Q + \frac{\omega_0}{m} n U^Q \square t = 0, \quad (35)$$

where we have defined the *quantum flux* as

$$J^Q_{\mu} \equiv -\frac{\omega_0}{4m^3} n \nabla_{\mu} (\nabla^0 \ln n). \quad (36)$$

If we sum Eqs. (26) and (33) for U^n and U^G , and Eq. (35), we obtain

$$\begin{aligned}
 &-\frac{\omega_0}{m} \nabla^0(n\mathcal{U}^s) + \nabla^\mu(nv_\mu\mathcal{U}^s) + \frac{\omega_0}{m} \nabla^i P^E{}_i + \frac{\omega_0}{m} \nabla^0 U^E \\
 &+ \nabla^\mu \left(J^Q{}_\mu + pv_\mu \right) + \frac{\omega_0}{m} n \nabla^0 U^G + nv^\mu \nabla_\mu U^G \\
 &+ \frac{\omega_0}{m} n \left(U^G + U^Q \right) \square_t = 0,
 \end{aligned} \tag{37}$$

where we have introduced the energy density of the scalar field, $\mathcal{U}^s \equiv K + U^G + U^Q + U^n$. Equation (37) is the total energy balance equation. The total flux associated with the energy density of the scalar field \mathcal{U}^s involves the energy flux $nv_\mu\mathcal{U}^s$, the quantum flux $J^Q{}_\mu$, the pressure flux pv_μ , and a contribution due to the gravitational interaction U^G . The flux associated with the electromagnetic energy density U^E is the Poynting vector $P^E{}_i$.

7 The energy–momentum tensor

By construction, in general relativity, the energy and momentum densities of matter are usually defined as the time-time and time-space of the expected value of the energy momentum tensor laying at the r.h.s. of Einstein’s equations. The goal of this section is to find out a relation between such components of the energy–momentum tensor and the different so-called “energy” contributions appearing in the balance equation for the boson gas studied here.

The energy–momentum tensor for a Bose gas reads

$$\begin{aligned}
 T_{\mu\nu} &= T_{\mu\nu}^\Phi + T_{\mu\nu}^A \\
 &= \frac{1}{2} \left[(\Phi_{,\mu} + iA_\mu) (\Phi_{,\nu}^* - iA_\nu) + (\Phi_{,\mu}^* - iA_\mu) (\Phi_{,\nu} + iA_\nu) \right. \\
 &\quad \left. - g_{\mu\nu} \left((\Phi_{,\sigma} + iA_\sigma) (\Phi_{,\sigma}^* - iA^\sigma) + V \right) \right].
 \end{aligned} \tag{38}$$

As we shall see shortly, for the sake of clearness, we split it in two terms:

$$\begin{aligned}
 T_{\mu\nu}^v &= n \left[(\ln \sqrt{n})_{,\mu} (\ln \sqrt{n})_{,\nu} - \frac{1}{2} g_{\mu\nu} (\ln \sqrt{n})_{,\sigma} (\ln \sqrt{n})^{,\sigma} \right. \\
 &\quad \left. + (\theta_{,\mu} + eA_\mu) (\theta_{,\nu} + eA_\nu) \right. \\
 &\quad \left. - \frac{1}{2} g_{\mu\nu} \left((\theta_{,\sigma} + eA_\sigma) (\theta^{,\sigma} + eA^\sigma) + \frac{V}{n} \right) \right],
 \end{aligned} \tag{39}$$

and

$$\begin{aligned}
 T_{\mu\nu}^\theta &= e\sqrt{n} \left[\left((\ln \sqrt{n})_{,\mu} A_\nu + (\ln \sqrt{n})_{,\nu} A_\mu - g_{\mu\nu} (\ln \sqrt{n})_{,\sigma} A^\sigma \right) \sin \theta \right. \\
 &\quad \left. + (\theta_{,\mu} A_\nu + \theta_{,\nu} A_\mu - g_{\mu\nu} \theta_{,\sigma} A^\sigma) (\cos \theta - \sqrt{n}) \right].
 \end{aligned} \tag{40}$$

Rewritten in terms of the velocities v_μ , Eq. (39) transforms into

$$\begin{aligned}
 T_{\mu\nu}^v &= n \left[(\ln \sqrt{n})_{,\mu} (\ln \sqrt{n})_{,\nu} - \frac{1}{2} g_{\mu\nu} (\ln \sqrt{n})_{,\sigma} (\ln \sqrt{n})^{,\sigma} \right. \\
 &\quad \left. - \frac{1}{2} g_{\mu\nu} \left(m^2 v_\sigma v^\sigma + \frac{V}{n} \right) \right. \\
 &\quad \left. + m^2 v_\mu v_\nu - \omega_0 \left(m v_\mu \delta_\nu^0 + m v_\nu \delta_\mu^0 + \omega_0 \delta_\mu^0 \delta_\nu^0 - g_{\mu\nu} \left(m v^0 - \frac{\omega_0}{2N^2} \right) \right) \right].
 \end{aligned}
 \tag{41}$$

At this point, the motivation to decompose $T_{\mu\nu}$ into $T_{\mu\nu}^v$ and $T_{\mu\nu}^\theta$ becomes clear, since it is not possible to write $T_{\mu\nu}^\theta$ in terms of the velocity alone:

$$\begin{aligned}
 T_{\mu\nu}^\theta &= e\sqrt{n} \left[\left((\ln \sqrt{n})_{,\mu} A_\nu + (\ln \sqrt{n})_{,\nu} A_\mu - g_{\mu\nu} (\ln \sqrt{n})_{,\sigma} A^\sigma \right) \sin \theta \right. \\
 &\quad \left. + \left(m \left(v_\mu A_\nu + v_\nu A_\mu - g_{\mu\nu} v_\sigma A^\sigma \right) - \omega_0 \left(\delta_\mu^0 A_\nu + \delta_\nu^0 A_\mu - g_{\mu\nu} A^0 \right) \right) \right. \\
 &\quad \left. - 2e \left(A_\mu A_\nu + \frac{1}{2} g_{\mu\nu} A_\sigma A^\sigma \right) (\cos \theta - \sqrt{n}) \right].
 \end{aligned}
 \tag{42}$$

Let us introduce the unit normal 4-vector n^μ to the spatial hypersurfaces, given by $n^\mu = \frac{1}{N}(1, -N^i)$ and $n_\mu = (-N, 0, 0, 0)$, such that $n_\mu n^\mu = -1$. Let us recall that N is the lapse function and N the shift vector of the 3 + 1 foliation (see Eq. (6)). Notice that this unit normal vector corresponds to the 4-velocity of the Eulerian observers. Additionally, let us introduce $h_i^\mu = \delta_i^\mu + n_i n^\mu$ which is a projector tensor. In our case, it corresponds to $h_i^\mu = \delta_i^\mu$. Thus, in terms of these quantities, the scalar field density is defined as

$$\begin{aligned}
 \rho^v &= n_\mu n_\nu T^{\Phi\mu\nu} \\
 &= n \left[N^2 \left(\nabla^0 \ln \sqrt{n} \right)^2 + \frac{1}{2} (\ln \sqrt{n})_{,\sigma} (\ln \sqrt{n})^{,\sigma} + \frac{V}{2n} \right. \\
 &\quad \left. + m^2 N^2 \left(v^0 \right)^2 + \frac{1}{2} m^2 v_\sigma v^\sigma + \omega_0 \left(m v^0 - \frac{\omega_0}{2N^2} \right) \right],
 \end{aligned}
 \tag{43}$$

and

$$\begin{aligned}
 \rho^\theta &= n_\mu n_\nu T^{A\mu\nu} \\
 &= e \left[N^2 \left(\nabla^0 \sqrt{n} \right) A^0 + (\sqrt{n})_{,k} A^k \right] \sin \theta + e\sqrt{n} \left[m \left(N^2 v^0 A^0 + v_k A^k \right) \right. \\
 &\quad \left. - \omega_0 \left(1 - \frac{2}{N} \right) A^0 - e \left(N^2 \left(A^0 \right)^2 + A_k A^k \right) (\cos \theta - \sqrt{n}) \right].
 \end{aligned}
 \tag{44}$$

In the same way, we can obtain the fluxes

$$\begin{aligned}
 J_i^v &= n^\mu h_i^v T_{v\mu}^\Phi \\
 &= n \left[-N \left(\nabla^0 \ln \sqrt{n} \right) (\ln \sqrt{n})_{,i} - m^2 N v^0 v_i - \frac{\omega_0 m v_i}{N} \right], \tag{45} \\
 S_{ij}^v &= h_i^v h_j^\mu T_{\mu v}^\Phi \\
 &= n \left[(\ln \sqrt{n})_{,i} (\ln \sqrt{n})_{,j} - \frac{1}{2} \gamma_{ij} (\ln \sqrt{n})_{,\sigma} (\ln \sqrt{n})^{,\sigma} - \frac{1}{2} \gamma_{ij} v_\sigma v^\sigma \right. \\
 &\quad \left. - \frac{1}{2} \gamma_{ij} \left(m^2 + \frac{\lambda}{2} n \right) + m^2 v_i v_j + \gamma_{ij} \omega_0 \left(m v^0 - \frac{\omega_0}{2N^2} \right) \right], \tag{46}
 \end{aligned}$$

and

$$\begin{aligned}
 J_i^\theta &= n^\mu h_i^v T_{v\mu}^A \\
 &= -eN \left[\left(\nabla^0 \sqrt{n} A_i + (\sqrt{n})_{,i} A^0 \right) \sin \theta \right. \\
 &\quad \left. + \sqrt{n} \left(m \left(v^0 A_i + v_i A^0 \right) + \frac{\omega_0}{N^2} A_i + \frac{2e}{N} A^0 A_i \right) (\cos \theta - \sqrt{n}) \right], \tag{47} \\
 S_{ij}^\theta &= h_i^\mu h_j^v T_{\mu v}^A \\
 &= \left[\left((\sqrt{n})_{,i} A_j + (\sqrt{n})_{,j} A_i - \gamma_{ij} (\sqrt{n})_{,\sigma} A^\sigma \right) \sin \theta \right. \\
 &\quad \left. + \sqrt{n} e \left(m \left(v_i A_j + v_j A_i - \gamma_{ij} v_\sigma v^\sigma \right) \right. \right. \\
 &\quad \left. \left. + \gamma_{ij} \omega_0 A^0 - 2e \left(A_i A_j - \frac{1}{2} \gamma_{ij} A_\sigma A^\sigma \right) \right) (\cos \theta - \sqrt{n}) \right]. \tag{48}
 \end{aligned}$$

The main goal of this section is to figure out the relation of the different energy contributions in the balance equation (37) to the actual densities of energy and momentum derived from the energy–momentum tensor. The motivation of doing so is to give physical meaning to such quantities. For convenience, we write the expressions above in terms of the energy potentials in the balance equation (37). In order to do so, notice that the Bernoulli equation (19) leads to

$$\frac{\omega_0}{m} v^0 = K + 2U^n + 2U^G + U^Q + \frac{1}{2}. \tag{49}$$

In addition, notice that the following relations are satisfied

$$(\ln \sqrt{n})_{,\sigma} (\ln \sqrt{n})^{,\sigma} = -N^2 \left(\nabla^0 \ln \sqrt{n} \right)^2 + \frac{n_{,k} n^{,k}}{4n^2}, \tag{50}$$

$$2m^2 U^Q = -\frac{\square n}{2n} + (\ln \sqrt{n})_{,\sigma} (\ln \sqrt{n})^{,\sigma}, \tag{51}$$

which can be used to write the actual energy density as

$$\rho^v = nm^2 \left(1 + 2\tilde{U} \right) - \frac{n}{4} \square \ln n - \left[U^n - 2U^Q + \frac{1}{4U^G} \left(K + \frac{1}{2} \right)^2 \right], \tag{52}$$

where we have defined $\tilde{U} \equiv K + 2U^n + 2U^G + U^Q$. We can follow a similar procedure in order to relate the fluxes to the energy potentials:

$$S_{ij}^v = \frac{n_i n_j}{4n} - \frac{1}{2} \gamma_{ij} \square n + m^2 n \left[v_i v_j - \gamma_{ij} \left(U^n + 4U^G \right) \right]. \tag{53}$$

In order to lay out an interpretation for the relations above notice that (52), which corresponds to the energy density of the system according to the ADM formalism, is related to the rest mass of the gas and the total energy. It is worth to mention that the influence of the rest of the components could be important. Equation (52) shows that the contribution of each class of energy is non-trivial and that the energy–momentum tensor contains a number of components whose relation with physical conserved quantities is not explicit a priori.

8 Newtonian limit

In the longitudinal Newtonian gauge in flat space the interval is given by

$$ds^2 = -(1 + 2\varphi)dt^2 + \delta_{ij}(1 - 2\varphi) dx^i dx^j, \tag{54}$$

which implies that the lapse function is $N^2 = 1 + 2\varphi$, and $\gamma_{ij} = (1 - 2\varphi)\delta_{ij}$. Within the non-relativistic limit, Eq. (8) becomes the traditional GP equation [52]

$$i \frac{\partial \Psi}{\partial t} = -\frac{1}{2m} (\nabla - ie\mathbf{A})^2 \Psi + \frac{\lambda}{m} |\Psi|^2 \Psi + m\varphi\Psi + e\varphi_E\Psi \tag{55}$$

and the hydrodynamic Eqs. (18), (19) and (25) reduce to

$$\partial_t n + \nabla \cdot (n\mathbf{v}) = 0, \tag{56}$$

$$\partial_t S + \frac{1}{2m} (\nabla S - e\mathbf{A})^2 = -m \left(U^Q + h + \varphi + \frac{e}{m} \varphi_E \right), \tag{57}$$

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla U^Q - \frac{1}{n} \nabla p - \nabla \varphi + \frac{e}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{58}$$

where $U^Q = -(1/2m^2)\Delta\sqrt{n}/\sqrt{n}$ is the Madelung classical quantum potential [53] and $(\mathbf{E}, \mathbf{B}) = (-\partial_t \mathbf{A} - \nabla \varphi_E, \nabla \times \mathbf{A})$ is the electromagnetic field.

Taking the scalar product of Eq. (58) with \mathbf{v} and using the continuity equation (56), we obtain

$$\partial_t (nK) + \nabla \cdot (nK\mathbf{v}) = -n\mathbf{v} \cdot \nabla U^Q - \mathbf{v} \cdot \nabla p - n\mathbf{v} \cdot \nabla \varphi + \mathbf{J}^E \cdot \mathbf{E}, \tag{59}$$

where $K = v^2/2$ is the density of kinetic energy and we have introduced the current of charge $\mathbf{J}^E = (e/m)n\mathbf{v}$. From the Maxwell equations $\nabla \times \mathbf{E} = -\partial_t \mathbf{B}$ and $\nabla \times \mathbf{B} = \mathbf{J}^E + \partial_t \mathbf{E}$, using the identity $\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B})$, one can show that $\mathbf{J}^E \cdot \mathbf{E} = -\partial_t U^E - \nabla \cdot \mathbf{P}^E$, where $U^E = (E^2 + B^2)/2$ is the electromagnetic energy and $\mathbf{P}^E = \mathbf{E} \times \mathbf{B}$ is the Poynting vector. From the continuity equation (56), we get

$$\partial_t(nU) + \nabla(nU\mathbf{v}) = n\partial_t U + n\mathbf{v} \cdot \nabla U. \tag{60}$$

Applying Eq. (60) to U^Q , U^n and $U^G = \varphi/2$, using the identity

$$n\partial_t U^Q = -\nabla \cdot \mathbf{J}^Q; \quad \mathbf{J}^Q = \frac{1}{4m^2}n \partial_t \nabla \ln n, \tag{61}$$

and introducing the energy density of the scalar field $\mathcal{U}^s = K + U^Q + U^n + U^G$, we obtain the local energy conservation equation

$$\begin{aligned} &\partial_t(n\mathcal{U}^s) + \nabla \cdot (n\mathcal{U}^s\mathbf{v}) + \partial_t U^E + \nabla \cdot \mathbf{P}^E + \nabla \cdot (p\mathbf{v}) \\ &+ \nabla \cdot \mathbf{J}^Q + \frac{1}{2}n\mathbf{v} \cdot \nabla \varphi - \frac{1}{2}n\partial_t \varphi = 0. \end{aligned} \tag{62}$$

Using the Poisson equation, $\nabla^2 \varphi = 4\pi Gm^2n$, and the continuity Eq. (56), one can easily show that (62) implies the global conservation of the total (scalar field + electromagnetic) energy contribution, $E_{\text{tot}} = \int (n\mathcal{U}^s + U^E) \, d\mathbf{x}$, i.e. $\dot{E}_{\text{tot}} = 0$. This equation relates different contributions of the energy of a set of bosons laying within a gravitational potential well deforming the Minkowski space-time. This approximation suffices to describe a host of astrophysical situations.

9 Relativistic first-order correction in time to the Newtonian approach

As shown in the previous section, the Newtonian limit of the KGM system in its hydrodynamical representation is obtained by choosing a particular foliation of the space-time, meanwhile terms of higher order in powers of v/c are neglected. In this section, we present an application of our generalized GP equation and its corresponding hydrodynamic representation in order to illustrate how a first-order correction in powers of v/c gives rise to new features of the different components in the balance equation. By now, we ignore the electromagnetic fields in order to focus on the effects due to general relativity. The modified behavior of the solutions presented here carries up new physics inherited by the general relativistic nature of the underlying KG system from which this limit is obtained.

In this section, we present two simple examples in the weak gravity and diluted density limits. Firstly, the simplest planar-wave-like solution and, secondly, we consider the simplest spherically symmetric solution assuming a uniform sphere of bosonic gas. Both cases are simple academic examples that pretend to illustrate the behavior of the potentials appearing in the balance equation.

9.1 Weak gravity limit

Let us start by assuming that the gravitational potential well generated by the boson gas is weak enough, i.e. $\varphi(t, r) \ll 1$, such that the space-time is Minkowskian with a good approximation. In this case the constraint Einstein equations reduce to the Poisson equation given by

$$\nabla^2 \varphi = 4\pi G V (\Phi \Phi^*). \tag{63}$$

By now, let us assume the simplest scenario in which the scalar configuration is made of a homogeneous gas configuration with $n = \Phi_0^2 = \text{constant}$. We start with a simple ansatz, by considering the case of a planar wave solution for Φ , i.e.

$$\Phi = \Phi_0 \exp(ik_\mu x^\mu) = \Phi_0 \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]. \tag{64}$$

At leading order, the KG equation (1) reduces to a dispersion relation

$$\omega^2 - k^2 - (m^2 + \lambda n) = 0, \tag{65}$$

for k and ω real constants and the corresponding Poisson equation (63) turns into

$$\nabla^2 \varphi = 4\pi G \left(m^2 + \frac{\lambda}{2} n \right) n. \tag{66}$$

Equation (65) provides a dispersion relation for a free plane-wave. At this point, it is worth to notice that a massless-boson-like behavior, $k = \omega$, might be obtained if the density of the boson gas satisfies $n = -m^2/\lambda$. Such situation is only possible for $\lambda < 0$, that is, for an attractive self-interaction. Under such conditions, the gravitational potential in spherical coordinates for the bosonic system is given by

$$\varphi(r) = \frac{1}{3}\pi G m^2 n r^2 - \frac{C_1}{r} + C_2, \tag{67}$$

where r is the radial coordinate. In this case, the velocity in the hydrodynamic formulation is given by $mv_\mu = \omega_0 \delta_\mu^0$, since $S = k_\mu x^\mu + \omega_0 t$. After making the previous assumptions, the kinetic, gravitational and internal energy potentials are given by

$$K = -\frac{\omega_0^2}{2m^2} (1 - 2\varphi), \tag{68}$$

$$U^G = -\frac{\omega_0^2}{4m^2} (1 - 2\varphi) = \frac{K}{2}, \tag{69}$$

$$U^n = -\frac{1}{4}. \tag{70}$$

The behavior of the scalar solution is linked to the evolution of the potentials above. Since Φ is a periodic function of time and space, the kinetic and gravitational potentials also are. As a consequence, these contributions should compensate each other into the

balance equation. On the other hand, U^n is constant. This is expected because we are dealing with an almost flat space-time.

9.2 Diluted density limit

Another simple case that might be relevant for different applications in astrophysics is to consider spherically symmetric solutions of the KG equation in a weak gravity regime $N \approx 1$. We also assume here that $\lambda = 0$. Consider that the density of the bosonic gas is diluted, so that $n = n(r, t) \ll 1$. Under such assumptions, the KG equation (1), using hydrodynamic variables, reduces to

$$\frac{\square\sqrt{n}}{\sqrt{n}} = \left[m^2 - (\mu - \omega_0)^2 \right] + i(\mu - \omega_0) \frac{n_t}{n}, \tag{71}$$

where the phase has been harmonically decomposed as $S = \mu t$; n_t, n_r denote derivatives of the density n with respect to time and radius, respectively. For consistency, we set $\mu = \omega_0$ in Eq. (71) to have a real solution for the density. This last equation is solved by the following radial profile:

$$n(t, r) = \frac{1}{r^2} \left[\sqrt{C_1^2 + C_2^2} + C_1 \sin(2kr) + C_2 \cos(2kr) \right] T(t). \tag{72}$$

We choose the integration constants in Eq. (72) in order to get a regular non-singular solution given by

$$n = n_0 \frac{\sin^2(kr)}{k^2 r^2} \cos^2(\omega t), \tag{73}$$

with the dispersion relation $\omega \equiv \sqrt{k^2 + m^2}$. Notice that the previous solution for the density is a modified version of the Newtonian solution given by $n = n_0 \sin^2(kr)/(kr)^2$, where a periodic time-dependence has arisen due to first-order relativistic corrections. At this point, the gravitational potential generated by such a solution for the scalar field can be computed by plugging the previous solution for n into Eq. (66), resulting in

$$\varphi = \frac{\varphi_0}{k^2} \left[\ln(2kr) + \frac{1}{2k} \sin(2kr) - \text{Ci}(2kr) \right] \cos^2(\omega t), \tag{74}$$

where $\varphi_0 \equiv 8\pi Gm^2 n_0^2$ and $\text{Ci}(x)$ is the cosine integral function given by

$$\text{Ci}(x) = \gamma + \ln(x) + \int_0^x \frac{\cos(y) - 1}{y} dy, \tag{75}$$

being $\gamma = 0.5772$.

It is worth to point out that the gravitational potential has a harmonic evolution in time in contrast to the Newtonian case which is static. However, the quantum potential of the bosonic system defined by Eq. (23) turns out to be constant, $U^Q = -1/2$, which

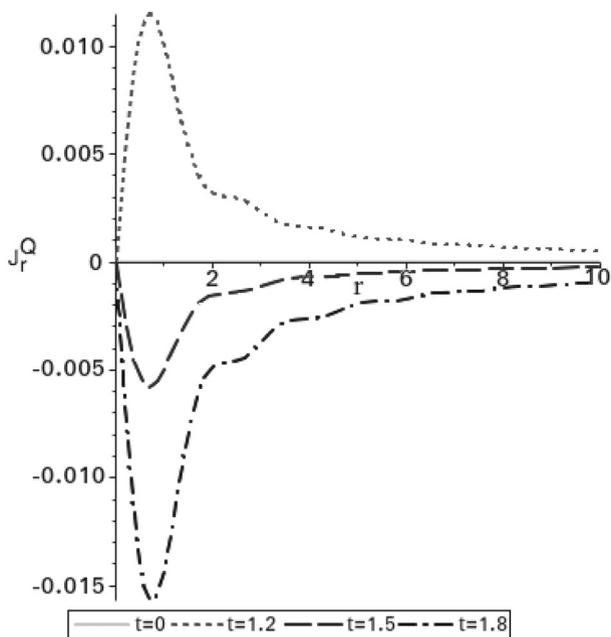


Fig. 1 Behavior of the radial component of the quantum flux. The values of the parameters used in this plot are: $\mu = 1$, $\omega = 0.1$, $k = 2$, $m = 1$, $n_0 = 1$

implies that the quantum force is equal to zero. Nevertheless, the quantum flux J_μ^Q (see Eq. (36)) evolves non-trivially and its components (assuming spherical symmetry) are given by

$$J_r^Q = \frac{(1 + \varphi) \omega}{4m^3 n} \left(\frac{n_t n_r}{n} - n_{tr} + n_t \varphi_r \right), \tag{76}$$

$$J_t^Q = \frac{(1 + \varphi) \omega}{4m^3 n} \left(\frac{n_t n_t}{n} - n_{tt} \right). \tag{77}$$

Snapshots of the time evolution and of the spatial structure of the components of the quantum flux are illustrated in Figs. 1 and 2.

On the other side, the Bernoulli equation, which is equivalent to the KG equation (71) after being expressed in hydrodynamic variables, is useful to compute the kinetic potential given by

$$2K = v_\mu v^\mu = -\frac{\omega^2}{m^2} (1 - \varphi), \tag{78}$$

which clearly evolves non-trivially in time (see Fig. 3) and has almost the same dependence on the gravitational potential as the kinetic potential for the planar wave. For comparison, Fig. 4 shows snapshots of the total energy contribution \mathcal{U}^S of the system.

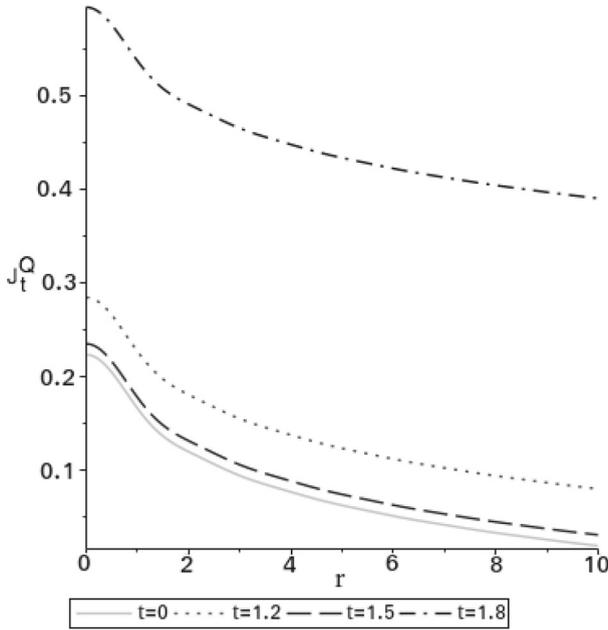


Fig. 2 Behavior of the time component of the quantum flux. The flux is bigger at the center of the system. The values of the parameters used in this plot are the same as in Fig. 1

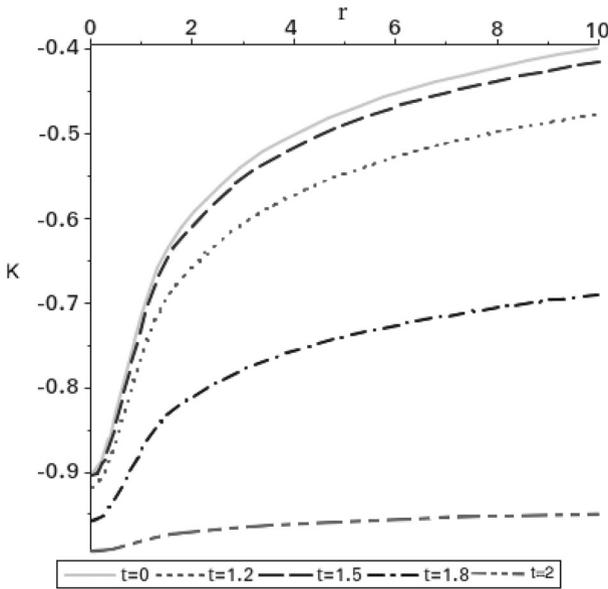


Fig. 3 Behavior of the kinetic potential K of the system. The potential is bigger at the outer side of the system. The values of the parameters used in this plot are the same as in Fig. 1

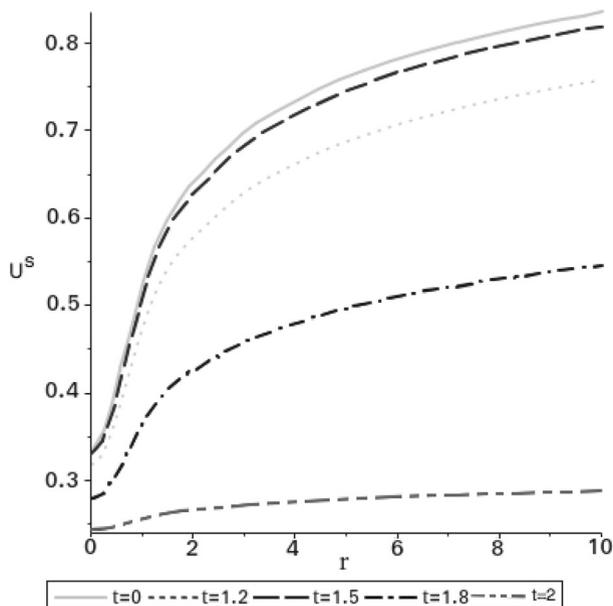


Fig. 4 Behavior of the total energy contribution U^S of the system. The total energy is bigger at the outer parts of the system. The values of the parameters used in this plot are the same as in Fig. 1

10 Conclusions

In this article, we have derived a generalized hydrodynamic formulation of a system of charged bosonic excitations laying in a curved space-time, which is governed by a general relativistic set of continuity, Hamilton–Jacobi, and Euler equations written up in Madelung variables, equivalent to the Klein–Gordon–Maxwell equations. By performing a $3 + 1$ foliation of the space-time we are able to handle curved geometries within this framework. We have shown that it is possible to split the total energy contribution of the hydrodynamical system associated to the boson gas into different contributions or potentials, specifically the kinetic, quantum and electromagnetic parts, and a term due to the gravitational field strength arising from the curvature of space-time. The main result of this article is the energy balance equation for the boson gas which plays the role of an hydrodynamical first law of thermodynamics for the system in the general relativistic regime. In addition, in order to relate the potentials involved in the balance equation, we compute some physical conserved quantities defined along the $3 + 1$ foliation such as the total energy and the projected momenta, which are written in terms of the energy–momentum tensor of the scalar and pure geometric entities associated with the foliation. In this way, we establish a mapping between the potentials and actual physical observables. It is worth remarking that this result is general and has not been derived before and it is an important result for models involving canonical scalar fields in astrophysics, such as models of dark matter and neutron stars. We believe that it is possible to carry out the same procedure by decomposing the matter equation for fermions, but it is beyond the scope of this

work. Finally, for illustrative purposes, we present a simple case of study consisting in a couple of bosonic systems—plane wave case and the spherically symmetric case – laying in flat space-time whose scalar equation is the Newtonian one plus a first order relativistic (post-Newtonian) correction which gives rise to non-static behavior of the potentials in the balance equation. The main aim of this work is to provide a general relativistic framework to study scalar field configurations in highly relativistic environments, such as scalar field dark matter laying in the vicinity of compact objects like black holes or neutron stars. Also this hydrodynamical framework is useful to handle the metric and field perturbations in a cosmological scenario in order to study structure formation. We have provided such a theoretical framework to be used in further research.

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