



# Hydrodynamic representation and energy balance for Dirac and Weyl fermions in curved space-times

Tonatiuh Matos<sup>1,a</sup>, Omar Gallegos<sup>1,b</sup> , Pierre-Henri Chavanis<sup>2,c</sup>

<sup>1</sup> Departamento de Física, Centro de Investigación y de Estudios Avanzados del IPN, A.P. 14-740, 07000 Mexico City, Mexico

<sup>2</sup> Laboratoire de Physique Théorique, Université de Toulouse, CNRS, UPS, Toulouse, France

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**Abstract** Using a generalized Madelung transformation, we derive the hydrodynamic representation of the Dirac equation in arbitrary curved space-times coupled to an electromagnetic field. We obtain Dirac–Euler equations for fermions involving a continuity equation and a first integral of the Bernoulli equation. Comparing between the Dirac and Klein–Gordon equations we obtain the balance equation for fermion particles. We also use the correspondence between fermions and bosons to derive the hydrodynamic representation of the Weyl equation which is a chiral form of the Dirac equation.

## 1 Introduction

The Standard Model of elementary particles establishes that there exist two kinds of particles, fermions and bosons. In previous works [1,2], the energy balance for bosons was derived starting from the general relativistic Klein–Gordon (KG) equation. In the present work, we study a system of fermions described by the Dirac equation in arbitrary curved space-times taking into account electromagnetic effects. We also use the Weyl equation which is a chiral form of the Dirac equation due to the relationship between the Lie algebras of the symmetry groups for both systems of particles. We give the hydrodynamic representation of the Dirac and Weyl equations for fermions using previous results obtained for boson particles. This representation is built analogously as in quantum mechanics (QM) and as in the bosonic case [1], where it was introduced by the Madelung transformation in order to find an alternative interpretation of a boson system. This interpretation has been very useful in astrophysics

[2]. In this article, we extend the previous transformation to the fermionic case, in the same way we pretend to give an alternative interpretation of the fermionic systems.

Many examples of fermion particles in strong gravitational fields can be found in nature. Indeed, the curvature of space-time plays an important role in a neutron star, in the early Universe, or in a fermion cloud (e.g. a dark matter halo) in the vicinity of a black hole. We need to develop a general framework to identify what are the different energy contributions in such systems. In this work we use the geometrical decomposition of the metric in 3+1 slices and the tetrad formalism to study the particle spin in an arbitrary space-time. We define the gamma matrices in curved space-times and derive the generalized Dirac and Weyl equations. Then, using the Madelung transformation, we introduce a hydrodynamic representation of the Dirac and Weyl spinors. This hydrodynamic representation can help us to describe the fermionic system in a general framework. We can highlight that this description is convenient because it is easier to make a physical interpretation, since the hydrodynamic representation is given in some variable such as number of particles, speed, potential or energy. In fact, a non-obvious result is the energy balance equation, which is the first law of thermodynamics, which comes from the Dirac equation with the Madelung transformation for spinors. Although the equations obtained from this representation are more complicated than in the usual way, it can help us to have a closer answer for interpretations of quantum theory, for example, the de Broglie–Bohm interpretation [3–5]. In addition, we can compare the hydrodynamics and energy balance in different frames for classical and quantum particles, as well as spin and spinless particles, such as bosons and fermions.

Gravitational effects on quantum fields have been rigorously studied for a few decades, particularly in the case of spinor fields. Standard books such as [6–9] delve into the mathematical structure of the spinor formalism. Spinor fields

<sup>a</sup> e-mail: [tonatiuh.matos@cinvestav.mx](mailto:tonatiuh.matos@cinvestav.mx)

<sup>b</sup> e-mail: [omar.gallegos@cinvestav.mx](mailto:omar.gallegos@cinvestav.mx) (corresponding author)

<sup>c</sup> e-mail: [chavanis@irsamc.ups-tlse.fr](mailto:chavanis@irsamc.ups-tlse.fr)

in curved space-times have been studied in several papers, and we make a brief review of these works. In [10] the authors develop the formalism of the Dirac equation in a curved space-time coupled to an electromagnetic field. In [11] the authors give the key to generalize the Dirac equation from flat space-time to general relativity via the tetrad formalism with the Lorentz invariant transformation. In [12, 13] the authors study the quantum mechanics of the hydrogen atom in a general relativistic context. In [12] the analog of the Stark effect is considered with the center of mass formalism. Paper [13] analyses the modifications in the eigenvalues of the energy spectrum that arise due to the curvature of space-time. Additionally, [14] compares the energy levels of neutrinos and electrons in a curved space-time with spherical symmetry, that is, the Schwarzschild metric. Moreover, the authors study thermodynamical processes and the creation of neutrino pairs. On the other hand, in paper [15] the authors write the Dirac and Weyl equations for neutrinos in a Kerr metric using the tetrad formalism and compare them with the results obtained in a spherical metric without rotation. We mentioned these references to place our work in a broader context. There are specific points that we shall discuss deeply in the next sections, one of them being the consistency conditions for the continuity equation. More information about the continuity equation can be found in [16–19].

This paper is organized as follows. In Sect. 2, we present the field equations and the formalism that we will use to describe the Dirac fermions in curved space-times. In Sect. 3, we introduce a generalized Madelung transformation for Dirac fermions, which implies a hydrodynamic representation for this case. Since, we can work using either the Dirac or Weyl representation for 1/2-spin fermions. In Sect. 4, we give a brief introduction to these both representations, further it is shown the field equations for the Weyl fermions (or the chiral form of the Dirac fermions). Analogously, for Weyl fermions we introduce the hydrodynamic representation from a generalized Madelung transformation in Sect. 5. For both kinds of fermions in Sect. 6, we explain what are the different contributions of the energy for a Fermi gas in a curved space-time coupled to an electromagnetic field and we show a generalized Gross–Pitaevskii equation for fermions. Moreover, the conclusions are indicated in Sect. 7. Finally, in Appendix A, we can find a solution for a simple example to the Dirac equation in a flat space-time.

## 2 Field equations

We start using the tetrad formalism for the space-time geometry, and the canonical expansion of the space-time in a 3+1 ADM decomposition [9, 20–24], such that the coordinate  $t$  is the parameter of evolution. The 3+1 metric reads

$$ds^2 = N^2 c^2 dt^2 - h_{ij} (dx^i + N^i c dt) (dx^j + N^j c dt), \quad (1)$$

where  $N$  represents the lapse function which measures the proper time of the observers traveling along the world line,  $N^i$  is the shift vector that measures the displacement of the observers between the spatial slices and  $h_{ij}$  is the 3-dimensional slice-metric. In what follows  $i, j, k, l = 1, 2, 3$  are the spatial indices;  $a, b, c = 0, 1, 2, 3$  and  $\mu, \nu, \alpha = 0, 1, 2, 3$  the space-time indices. We write Eq. (1) in the tetrad formalism as  $ds^2 = \eta_{ab} e^a{}_\mu e^b{}_\nu dx^\mu dx^\nu$ , where  $\eta_{ab} = \text{diag}(1, -1, -1, -1)$ . Here  $e^a = e^a{}_\mu dx^\mu$  is the set of one-forms base of the cotangent space at the space-time manifold given by

$$\begin{aligned} e^0 &= N c dt, \\ e^k &= \hat{e}^k{}_i (dx^i + N^i c dt), \end{aligned} \quad (2)$$

with inverse

$$\begin{aligned} e_0 &= \frac{1}{N} \left( \frac{\partial}{c \partial t} - N^j \frac{\partial}{\partial x^j} \right), \\ e_k &= \hat{e}_k{}^j \frac{\partial}{\partial x^j}, \end{aligned} \quad (3)$$

where  $\hat{e}^k = \hat{e}^k{}_i dx^i$  are the one-form base to the three-dimensional slice of the cotangent manifold, such that  $h_{ij} = \delta_{kl} \hat{e}^k{}_i \hat{e}^l{}_j$ . We can also define the set of vectors base of the tangent-space to the space-time as  $e_a = e_a{}^\mu \partial_\mu$ , such that  $e^a e_b = \delta^a_b$ . We will use the tetrad formalism [7, 9, 22–25] to describe the space-time geometry where the fermion particles are located.

The action of a fermion system in curved space-times coupled to an electromagnetic field  $A_\mu$  is given by  $S[\psi(x^\mu), \partial_\mu \psi(x^\mu)] = \int \mathcal{L}(\psi(x^\mu), \partial_\mu \psi(x^\mu)) d^4x$ , where  $\mathcal{L} = \mathcal{L}(\psi(x^\mu), \partial_\mu \psi(x^\mu))$  is the Lagrangian density [16–18]:

$$\begin{aligned} \mathcal{L} &= \sqrt{-g} \frac{i\hbar c}{2} \left[ \psi^\dagger B \gamma^\mu (D_\mu \psi) - (D_\mu \psi)^\dagger B \gamma^\mu \psi \right. \\ &\quad \left. + \frac{2imc}{\hbar} \psi^\dagger B \psi \right]. \end{aligned} \quad (4)$$

Here,  $D_\mu = \nabla_\mu + \frac{iq}{\hbar c} A_\mu$  is the total covariant derivative accounting for electromagnetic effects. The covariant derivative of a spinor  $\psi = (\psi_{\dot{\nu}})$  is given by  $\nabla_\mu (\psi_{\dot{\nu}}) = \partial_\mu (\psi_{\dot{\nu}}) + \Gamma_{\mu\dot{\nu}}^{\dot{\alpha}} (\psi_{\dot{\alpha}})$ , where  $\Gamma_{\mu\dot{\nu}}^{\dot{\alpha}}$  is the spin connection [9, 26]. Observe the internal indices as dot indices. Using the least action principle it is possible to obtain from Eq. (4) the corresponding Dirac equation. This equation is given by

$$[i\hbar \gamma^\mu (\nabla_\mu + iq A_\mu) - mc] \psi = 0, \quad (5)$$

where  $\hbar, c$  are the Planck constant and the speed of light respectively, while  $q, m$  are the charge and mass of the

fermion particle and  $\psi$  is its spinor. Besides, the gamma matrices  $\gamma^\mu$  are related to the spin and space-time geometry. They can be written as  $\gamma^\mu = e^\mu_a \tilde{\gamma}^a$ , where  $\tilde{\gamma}^a$  are the gamma matrices in flat space-time, which are well-know from standard Quantum Field Theory (QFT) [27–29] Henceforth, to simplify the notation, we use the natural units ( $c = \hbar = 1$ ), instance,  $mc/\hbar \rightarrow m$ . Therefore,

$$\begin{aligned} \gamma^0 &= N\tilde{\gamma}^0, \\ \gamma^k &= \tilde{e}^k_j(\tilde{\gamma}^j + N^j\tilde{\gamma}^0). \end{aligned} \tag{6}$$

In general, these matrices fulfill the following anti-commutation relation [6,9]

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbb{I}, \tag{7}$$

where  $g_{\mu\nu}$  represents the metric that describes the space-time geometry. Furthermore, as we know, the gamma matrices in flat space-time are related to the Pauli matrices, which describe the spin of the fermion particles. In addition, due to the Lorentz invariance that spinors follow, we note that  $\psi\psi^\dagger$  is not a Lorentz scalar and neither  $\psi\gamma^\mu\psi^\dagger$  is a Hermitian. On the other hand, we observe that, in general, the gamma matrices obey the following relation [16–19]

$$(\gamma^\mu)^\dagger = B\gamma^\mu B^{-1}, \tag{8}$$

where  $B$  is a hermitian matrix, i.e.  $B^\dagger = B$ , that is uniquely determined by the gamma matrices  $\gamma^\mu$ . As usual, we denote by  $B^\dagger$  the conjugate (or Hermitian) transpose of  $B$ . In contrast, using Eq. (8) it is straightforward to observe the invariant quantities under the Lorentz transformation are  $\psi\bar{\psi}$  as scalar and  $\psi\gamma^\mu\bar{\psi}$  as a four-vector, where  $\bar{\psi} = \psi^\dagger B$  is named the adjoint spinor (see more in [6,7,9,27]).

Furthermore, we note that in QFT the relation (8) is fulfilled when  $B = \tilde{\gamma}^0$  and the gamma matrices are in flat space-time. From the action (4) of the fermion system we can find the equation for the transpose conjugated spinor by making an infinitesimal variation of this action with respect to  $\psi$ . Another way of getting this equation of motion is to take the transpose conjugate of the Dirac equation (5) and using (8). In this manner we find that the transpose conjugated Dirac equation in curved space-time is given by

$$i(\nabla_\mu\bar{\psi})\gamma^\mu - i\psi^\dagger\nabla_\mu(B\gamma^\mu) + i\bar{\psi}\nabla_\mu\gamma^\mu + \bar{\psi}A_\mu\gamma^\mu + m\bar{\psi} = 0. \tag{9}$$

We consider  $(\nabla_\mu\psi)^\dagger = \nabla_\mu\psi^\dagger$  and denote the adjoint spinor as  $\bar{\psi} = \psi^\dagger B$ . Using the gamma matrices in flat space-time and the fact that  $B = \tilde{\gamma}^0$ , we recover the definition of  $\bar{\psi}$  in QFT and the transpose conjugated Dirac equation. However, in an arbitrary space-time  $\nabla_\mu\gamma^\mu$  is distinct from zero, since  $\gamma^\mu = e^\mu_a\tilde{\gamma}^a$ . Therefore, in general  $\nabla_\mu e^\mu_a$  is non-zero.

We can get the conserved charge from the Noether theorem [30]. The Dirac current is

$$J^\mu = \bar{\psi}\gamma^\mu\psi = \psi^\dagger B\gamma^\mu\psi. \tag{10}$$

To obtain the continuity equation

$$\nabla_\mu J^\mu = 0, \tag{11}$$

for the Dirac current, we take the covariant derivative of Eq. (10). This gives

$$\nabla_\mu J^\mu = (\nabla_\mu\bar{\psi})\gamma^\mu\psi + \bar{\psi}(\nabla_\mu\gamma^\mu)\psi + \bar{\psi}\gamma^\mu\nabla_\mu\psi. \tag{12}$$

If we multiply the Dirac equation (5) by  $\bar{\psi}$  and its transpose conjugate (9) by  $\psi$  and sum both equations, it follows that

$$\nabla_\mu J^\mu = \psi^\dagger\nabla_\mu(B\gamma^\mu)\psi. \tag{13}$$

If we require that the continuity equation (11) is fulfilled, i.e., that the number of particles is conserved, then we need  $\nabla_\mu(B\gamma^\mu) = 0$ , or equivalently

$$(\nabla_\mu B)\gamma^\mu = -B\nabla_\mu\gamma^\mu. \tag{14}$$

At this point, we want to emphasize the consistency conditions for the continuity equation (11). Some authors in [14] impose  $\nabla_\mu\gamma^\nu = 0$  while others, [13], impose  $\nabla_\mu B = 0$ . These conditions are independent of each other. Instead, in references [17,18], the authors conclude that the condition  $\nabla_\mu(B\gamma^\nu) = 0$  is the most convenient because it is implied by  $\nabla_\mu\gamma^\nu = 0$  and  $\nabla_\mu B = 0$ .

In addition, we can note that the matrix  $B$  can be obtained for a general metric (1) by solving the differential equation

$$\left(\nabla_0(BN) + \nabla_j(B\tilde{e}^j_i N^i)\right)\tilde{\gamma}^0 - \nabla_j(B\tilde{e}^j_i)\tilde{\gamma}^i = 0, \tag{15}$$

which follows from Eq. (14). Using the condition (14), it is possible to rewrite the transpose conjugated Dirac equation (9) as

$$i(\nabla_\mu\bar{\psi})\gamma^\mu + i\bar{\psi}\nabla_\mu\gamma^\mu + \bar{\psi}A_\mu\gamma^\mu + m\bar{\psi} = 0. \tag{16}$$

In order to find the conserved quantity resulting from the continuity equation, we take an arbitrary surface  $\mathcal{S}$  enclosing the volume  $\mathcal{V}$  which contains the whole system. Let  $k^j$  be an orthonormal vector to  $\mathcal{S}$  such that

$$\int_{\mathcal{V}} \nabla_\mu J^\mu dV = \int_{\mathcal{V}} \nabla_0 J^0 dV + \int_{\mathcal{S}} k_j J^j \sqrt{h} d^3x = 0. \tag{17}$$

where  $h$  is the determinant of the slice-metric  $h_{ij}$ . We assume that far away from the source spinor  $\psi$  goes to zero, that means that in this region  $J^\mu$  is negligible. Then, the surface integral in Eq. (17) vanishes, and we obtain

$$\frac{dQ}{dt} = \int_{\mathcal{V}} \nabla_0 J^0 dV = 0, \tag{18}$$

where  $Q = \int_{\mathcal{V}} J^0 dV$  is the conserved charge,  $dV$  is the curved volumen element  $dV = \sqrt{-g}d^4x$ . In QFT this charge is identified with the number of fermions or with the electric charge of the system. In flat space-time we have  $B = \tilde{\gamma}^0$ , so that  $J^0 = \psi^\dagger\psi = n$  represents the number density of fermion particles. In curved space-time  $J^0$  (which is determined by  $\gamma^0$  and by the generalized gamma matrices) has a

different interpretation. The form of  $B$  given by Eqs. (8) and (14) for each metric is related to the gamma matrices and to the tetrad formalism.

Finally, since the spinor field used is coupled to an electromagnetic field, we show the equations that describe the electromagnetic field. Thus, with the Maxwell four-potential we can define the Faraday tensor

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu. \tag{19}$$

In the electromagnetic theory, the Faraday tensor  $F_{\mu\nu}$  satisfies the Maxwell field equations

$$\nabla_\nu F^{\nu\mu} = J^{E\mu}, \tag{20}$$

where  $J^{E\mu}$  is the four-electromagnetic current.

At this point, we gave the most general form for standard Dirac fermions in an arbitrary framework coupled to an electromagnetic field. In fact, for quantities like  $B$  and  $\gamma^\mu$  we have not yet adopted any representation. Nevertheless, we will have to make this decision to give some examples and results in the sections below.

### 3 Dirac hydrodynamic representation

Analogously to the hydrodynamic representation of the Schrödinger equation, which was introduced by Madelung [31], we derive the hydrodynamic representation of the Dirac equation. We carry out the following generalized Madelung transformation for each component of the spinor  $\psi = \psi(x^\mu)$  as follows

$$\psi = \exp(i\theta\mathbb{I})R, \tag{21}$$

where  $\mathbb{I}$  is the identity matrix,  $R$  is a spinor and  $\theta$  is a complex function. Observe that the spinor  $\psi$  has eight degrees of freedom and the spinor  $R \exp(i\theta\mathbb{I})$  has ten. A similar situation appeared for the case of the boson case, where the scalar field  $\Phi = \Psi \exp(i\theta)$  has two degrees of freedom and the right hand side has three. This extra degree of freedom is interpreted as the velocity potential. Here it will be a similar situation. In what follows we will denote  $\theta\mathbb{I} \rightarrow \theta$ , unless it is specify. For the case where we consider a Dirac electron-like fermion,  $\theta = \theta(x^\mu)$ , the spinor  $\psi$  reads

$$\psi = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix} \exp(i\theta) = R \exp(i\theta), \tag{22}$$

where we use the notation  $\dot{\mu}, \dot{\nu}, \dots = \dot{1}, \dots, \dot{4}$  for the spinor indices such that

$$R = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix} = \begin{pmatrix} \sqrt{n_1} \\ \sqrt{n_2} \\ \sqrt{n_3} \\ \sqrt{n_4} \end{pmatrix}. \tag{23}$$

where we will use  $n_{\dot{\mu}} = |R_{\dot{\mu}}|^2$ , here  $n_{\dot{\mu}}$  is the number density which represents the modulus of  $\psi_{\dot{\mu}}$  and  $\theta$  is its phase (both are complex variables). In general,  $n_{\dot{\mu}}$  is different for each component of the spinor. Note that the covariant derivative of the spinor  $\psi$  in terms of its decomposition (22) is  $\nabla_\mu(\psi_{\dot{\nu}}) = \partial_\mu(R_{\dot{\nu}}e^{i\theta}) + \Gamma_{\mu\dot{\nu}}^{\dot{\alpha}}(R_{\dot{\alpha}}e^{i\theta}) = (\partial_\mu R_{\dot{\nu}})e^{i\theta} + i(\partial_\mu\theta)R_{\dot{\nu}}e^{i\theta} + \Gamma_{\mu\dot{\nu}}^{\dot{\alpha}}(R_{\dot{\alpha}}e^{i\theta})$ , implying that  $\nabla_\mu\theta = \partial_\mu\theta$ . In the Appendix, we show some exact solutions of the Dirac equation with this ansatz in flat space-time.

Using the transformation (22) in Eq. (5), the Dirac equation in terms of the variables  $R$  and  $\theta$  reads

$$\exp(i\theta)\gamma^\mu \left( i\nabla_\mu R - (\nabla_\mu\theta)R - qA_\mu R - \frac{m}{4}\gamma_\mu R \right) = 0. \tag{24}$$

To get the last term, we used the property of the gamma matrices that  $\gamma_\mu\gamma^\mu = 4\mathbb{I}$ , where  $\mathbb{I}$  is the  $4 \times 4$  identity matrix. This property results from the anti-commutation relation of the gamma matrices.

Similarly, the continuity equation (11) with (10) can be written with these new variables as

$$\left( \nabla_\mu R^\dagger \right) K^\mu R + R^\dagger K^\mu (\nabla_\mu R) = 0, \tag{25}$$

where  $R^\dagger$  denotes the conjugated transpose of  $R$  and  $K^\mu = B\gamma^\mu$ . Observe that  $K^\mu$  is hermitian ( $K^{\mu\dagger} = K^\mu$ ).

Summarizing, we have introduced the Madelung transformation for the Dirac equation (24) and the continuity relation (25) by making the change of variables from Eq. (21). With this new form to write the Dirac equation, we can introduce variables that have a more plausible physical interpretation in quantum theory.

To see this, we apply the operator  $i\gamma^\mu D_\mu = i\gamma^\mu \nabla_\mu - q\gamma^\mu A_\mu$  to the Dirac equation (5) written under the form  $i\gamma^\mu \nabla_\mu \psi = q\gamma^\mu A_\mu \psi + m\psi$ . This yields

$$\begin{aligned} & -\gamma^\mu\gamma^\nu \left( \nabla_\mu \nabla_\nu \psi + iq(\nabla_\mu A_\nu)\psi + iqA_\nu(\nabla_\mu \psi) \right. \\ & \quad \left. + iqA_\mu(\nabla_\nu \psi) - q^2 A_\mu A_\nu \psi \right) \\ & - m^2 \psi - \gamma^\mu (\nabla_\mu \gamma^\nu) (\nabla_\nu \psi + iqA_\nu \psi) = 0. \end{aligned} \tag{26}$$

Using the relation (7) in Eq. (26), we obtain

$$\square_E \psi + m^2 \psi + \frac{i}{2} q \gamma^\mu \gamma^\nu F_{\mu\nu} \psi + \gamma^\mu (\nabla_\mu \gamma^\nu) (D_\nu \psi) = 0, \tag{27}$$



where we have defined the D’Alambertian operator in the presence of an electromagnetic field by  $\square_E = (\nabla_\mu + iqA_\mu)(\nabla^\mu + iqA^\mu)$  and the anti-symmetric Faraday tensor by  $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ . Equation (27) is similar to the Klein–Gordon equation with an electromagnetic source except that here  $\psi$  is a spinor instead of a complex scalar field. Note that the first two terms in (27) are the Klein–Gordon equation, but the the electromagnetic field and the spinorial character of the equation add two more terms. The difference here is that if you “square” the Dirac equation in flat space-time, you obtain the Klein–Gordon equation, for an arbitrary curved space this does not happen. The last term of Eq. (27) contains the covariant derivative of  $\gamma^\mu$  which vanishes in a flat space-time.

As for the Klein–Gordon equation [1,2], we define the diagonal matrix 4-velocity  $v_\mu$  by

$$mv_\mu = \nabla_\mu S + qA_\mu \mathbb{I}. \tag{28}$$

Here,  $S(x^\mu)$  is a phase with components  $S = (\theta - \omega t)\mathbb{I}$ , where  $\omega$  are constants that can be related to the mass of the fermion particle by  $\omega = mc^2/\hbar$ . In this manner we can write

$$\nabla_\mu \theta \mathbb{I} = mv_\mu - \omega \delta^0_\mu \mathbb{I} - qA_\mu \mathbb{I}. \tag{29}$$

We interpret  $n_{\dot{v}}$  as the density number of fermions and  $v_\mu$  as its velocity. In what follow we denote  $\omega \rightarrow \omega \mathbb{I}$  unless otherwise stated. Additionally, we will show that Eq. (24) can be interpreted as the first integral of the Bernoulli equation for fermions in an arbitrary space-time. For doing so, we will use this new interpretation using variables  $n_{\dot{v}}$  and  $v_\mu$  in the Dirac equation, instead of  $\psi$  in order to write a Navier–Stokes-like equation for fermions, in the same way a it was done for bosons in [1]. Then, we will see that Eq. (25) can be interpreted as the generalized first integral of the Bernoulli equation in the sense that, for obtaining the Navier–Stokes-like equation, we need to differentiate Eq. (24).

According to [1,2] if we apply the transformation (21) to Eq. (27), we could expect to obtain the continuity equation for the imaginary part and the Bernoulli equation for the real part. However, in the case of the Dirac equation, the four components are mixed by the presence of the four dimensional spinor  $\psi$ . Hence, we obtain the following expression

$$\begin{aligned} & i [2(mv^\mu - \omega \delta^0_\mu) \nabla_\mu R - qA_\mu + q \nabla_\mu (A^\mu R) \\ & + \nabla_\mu (mv^\mu - \omega \delta^0_\mu - qA^\mu) R] \\ & + \left( m^2 v_\mu v^\mu + 2m\omega v^0 + \frac{\omega^2}{N^2} + m^2 \right) R - \square R \\ & + \frac{i}{2} q \gamma^\mu \gamma^\nu F_{\mu\nu} R + \gamma^\mu (\nabla_\mu \gamma^\nu) \\ & (i(mv_\nu + \omega \nabla_\nu t) R + D_\nu R) = 0. \end{aligned} \tag{30}$$

Here, we have defined  $\square = \nabla^\nu \nabla_\nu$ . For bosons, the real and imaginary parts are separated into two independent equations, namely, the continuity equation and the Bernoulli equation

[1,2]. But in the spinor case, the last line of Eq. (30) mixes both the imaginary and real parts and there is no natural separation into real and imaginary parts. The system remains coupled.

### 4 Weyl representation

The Dirac equation for 1/2-spin particles is associated with the  $SO(1, 3)$  symmetry group. Nevertheless, we can introduce a new representation as in standard QFT, since there exists a surjective homomorphism between the  $SO(1, 3)$  and  $SU(2) \otimes SU(2)$  Lie groups.

As we know, the special unitary group  $SU(2)$  is formed by the set of  $2 \times 2$  complex matrices  $A$ , which satisfy  $\det(A) = 1$ . Explicitly, we have

$$A = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}, \tag{31}$$

with  $\det(A) = |a|^2 + |b|^2 = 1$ , where  $a$  and  $b$  are complex parameters. Equivalently, we have the identity  $A^\dagger = A^{-1}$ .

The Lie algebra  $\mathfrak{su}(2)$  associated to the  $SU(2)$  Lie group is given by the exponential map

$$\exp(\mathfrak{su}(2)) \rightarrow SU(2). \tag{32}$$

For any element  $X$  of the Lie algebra, we have  $\exp(X) \exp(X)^\dagger = \mathbb{I}$ , implying that  $X + X^\dagger = 0$ . In what follows, we will indistinctly use  $\exp(X)$  and  $e^X$  as the exponential map.

In terms of the Pauli matrices  $\sigma^\mu$  the  $4 \times 4$  gamma matrices  $\gamma^\mu$  can be written as two  $2 \times 2$  block matrices

$$\gamma^0 = N \tilde{\gamma}^0 = N \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \tag{33}$$

$$\begin{aligned} \gamma^j &= \hat{e}^j_i (\tilde{\gamma}^i + N^i \tilde{\gamma}^0) \\ &= \begin{pmatrix} 0 & -\hat{e}^j_i (\tilde{\sigma}^i - N^i \mathbb{I}) \\ \hat{e}^j_i (\tilde{\sigma}^i + N^i \mathbb{I}) & 0 \end{pmatrix}, \end{aligned} \tag{34}$$

where  $\tilde{\sigma}^i$  are the  $2 \times 2$  Pauli matrices in flat space-time

$$\tilde{\sigma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{\sigma}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tilde{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{35}$$

and  $\mathbb{I}$  is the  $2 \times 2$  identity matrix. The  $\gamma^\mu$  matrices satisfy  $(\gamma^0)^\dagger = \gamma^0$  and  $(\gamma^j)^\dagger = -\gamma^j + 2N^j \gamma^0/N$ . At this point, we need to adopt the standard representation for the gamma matrices in a flat space-time  $\tilde{\gamma}^\mu$  as follows

$$\tilde{\gamma}^0 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \quad \tilde{\gamma}^j = \begin{pmatrix} 0 & -\tilde{\sigma}^j \\ \tilde{\sigma}^j & 0 \end{pmatrix}. \tag{36}$$

This representation helps us to build the Weyl representation. Additionally, in the Weyl representation we can write a Dirac

fermion as a four-spinor  $\psi$  made of two spinors, each of which having two components, for instance

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \tag{37}$$

where  $\psi_R$  and  $\psi_L$  are the right- and the left- handed Weyl spinors, respectively. If we write the adjoint spinor  $\bar{\psi}$  and use the Weyl representation, it follows that

$$\bar{\psi} = \psi^\dagger B = (\psi_R^\dagger, \psi_L^\dagger) B, \tag{38}$$

where  $B$  is the matrix from Eqs. (8) and (14). If we use the relation (8) it is straightforward to see that the matrix  $B$  must have the following form

$$B = \begin{pmatrix} 0 & B_\zeta \\ B_\zeta & 0 \end{pmatrix}, \tag{39}$$

where the  $2 \times 2$  matrix  $B_\zeta$  is a diagonal matrix,  $B_\zeta = b\mathbb{I}$ , with  $b = b(x^\mu)$ . Therefore, we get  $B = b\tilde{\gamma}^0$  and Eq. (15) transforms into

$$\nabla_0(Nb) + \nabla_j(\hat{e}_i^j N^i b) = 0, \tag{40}$$

$$\nabla_j(\hat{e}_i^j b)\tilde{\sigma}^i = 0. \tag{41}$$

Note that in Eq. (40), we assume also a representation to  $B$  matrix. Adopt a specific representation for the symmetry group, which is done without loss of generality. In fact, it shall make this choice to build the Weyl fermions and its field equations. Hence, using the definition of the spinor and its adjoint we can write the Dirac quadricurrent  $J^\mu$  from Eq. (10) as

$$J^\mu = (\psi_R^\dagger, \psi_L^\dagger) B \gamma^\mu \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \tag{42}$$

where the gamma matrices are defined by Eqs. (33) and (34) and, in general,  $B$  is given by the previously mentioned conditions. This yields

$$J^0 = Nb(\psi_R^\dagger \psi_R + \psi_L^\dagger \psi_L), \tag{43}$$

$$J^j = b\hat{e}_i^j(\psi_R^\dagger(\tilde{\sigma}^i + N^i\mathbb{I})\psi_R - \psi_L^\dagger(\tilde{\sigma}^i - N^i\mathbb{I})\psi_L). \tag{44}$$

In order to simplify the notation, we now define the vectors of  $2 \times 2$  matrices  $\mathbb{S}^a = (\mathbb{I}, \tilde{\sigma}^j + N^j\mathbb{I})$  and  $\tilde{\mathbb{S}}^a = (-\mathbb{I}, \tilde{\sigma}^j - N^j\mathbb{I})$  in terms of the Pauli matrices.  $\mathbb{S}^a$  and  $\tilde{\mathbb{S}}^a$  are the (generalized) Pauli matrices in flat space-time. In terms of these new definitions, the density currents read

$$\begin{aligned} J^\mu &= b\hat{e}_i^\mu(\psi_R^\dagger \mathbb{S}^i \psi_R - \psi_L^\dagger \tilde{\mathbb{S}}^i \psi_L) \\ &= b(\psi_R^\dagger \sigma^\mu \psi_R - \psi_L^\dagger \bar{\sigma}^\mu \psi_L), \end{aligned} \tag{45}$$

where we have defined the  $2 \times 2$  Pauli matrices in a curved space-time by  $\sigma^\mu = e_a^\mu \mathbb{S}^a$  and  $\bar{\sigma}^\mu = e_a^\mu \tilde{\mathbb{S}}^a$ . With this definition, the matrices  $\gamma^j$  read

$$\gamma^j = \begin{pmatrix} 0 & -\bar{\sigma}^j \\ \sigma^j & 0 \end{pmatrix}. \tag{46}$$

Furthermore, observe that the  $\sigma^j$  matrices follow the same commutation relations as the flat space-time Pauli matrices. This means that  $[\sigma^i, \bar{\sigma}^j] = -\hat{e}_k^i \hat{e}_l^j [\bar{\sigma}^k, \bar{\sigma}^l]$ . For the Weyl representation we have to obtain two equations for each Dirac fermion. Thus, we need to redefine the covariant derivative  $\nabla_\mu$  and the spinor affine connection  $\Gamma_\mu$  [26] [32], which can be written as  $\nabla_\mu = \partial_\mu + \Gamma_\mu$  and  $\Gamma_\mu = \frac{1}{4} \bar{\sigma}_\nu \sigma_{;\mu}^\nu$ , where  $\sigma_{;\nu}^\mu = \partial_\nu \sigma^\mu + \Gamma_{\alpha\nu}^\mu \sigma^\alpha$ . Nevertheless, in this representation we need to introduce two other notations due to the presence of  $\bar{\sigma}^\mu$ . Let  $\bar{\nabla}_\mu$  and  $\bar{\Gamma}_\mu$  be the bar covariant derivative and the bar spinor affine connection, respectively, defined by  $\bar{\nabla}_\mu = \partial_\mu + \bar{\Gamma}_\mu$ , where  $\bar{\Gamma}_\mu = \frac{1}{4} \sigma_\nu \bar{\sigma}_{;\mu}^\nu$  (we stress that we use the greek indices for denoting the objects in curved space-time as the gamma and Pauli matrices).

We can now apply the Weyl representation to rewrite the Dirac equation (5) for a spinor with four components as

$$\begin{pmatrix} i\sigma^\mu (\bar{\nabla}_\mu + iqA_\mu) \psi_R - m\psi_L \\ i\bar{\sigma}^\mu (\nabla_\mu + iqA_\mu) \psi_L - m\psi_R \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{47}$$

These are the Weyl equations for a spinor in a curved space-time coupled to an electromagnetic field. If we apply the Weyl representation to the transpose conjugated Dirac equation (16), it is straightforward to obtain the Weyl equation for the adjoint spinor (38). However, we shall not write the adjoint spinor equation explicitly because the results are analogous to the spinor equation as we have seen in the previous sections.

If we set  $B = b\tilde{\gamma}^0$ , the current density now reads

$$J^\mu = b(\psi_R^\dagger \sigma^\mu \psi_R - \psi_L^\dagger \bar{\sigma}^\mu \psi_L). \tag{48}$$

Explicitly, we have for the spatial part

$$J^j = b\hat{e}_i^j \left( \psi_R^\dagger \tilde{\sigma}^i \psi_R - \psi_L^\dagger \tilde{\sigma}^i \psi_L + \frac{N^i}{Nb^2} J^0 \right). \tag{49}$$

On the other hand, the last line of Eq. (30) can be obtained from the identities

$$\gamma^\mu \gamma^\nu F_{\mu\nu} \psi = \begin{cases} (2NN^k F_{0k} + i\hat{F}_{ij}\epsilon^{ij}_k \tilde{\sigma}^k) \psi_R \\ -(2NN^k F_{0k} - i\hat{F}_{ij}\epsilon^{ij}_k \tilde{\sigma}^k) \psi_L \end{cases}, \tag{50}$$

and using definition (46), we find that

$$\begin{aligned} &\gamma^\mu (\nabla_\mu \gamma^\nu) (D_\nu \psi) \\ &= \begin{cases} -\tilde{\mathbb{S}}^a \mathbb{S}^b (\hat{\nabla}_a \hat{e}_b^\nu) (D_\nu \psi_R) \\ -\mathbb{S}^a \tilde{\mathbb{S}}^b (\hat{\nabla}_a \hat{e}_b^\nu) (D_\nu \psi_L) \end{cases} \\ &= \begin{cases} (N(\nabla_0 N) - \tilde{\sigma}^j (\nabla_j N)) (D_0 \psi_R) + (N(\nabla_0 \sigma^i) \\ \quad - \tilde{\sigma}^j (\nabla_j \sigma^i)) (D_i \psi_R) \\ (N(\nabla_0 N) + \sigma^j (\nabla_j N)) (D_0 \psi_L) - (N(\nabla_0 \bar{\sigma}^i) \\ \quad - \sigma^j (\nabla_j \bar{\sigma}^i)) (D_i \psi_L) \end{cases} \\ &= \begin{cases} (\hat{\nabla}_0 N - \tilde{\mathbb{S}}^k (\hat{\nabla}_k N)) (D_0 \psi_R) + (\mathbb{S}^k \hat{\nabla}_0 \hat{e}_k^i \\ \quad - \tilde{\mathbb{S}}^k \mathbb{S}^l \hat{\nabla}_k \hat{e}_l^i) (D_i \psi_R) \\ (\hat{\nabla}_0 N + \mathbb{S}^k (\hat{\nabla}_k N)) (D_0 \psi_L) - (\tilde{\mathbb{S}}^k \hat{\nabla}_0 \hat{e}_k^i \\ \quad - \mathbb{S}^k \tilde{\mathbb{S}}^l (\hat{\nabla}_k \hat{e}_l^i)) (D_i \psi_L), \end{cases} \end{aligned} \tag{51}$$

where  $\epsilon^{ij}_k$  is the usual Levi-Civita tensor,  $\hat{F}_{ij} = \hat{e}^l_i \hat{e}^m_j F_{lm}$  is the directional Maxwell tensor  $\hat{F}_{ij} = (\hat{e}^l_i \hat{\nabla}_j - \hat{e}^l_j \hat{\nabla}_i) A_l$ , and  $\hat{\nabla}_a = \hat{e}^\alpha_a \nabla_\alpha$  is the directional covariant derivative which defines the Cartan connection  $\hat{\nabla}_c \hat{e}^b_a = \Gamma^a_{bc} \hat{e}^b_a$ . The Cartan connection  $\Gamma^a_{bc} = \hat{e}^a_\nu \hat{\nabla}_c \hat{e}^b_\nu$  determines the Cartan first fundamental form  $d\hat{e}^a + \Gamma^a_b \wedge \hat{e}^b$  for the connections  $\Gamma^a_b = \Gamma^a_{bd} \hat{e}^d$  with the property that  $\Gamma_{ab} + \Gamma_{ba} = 0$ , where  $\Gamma_{ab} = \eta_{ad} \Gamma^d_b$ .

In this section, we have introduced the field equations for Weyl fermions using the relation with the Dirac fermion equations. Moreover, we assume a certain representation for the symmetry Lie group to describe the Weyl spinors. In the next section, we will use the field equations found here to get a hydrodynamic representation as in the Dirac spinor case.

### 5 Weyl hydrodynamic representation

We now have all the ingredients to propose a hydrodynamic representation for the Weyl fermions, following the same procedure as the one developed for the Schrödinger and KG equations in Refs. [1,2].

We start to propose our Madelung transformation in the Weyl spinor, using the exponential map, that is

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = \begin{pmatrix} R_R \\ R_L \end{pmatrix} e^{i\theta}. \tag{52}$$

Since  $\psi_R$  and  $\psi_L$  are two spinors, we observe that  $R_R$  and  $R_L$  are two two-dimensional vectors. The Weyl representation of the adjoint spinor  $\bar{\psi}$  when  $B = b\tilde{\gamma}^0$  is

$$\bar{\psi} = b \left( \psi^\dagger_R, \psi^\dagger_L \right) \tilde{\gamma}^0 = \left( R^\dagger_R, R^\dagger_L \right) e^{-i\theta}. \tag{53}$$

As in Sect. 3, we use  $R_L$  and  $R_R$  as complex two-spinors and  $\theta$  as a complex function. Therefore, using the Madelung transformation (52) in the Weyl equations (47) and applying the Lie algebra and the Lie group, we can get the following expression

$$\begin{pmatrix} -\sigma^\mu (\bar{\nabla}_\mu \theta) R_R + i\sigma^\mu (\bar{\nabla}_\mu R_R) - q\sigma^\mu A_\mu R_R \\ -\bar{\sigma}^\mu (\nabla_\mu \theta) R_L + i\bar{\sigma}^\mu (\nabla_\mu R_L) - q\bar{\sigma}^\mu A_\mu R_L \end{pmatrix} = \begin{pmatrix} mR_L \\ mR_R \end{pmatrix}. \tag{54}$$

These are the Weyl equations in curved space-time with the Madelung transformation. We can also apply the Madelung transformation (52) and (53) to the current density (48), thereby obtaining

$$J^\mu = b \left( R^\dagger_R \bar{\sigma}^\mu R_R - R^\dagger_L \sigma^\mu R_L \right). \tag{55}$$

Its components are

$$J^0 = Nb(R^\dagger_R R_R + R^\dagger_L R_L) = Nbn, \tag{56}$$

$$J^j = b \left( \hat{e}^j_3 (n_1 - n_2 - n_3 + n_4) + 2\hat{e}^j_1 (\sqrt{n_1 n_2} - \sqrt{n_3 n_4}) + \hat{e}^j_i N^i n \right). \tag{57}$$

We note that the zero component, where  $n = \sum_{\dot{\nu}=1}^4 n_{\dot{\nu}}$  is the density number of fermions in the system, gives the number of both right- and left-handed particles. We can write the following expressions  $|\psi_R|^2 = \psi^\dagger_R \psi_R = R^\dagger_R R_R = n_R$  and  $|\psi_L|^2 = \psi^\dagger_L \psi_L = R^\dagger_L R_L = n_L$  for the right- and left-handed spinors, as in the Dirac case. Thus,  $n_R, n_L$  are the right- and left-handed particle number and  $n = n_R + n_L$  is the total density number.

Furthermore, Eq. (30) using the Weyl representation, which has been discussed in this section, it becomes

$$\begin{aligned} & i \left[ 2(mv^\mu - \omega\delta^\mu_0) \nabla_\mu R_R - qA_\mu + q\nabla_\mu (A^\mu R_R) \right. \\ & \quad \left. + \nabla_\mu (mv^\mu - \omega\delta^\mu_0 - qA^\mu) R_R \right] \\ & \quad + \left( m^2 v_\mu v^\mu + 2m\omega v^0 + \frac{\omega^2}{N^2} + m^2 \right) R_R - \square R_R \\ & \quad + (2NN^k F_{0k} + i\epsilon^{lj}_k \hat{F}_{lj} \tilde{\sigma}^k) R_R \\ & \quad + (N(\nabla_0 N) - \bar{\sigma}^j (\nabla_j N)) (mv_0 - \omega) R_R + D_0 R_R \\ & \quad + (N(\nabla_0 \sigma^k) - \bar{\sigma}^j (\nabla_j \sigma^k)) (imv_k R_R + D_k R_R) = 0. \end{aligned} \tag{58}$$

A similar equation is obtained for the left-handed spinor  $R_L$  with the substitution  $R \rightarrow L$  and  $\mathbb{S} \leftrightarrow \bar{\mathbb{S}}$  in Eq. (58). Simplifying the first line in this equation for  $\dot{\nu} = 1, 2$  corresponding to right-handed components, we get

$$\begin{aligned} & i \frac{m}{\sqrt{n_{\dot{\nu}}}} \left[ -\frac{\omega}{m} \nabla_0 n_{\dot{\nu}} + \nabla_\mu (n_{\dot{\nu}} v^\mu) + \frac{\omega}{m} \square t \right] \\ & \quad + \sqrt{n_{\dot{\nu}}} \left[ m^2 v_\mu v^\mu + 2m\omega v^0 + \frac{\omega^2}{N^2} + m^2 - \frac{\square \sqrt{n_{\dot{\nu}}}}{\sqrt{n_{\dot{\nu}}}} \right] \\ & \quad + (2NN^k F_{0k} + i\epsilon^{lj}_k \hat{F}_{lj} \tilde{\sigma}^k) R_R \\ & \quad + -(\hat{\nabla}_a \hat{e}^\alpha_b) \bar{\mathbb{S}}^a \mathbb{S}^b ((mv_\alpha - \omega\delta^\alpha_0) R_R + D_\alpha R_R) = 0. \end{aligned} \tag{59}$$

The equation for the left-handed components  $\dot{\nu} = 3, 4$  is obtained by changing  $R_R \rightarrow R_L$  and  $\mathbb{S} \leftrightarrow \bar{\mathbb{S}}$ . Note that, although in Eq. (59) the first line is multiplied by  $i$ , we cannot consider the separation between the real and imaginary part, since from the Madelung transformation (21) we assume  $R$  and  $\theta$  as complex parameters. Additionally, the first line of Eq. (59) represents the hydrodynamic part of the fermionic fluid. The second line in Eq. (59) is written the Bernoulli equation. In this respect, we note that Eq. (24) is the first integral of this equation. Then, the last lines of Eq. (59) are the source of the fermionic fluid, something that is not present in the case of bosons. This is because the Dirac equation was

introduced [33] in order to eliminate the negative probability problem of the KG equation. As a result, the Dirac equation involves only first derivatives while the KG equation is a second order equation. We will identify the terms in Eq. (59) as terms of the first law of thermodynamics in the next section.

Writing explicitly each component of Eq. (59), we can obtain for  $\dot{v} = \dot{1}$ :

$$\begin{aligned}
 & i \frac{m}{\sqrt{n_1}} \left[ -\frac{\omega}{m} \nabla_0 n_1 + \nabla_\mu (n_1 v^\mu) + \frac{\omega}{m} \square t \right] \\
 & + \sqrt{n_1} \left[ m^2 v_\mu v^\mu + 2m\omega v^0 + \frac{\omega^2}{N^2} + m^2 - \frac{\square \sqrt{n_1}}{\sqrt{n_1}} \right] \\
 = & i \left[ F_{12} \sqrt{n_1} + F_{23} \sqrt{n_2} \right. \\
 & - 2\Gamma_{21}^a ((m\hat{v}_a - \omega\hat{\delta}_a^0) \sqrt{n_1} + \hat{D}_a \sqrt{n_1}) \\
 & - 2i(\Gamma_{21}^a N^1 - \Gamma_{32}^a N^3 + \Gamma_{20}^a + \Gamma_{32}^a) \\
 & \times ((m\hat{v}_a - \omega\hat{\delta}_a^0) \sqrt{n_2} + \hat{D}_a \sqrt{n_2}) \\
 & + 2N(F_{01} N^1 + F_{02} N^2 + F_{03} N^3) \sqrt{n_1} - F_{13} \sqrt{n_2} \\
 & + \left[ \Gamma_{11}^a (1 - (N^1)^2) + \Gamma_{22}^a (1 - (N^2)^2) \right. \\
 & \left. + \Gamma_{33}^a (1 - (N^3)^2) \right] 2\Gamma_{31}^a N^1 + 2\Gamma_{32}^a N^2 - \Gamma_{00}^a + 2\Gamma_{30}^a \\
 & \times ((m\hat{v}_a - \omega\hat{\delta}_a^0) \sqrt{n_1} + \hat{D}_a \sqrt{n_1}) \\
 & + (-2\Gamma_{21}^a N^2 - 2\Gamma_{31}^a N^3 + 2\Gamma_{10}^a + 2\Gamma_{31}^a) \\
 & \left. \times ((m\hat{v}_a - \omega\hat{\delta}_a^0) \sqrt{n_2} + \hat{D}_a \sqrt{n_2}), \right. \tag{60}
 \end{aligned}$$

for  $\dot{v} = \dot{2}$ :

$$\begin{aligned}
 & i \frac{m}{\sqrt{n_2}} \left[ -\frac{\omega}{m} \nabla_0 n_2 + \nabla_\mu (n_2 v^\mu) + \frac{\omega}{m} \square t \right] \\
 & + \sqrt{n_2} \left[ m^2 v_\mu v^\mu + 2m\omega v^0 + \frac{\omega^2}{N^2} + m^2 - \frac{\square \sqrt{n_2}}{\sqrt{n_2}} \right] \\
 = & i \left[ -F_{12} \sqrt{n_2} + F_{23} \sqrt{n_1} \right. \\
 & + 2\Gamma_{21}^a ((m\hat{v}_a - \omega\hat{\delta}_a^0) \sqrt{n_2} + \hat{D}_a \sqrt{n_2}) \\
 & + 2i(\Gamma_{21}^a N^1 - \Gamma_{32}^a N^3 + \Gamma_{20}^a - \Gamma_{32}^a) \\
 & \times (m\hat{v}_a - \omega\hat{\delta}_a^0) \sqrt{n_1} + \hat{D}_a \sqrt{n_1} \\
 & + 2N(F_{01} N^1 + F_{02} N^2 + F_{03} N^3) \sqrt{n_2} + F_{13} \sqrt{n_1} \\
 & + \left[ \Gamma_{11}^a (1 - (N^1)^2) + \Gamma_{22}^a (1 - (N^2)^2) \right. \\
 & \left. + \Gamma_{33}^a (1 - (N^3)^2) \right] \\
 & + -2\Gamma_{31}^a N^1 - 2\Gamma_{32}^a N^2 - \Gamma_{00}^a - 2\Gamma_{30}^a \\
 & \times ((m\hat{v}_a - \omega\hat{\delta}_a^0) \sqrt{n_2} + \hat{D}_a \sqrt{n_2}) \\
 & + (-2\Gamma_{21}^a N^2 - 2\Gamma_{31}^a N^3 + 2\Gamma_{10}^a - 2\Gamma_{31}^a) \\
 & \left. \times ((m\hat{v}_a - \omega\hat{\delta}_a^0) \sqrt{n_1} + \hat{D}_a \sqrt{n_1}), \right. \tag{61}
 \end{aligned}$$

for  $\dot{v} = \dot{3}$ :

$$\begin{aligned}
 & i \frac{m}{\sqrt{n_3}} \left[ -\frac{\omega}{m} \nabla_0 n_3 + \nabla_\mu (n_3 v^\mu) + \frac{\omega}{m} \square t \right] \\
 & + \sqrt{n_3} \left[ m^2 v_\mu v^\mu + 2m\omega v^0 + \frac{\omega^2}{N^2} + m^2 - \frac{\square \sqrt{n_3}}{\sqrt{n_3}} \right] \\
 = & i \left[ F_{12} \sqrt{n_3} + F_{23} \sqrt{n_4} \right. \\
 & - 2\Gamma_{21}^a ((m\hat{v}_a - \omega\hat{\delta}_a^0) \sqrt{n_3} + \hat{D}_a \sqrt{n_3}) \\
 & + 2i(\Gamma_{21}^a N^1 - \Gamma_{32}^a N^3 + \Gamma_{20}^a - \Gamma_{32}^a) \\
 & \times ((m\hat{v}_a - \omega\hat{\delta}_a^0) \sqrt{n_4} + \hat{D}_a \sqrt{n_4}) \\
 & + 2N(F_{01} N^1 + F_{02} N^2 + F_{03} N^3) \sqrt{n_3} - F_{13} \sqrt{n_4} \\
 & + \left[ \Gamma_{11}^a (1 - (N^1)^2) + \Gamma_{22}^a (1 - (N^2)^2) \right. \\
 & \left. + \Gamma_{33}^a (1 - (N^3)^2) \right] \\
 & + -2\Gamma_{31}^a N^1 - 2\Gamma_{32}^a N^2 - \Gamma_{00}^a - 2\Gamma_{30}^a \\
 & \times ((m\hat{v}_a - \omega\hat{\delta}_a^0) \sqrt{n_3} + \hat{D}_a \sqrt{n_3}) \\
 & + (2\Gamma_{21}^a N^2 + 2\Gamma_{31}^a N^3 - 2\Gamma_{10}^a + 2\Gamma_{31}^a) \\
 & \left. \times ((m\hat{v}_a - \omega\hat{\delta}_a^0) \sqrt{n_4} + \hat{D}_a \sqrt{n_4}), \right. \tag{62}
 \end{aligned}$$

and for  $\dot{v} = \dot{4}$ :

$$\begin{aligned}
 & i \frac{m}{\sqrt{n_4}} \left[ -\frac{\omega}{m} \nabla_0 n_4 + \nabla_\mu (n_4 v^\mu) + \frac{\omega}{m} \square t \right] \\
 & + \sqrt{n_4} \left[ m^2 v_\mu v^\mu + 2m\omega v^0 + \frac{\omega^2}{N^2} + m^2 - \frac{\square \sqrt{n_4}}{\sqrt{n_4}} \right] \\
 = & i \left[ -F_{12} \sqrt{n_4} + F_{23} \sqrt{n_3} + 2\Gamma_{21}^a \right. \\
 & \times ((m\hat{v}_a - \omega\hat{\delta}_a^0) \sqrt{n_4} + \hat{D}_a \sqrt{n_4}) \\
 & - 2i(\Gamma_{21}^a N^1 - \Gamma_{32}^a N^3 + \Gamma_{20}^a + \Gamma_{32}^a) \\
 & \times ((m\hat{v}_a - \omega\hat{\delta}_a^0) \sqrt{n_3} + \hat{D}_a \sqrt{n_3}) \\
 & + 2N(F_{01} N^1 + F_{02} N^2 + F_{03} N^3) \sqrt{n_4} + F_{13} \sqrt{n_3} \\
 & + \left[ \Gamma_{11}^a (1 - (N^1)^2) + \Gamma_{22}^a (1 - (N^2)^2) \right. \\
 & \left. + \Gamma_{33}^a (1 - (N^3)^2) \right] \\
 & 2\Gamma_{31}^a N^1 + 2\Gamma_{32}^a N^2 - \Gamma_{00}^a + 2\Gamma_{30}^a \\
 & \times ((m\hat{v}_a - \omega\hat{\delta}_a^0) \sqrt{n_4} + \hat{D}_a \sqrt{n_4}) \\
 & + (2\Gamma_{21}^a N^2 + 2\Gamma_{31}^a N^3 - 2\Gamma_{10}^a - 2\Gamma_{31}^a) \\
 & \left. \times ((m\hat{v}_a - \omega\hat{\delta}_a^0) \sqrt{n_3} + \hat{D}_a \sqrt{n_3}), \right. \tag{63}
 \end{aligned}$$

where we have used that  $\Gamma_{ab} + \Gamma_{ba} = 0$  and defined the directional quantities  $v_a = v_\alpha \hat{e}_a^\alpha$ ,  $\hat{\delta}_a^0 = \delta_\alpha^0 \hat{e}_a^\alpha = N \delta_a^0$  and  $\hat{D}_a = \hat{e}_a^\alpha D_\alpha$ .



Observe that the structure of Eqs. (60)-(63) is

$$\begin{aligned}
 & i \frac{m}{\sqrt{n_{\dot{\nu}}}} \left[ -\frac{\omega}{m} \nabla_0 n_{\dot{\nu}} + \nabla_{\mu} (n_{\dot{\nu}} v^{\mu}) + \frac{\omega}{m} \square t \right] \\
 & + \sqrt{n_{\dot{\nu}}} \left[ m^2 v_{\mu} v^{\mu} + 2m\omega v^0 + \frac{\omega^2}{N^2} + m^2 - \frac{\square \sqrt{n_{\dot{\nu}}}}{\sqrt{n_{\dot{\nu}}}} \right] \\
 & = i \left[ e_{1\dot{\nu}} F_{12} \sqrt{n_{\dot{\nu}}} + F_{23} \sqrt{n_{\dot{\nu}}} \right. \\
 & \quad \left. - 2e_{1\dot{\nu}} \Gamma_{21}^a ((m\hat{v}_a - \omega\hat{\delta}_a^0) \sqrt{n_{\dot{\nu}}} + \hat{D}_a \sqrt{n_{\dot{\nu}}}) \right] \\
 & \quad - 2i (\Gamma_{21}^a N^1 - \Gamma_{32}^a N^3 + \Gamma_{20}^a + e_{2\dot{\nu}} \Gamma_{32}^a) \\
 & \quad \times ((m\hat{v}_a - \omega\hat{\delta}_a^0) \sqrt{n_{\dot{\nu}}} + \hat{D}_a \sqrt{n_{\dot{\nu}}}) \\
 & \quad + 2N (F_{01} N^1 + F_{02} N^2 + F_{03} N^3) \sqrt{n_{\dot{\nu}}} - e_{1\dot{\nu}} F_{13} \sqrt{n_{\dot{\nu}}} \\
 & \quad + \left[ \Gamma_{11}^a (1 - (N^1)^2) + \Gamma_{22}^a (1 - (N^2)^2) \right. \\
 & \quad \left. + \Gamma_{33}^a (1 - (N^3)^2) \right. \\
 & \quad \left. + 2e_{2\dot{\nu}} (\Gamma_{31}^a N^1 + \Gamma_{32}^a N^2 + \Gamma_{30}^a) - \Gamma_{00}^a \right] \\
 & \quad \times ((m\hat{v}_a - \omega\hat{\delta}_a^0) \sqrt{n_{\dot{\nu}}} + \hat{D}_a \sqrt{n_{\dot{\nu}}}) \\
 & \quad + (-2e_{3\dot{\nu}} (\Gamma_{21}^a N^2 + \Gamma_{31}^a N^3 - \Gamma_{10}^a) + 2e_{1\dot{\nu}} \Gamma_{31}^a) \\
 & \quad \times ((m\hat{v}_a - \omega\hat{\delta}_a^0) \sqrt{n_{\dot{\nu}}} + \hat{D}_a \sqrt{n_{\dot{\nu}}}), \tag{64}
 \end{aligned}$$

where the coefficients  $e_{i\dot{\nu}}$  are  $\pm 1$  with  $e_{1\dot{\nu}} = (+, -, +, -)$ ,  $e_{2\dot{\nu}} = (-, +, -, +)$  and  $e_{3\dot{\nu}} = (+, +, -, -)$ , and the sub-index  $\dot{\nu}$  are the conjugate of the sub-index  $\dot{\nu}$ , such that  $\dot{1} = \dot{2}$ ,  $\dot{2} = \dot{1}$ ,  $\dot{3} = \dot{4}$  and  $\dot{4} = \dot{3}$ . In comparison with the boson case, we cannot separate them in real and imaginary part. Due to, the generalized transformation, that we assume, has complex parameters. Therefore, we shall work with the full equations, which are more complicated than the standard equations for fermions in curved space-time, that themselves are complicated. An advantage for the hydrodynamic representation, that we found, is to give directly an interpretation of quantum theory through the De Broglie–Bohm interpretation.

### 6 Energy balance

From Eq. (59), we can identify the different energy contributions to the Fermi gas, and obtain an energy balance equation for fermions analogous to the one obtained for bosons in [1, 2]. In order to simplify the notations, we can re-write the Eq. (59) in terms of the  $\dot{\nu}$  coefficients with the understanding that the subindex  $R$  refers to each component  $R = \dot{1}, \dot{2}$  individually. We get

$$\begin{aligned}
 & i \left[ -\omega \nabla_0 \ln(n_{\dot{\nu}}) + \frac{m \nabla_{\mu} (n_{\dot{\nu}} v^{\mu})}{n_{\dot{\nu}}} + \frac{\omega}{n_{\dot{\nu}}} \square t \right] \\
 & + 2m^2 \left( K + \frac{1}{m} \omega v^0 + \frac{1}{2} U^N + U^Q \right) + E + U^S = 0. \tag{65}
 \end{aligned}$$

The first line in Eq. (65) describes the free density evolution of the fermions, while the contribution of the different energy

terms appears in the second line. The first one is the kinetic energy  $K_{\dot{\nu}}$  defined as

$$K = \frac{1}{2} v_{\mu} v^{\mu}. \tag{66}$$

The lapse potential  $U^N$  is given by

$$U^N = \frac{\omega^2}{m^2} \frac{1}{N^2} + 1. \tag{67}$$

It represents the energy contribution due to the chosen lapse function  $N$ . The quantum potential  $U^Q$  is defined as

$$U^Q = -\frac{1}{2m^2} \frac{\square \sqrt{n_{\dot{\nu}}}}{\sqrt{n_{\dot{\nu}}}}. \tag{68}$$

The contribution of the electromagnetic interaction  $E$  is given by

$$\begin{aligned}
 E & = (2NN^k F_{0k} + i\epsilon^{Lj}_k \hat{F}_{lj} \tilde{\sigma}^k), \\
 & = 2N (F_{01} N^1 + F_{02} N^2 + F_{03} N^3) - e_{1\dot{\nu}} F_{13} \sqrt{\frac{n_{\dot{\nu}}}{n_{\dot{\nu}}}} \\
 & \quad + i \left( e_{1\dot{\nu}} F_{12} + F_{23} \sqrt{\frac{n_{\dot{\nu}}}{n_{\dot{\nu}}}} \right). \tag{69}
 \end{aligned}$$

It depends on the Faraday tensor, shift vector and lapse function that are related to the Pauli matrices. This relationship is due to the interaction between the electromagnetic field and the fermionic spin. Finally, the potential  $U^S_{\dot{\nu}}$  describes the interaction between the spin and the geometry of space-time. It is given by

$$\begin{aligned}
 U^S & = - \left( (m\hat{v}_{Rd} - \omega_{\dot{\nu}} \hat{\delta}_d^0) + \frac{\hat{D}_a \sqrt{n_{\dot{\nu}}}}{\sqrt{n_{\dot{\nu}}}} \right) \Gamma_{ba}^d \bar{S}^a S^b, \tag{70} \\
 & = \left[ \Gamma_{11}^a (1 - (N^1)^2) + \Gamma_{22}^a (1 - (N^2)^2) \right. \\
 & \quad \left. + \Gamma_{33}^a (1 - (N^3)^2) \right. \\
 & \quad \left. + 2e_{2\dot{\nu}} (\Gamma_{31}^a N^1 + \Gamma_{32}^a N^2 + \Gamma_{30}^a) - \Gamma_{00}^a \right] \\
 & \quad \times \left( (m\hat{v}_a - \omega\hat{\delta}_a^0) + \frac{\hat{D}_a \sqrt{n_{\dot{\nu}}}}{\sqrt{n_{\dot{\nu}}}} \right) \\
 & \quad + (-2e_{3\dot{\nu}} (\Gamma_{21}^a N^2 + \Gamma_{31}^a N^3 - \Gamma_{10}^a) + 2e_{1\dot{\nu}} \Gamma_{31}^a) \\
 & \quad \times \left( (m\hat{v}_a - \omega\hat{\delta}_a^0) \sqrt{\frac{n_{\dot{\nu}}}{n_{\dot{\nu}}}} + \frac{\hat{D}_a \sqrt{n_{\dot{\nu}}}}{\sqrt{n_{\dot{\nu}}}} \right) \\
 & \quad + i \left[ -2e_{1\dot{\nu}} \Gamma_{21}^a \left( (m\hat{v}_a - \omega\hat{\delta}_a^0) + \frac{\hat{D}_a \sqrt{n_{\dot{\nu}}}}{\sqrt{n_{\dot{\nu}}}} \right) \right. \\
 & \quad \left. - 2(\Gamma_{21}^a N^1 - \Gamma_{32}^a N^3 + \Gamma_{20}^a + e_{2\dot{\nu}} \Gamma_{32}^a) \right. \\
 & \quad \left. \times \left( (m\hat{v}_a - \omega\hat{\delta}_a^0) \sqrt{\frac{n_{\dot{\nu}}}{n_{\dot{\nu}}}} + \frac{\hat{D}_a \sqrt{n_{\dot{\nu}}}}{\sqrt{n_{\dot{\nu}}}} \right) \right], \tag{71}
 \end{aligned}$$

Note that  $U^S$  disappears if we assume a flat space-time or if we consider particles without spin. Furthermore,  $U^S$  is constructed with the generalized gamma matrices (46), which

are related to the spin (the Pauli matrices) and to the space-time geometry (tetrads).

Finally, we can also write Eq. (27) as a Gross–Pitaevskii-like equation. If we perform the transformation  $\psi = \Psi e^{i\omega_0 t}$ , where  $\Psi$  is a four spinor that depends on all the variables  $x^\mu$ , Eq. (27) becomes

$$i\nabla^0\Psi - \frac{1}{2\omega_0}\square_E\Psi + \frac{m^2}{2\omega_0}\Psi + \left(-\frac{\omega_0}{N^2} - 2qA^0 + i\square_t\right)\Psi + \frac{1}{2\omega_0}\begin{pmatrix} 2NN^k F_{0k} + i\hat{F}_{ij}\epsilon^{ij}{}_k\tilde{\sigma}^k & 0 \\ 0 & -2NN^k F_{0k} + i\hat{F}_{ij}\epsilon^{ij}{}_k\tilde{\sigma}^k \end{pmatrix}\Psi - \frac{1}{2\omega_0}\begin{pmatrix} \bar{S}^a\bar{S}^b & 0 \\ 0 & S^a\bar{S}^b \end{pmatrix}\Gamma_{ba}^d(\hat{D}_d\Psi + i\omega_0 N\delta_d^0\Psi) = 0. \tag{72}$$

Equation (72) is the generalization of the Gross–Pitaevskii equation [34] for fermions with electromagnetic field interaction in an arbitrary space-time.

### 7 Conclusions

A non-standard representation for fermions was worked using an analogy as in the boson and quantum mechanics case, where it was proposed the Madelung transformation. We extended this transformation for the spinor case, either Dirac or Weyl fermions. Thus, it was possible to get a successful hydrodynamic representation for fermions in an arbitrary framework coupled to an electromagnetic field. Although, the full equations that describe the Fermi gas behaviour are more complicated than in standard description. This is closer to the De Broglie–Bohm interpretation in quantum theory, where the measure problem can be solved by a statistic way. Furthermore, a non-obvious result using this new description was the first law of the thermodynamics or the energy balance equation, where different energy contributions of these kind of particles were found.

The main difference between the hydrodynamic representation of bosons [1] [2] and fermions, concerns the form of the Bernoulli equation. For bosons, after doing the Madelung transformation, we can separate the KG equation into real and imaginary parts. By contrast, for fermion particles we have to work with the complete equations of motion because the real and imaginary parts cannot be easily separated. This is related to the fact that the gamma matrices are a representation of the SO(1, 3) group and the generalized Madelung transformation used, because it only admits complex parameter to fulfill the Lorentz invariance.

The spin is a fundamental outcome of the Dirac equation [33], which combines both elements of special relativity and quantum mechanics, that was introduced to solve the problem of negative probability present in the KG equation – first proposed as a relativistic generalization of the Schrödinger equation. Here, we observe that the general relativistic Dirac equation involves an additional contribution due to geometry and spin through the generalized gamma and Pauli matrices.

These terms arise from endowing a quantum field with a curvature (geometry) given by a metric in General Relativity. Such a contribution is absent in a flat space-time and in a system without spin as for a scalar field.

With this work we open the possibility of studying in detail the behavior of fermions in different situations (such as massive stars or dark matter halos harboring a central black hole), where general relativity effects may be important. We solved the problem of energy balance for both bosons and fermions. In this manner, we can compare the result of the hydrodynamic representation for classical and quantum fluids in the various geometries mentioned above.

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### A solutions to the Dirac equation in flat space-time

Equation (5) in flat space-time, using the Pauli matrices (35), reads

$$\begin{bmatrix} \frac{\partial}{\partial t}\psi_y - \frac{\partial}{\partial x}\psi_z + i\frac{\partial}{\partial y}\psi_z - \frac{\partial}{\partial z}\psi_y - m\psi_t \\ \frac{\partial}{\partial t}\psi_z - \frac{\partial}{\partial x}\psi_y - i\frac{\partial}{\partial y}\psi_y + \frac{\partial}{\partial z}\psi_z - m\psi_x \\ \frac{\partial}{\partial t}\psi_t + \frac{\partial}{\partial x}\psi_x - i\frac{\partial}{\partial y}\psi_x + \frac{\partial}{\partial z}\psi_t - m\psi_y \\ \frac{\partial}{\partial t}\psi_x + \frac{\partial}{\partial x}\psi_t + i\frac{\partial}{\partial y}\psi_t - \frac{\partial}{\partial z}\psi_x - m\psi_z \end{bmatrix} = 0, \tag{73}$$

where we have defined the spinor as  $\psi = (\psi_\mu) = (\psi_x, \psi_y, \psi_z, \psi_t)^T$ . In order to find an exact solution of the previous equation, we use the ansatz  $\psi_\mu = R_{0\mu} \exp(i(x_0x + y_0y + z_0z + t_0t))$ , where  $x_0 \dots t_0$  and  $R_{0\mu}$  are constants. Here,

we have the simplest solutions of the Dirac equation where the exponential is the same for all components. We obtain four linear equations

$$\begin{aligned} i R_{0z} \zeta_0^* + i R_{0y} \eta_0 + m R_{0t} &= 0, \\ i R_{0y} \zeta_0 - i R_{0z} \xi_0 + m R_{0x} &= 0, \\ R_{0x} \zeta_0^* + R_{0t} \xi_0 + i m R_{0y} &= 0, \\ R_{0t} \zeta_0 - R_{0x} \eta_0 + i m R_{0z} &= 0, \end{aligned} \tag{74}$$

where  $\zeta_0 = x_0 + iy_0$ ,  $\eta_0 = z_0 - t_0$ , and  $\xi_0 = z_0 + t_0$ . The solutions of these equations are

$$\begin{aligned} R_{0t} &= -\frac{1}{m}(i R_{0y} \eta_0 + i R_{0z} \zeta_0), \\ R_{0x} &= \frac{1}{m}(i R_{0z} \xi_0 - i R_{0y} \zeta_0^*), \end{aligned} \tag{75}$$

where  $x_0^2 + y_0^2 + z_0^2 - t_0^2 = m^2$ .

Now, we use the ansatz  $\psi_\mu = R_{0\mu} \exp(i\theta)$ , where  $\theta$  is an arbitrary function of the coordinates. Substituting this ansatz into (73), we obtain

$$\begin{aligned} i R_{0z} Z_0^* + i R_{0y} E_0 + m R_{0t} &= 0, \\ i R_{0y} Z_0 - i R_{0z} F_0 + m R_{0x} &= 0, \\ R_{0x} Z_0^* + R_{0t} F_0 + i m R_{0y} &= 0, \\ R_{0t} Z_0 - R_{0x} E_0 + i m R_{0z} &= 0, \end{aligned} \tag{76}$$

where  $Z_0 = \theta_{,x} + i\theta_{,y}$ ,  $E_0 = \theta_{,z} - \theta_{,t}$ , and  $F_0 = \theta_{,z} + \theta_{,t}$ . The solution of the previous system of differential equations is

$$\begin{aligned} \theta &= F(X) - it + \frac{m}{2R_{0t}R_{0z} + 2R_{0x}R_{0y}} \\ &\times \left( i\zeta_0^*(R_{0x}^2 - R_{0z}^2) - i\zeta_0(R_{0y}^2 - R_{0t}^2) \right), \end{aligned} \tag{77}$$

where  $F(X)$  is an arbitrary function of

$$X = \frac{R_{0t}(-\zeta R_{0y} - \zeta^* R_{0x} + \xi R_{0y} - \eta R_{0z})}{2R_{0t}R_{0z} + 2R_{0x}R_{0y}}. \tag{78}$$

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