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Exact Solutions of N-Dimensional Stationary Kaluza-Klein Field Equations.

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Summary. — Supposing that the components of the n-dimensional metric of the Kaluza-Klein field equations depend on two variables ρ and ζ, we develop a method for integrating them, assuming that one part of the field equations can be considered as a function of one variable $\lambda = \lambda(\rho, \zeta)$, which satisfies the Laplace equation.

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1. – Introduction.

The idea of working in higher dimensions in order to unify all interactions in nature is becoming more interesting among the physicists. The earliest attempt to work in more than four dimensions was the Kaluza-Klein theory [3, 4]. This theory unified gravitation with electromagnetism taking a five-dimensional manifold and assuming that the five-metric does not depend on the fifth coordinate. A more interesting approach was made in the 60's and 70's when the Kaluza-Klein idea was generalized to $N + 4$ dimensions unifying gravitation with electroweak and strong interactions [1]. String's theory is also a candidate for unifying all interactions in one space of 26 dimensions, and the superstring theory is one in a 10-dimensional space [2]. Even if the standard string and superstring theory have a flat underground, there are some approaches for studying strings in a curved space [10]. Anyway, it is of great interest to find exact solutions of the Einstein's equation in a n-dimensional Riemannian space and that is just what we intend to do in this work.

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The problem of finding exact solutions of the 4-dimensional Einstein's equation is so difficult that mathematicians and physicists had spent a lot of time trying to solve it. Its generalization to $N+4$ dimensions is obviously much more complicated. Therefore, we have to make some simplifications in order to be able to use methods developed to find exact solutions in four dimensions [5]. We start assuming that the $(N+4)$-dimensional metric depends only on two coordinates x^1 and x^2. This is not a very strong restriction since many physical solutions, such as Schwarzschild's or Kerr's solution, belong to this class. Furthermore we assume that the $(N+4)$-dimensional metric can be written as

$$ds^2 = f(dx^{1^2} + dx^{2^2}) + g_{\mu\nu}\, dx^\mu dx^\nu, \qquad \mu, \nu = 3, \ldots, N+4, \tag{1}$$

where the functions f and $g_{\mu\nu}$ depend only on x^1 and x^2. In [7] it was shown that the $(N+4)$-dimensional Einstein's field equations $\overset{N+4}{R}_{ab} = 0$, a, $b = 1, \ldots, N+4$ with the metric (1) reduce to two systems of nonlinear partial differential equations:

$$(\ln f)_{,z} = -\frac{1}{z+\bar{z}} + \frac{z+\bar{z}}{4}\,\mathrm{tr}\,(g_{,z}g^{-1})^2, \tag{2a}$$

$$(x^1 g_{,z} g^{-1})_{,\bar{z}} + (x^1 g_{,\bar{z}} g^{-1})_{,z} = 0, \quad z = x^1 + ix^2, \;\; (g)_{\mu\nu} = g_{\mu\nu}. \tag{2b}$$

Observe that in order to solve the field equations (2) we have to find first a solution of (2*b*), then substitute it in (2*a*). In this work we find exact solutions only of the second system of differential equations, namely (2*b*). We procede in the following form: assume that the metric g depends only on one parameter $\lambda = \lambda(z, \bar{z})$, *i.e.* $g_{\mu\nu} = g_{\mu\nu}(\lambda(z, \bar{z}))$, and suppose that λ fulfills generalized Laplace's equations $(x^1 \lambda_{,z})_{,\bar{z}} + (x^1 \lambda_{,\bar{z}})_{,z} = 0$ [8, 9]. Then, eq. (2*b*) reduces to

$$g_{,\lambda} = Ag, \qquad A \text{ is constant}, \tag{3}$$

g being real and symmetric. One can normalize g in such a way that $\det g = 1$, then g belongs to the matrix group $SL(N, R)$. It is clear that the transformation $g \to CgC^{\mathrm{T}}$ let eq. (2*b*) unchanged if C is a constant matrix that belongs also to the group $SL(N, R)$. Under this transformation, the matrix A changes to $A \to CAC^{-1}$. This last relation induces a partition in the set of matrices into equivalent classes; *i.e.* it is enough to work with a representative matrix for each class. Then, the first step of our work will be to write down such representatives.

Observe that the group conditions $g = \bar{g}$, $\det g = 1$ and the symmetry condition $g = g^{\mathrm{T}}$ are translated to the matrix A in the form

$$\begin{cases} a) & g = \bar{g} & \to & A = \bar{A}, \\ b) & \det\, g = 1 & \to & \mathrm{tr}\, A = 0, \\ c) & g = g^{\mathrm{T}} & \to & g A^{\mathrm{T}} = Ag, \end{cases} \tag{4}$$

that means that we have to find the representative of the classes using the conditions *a*) and *b*) in (4). These conditions restrict the representatives of the classes in such a form that we will have a reduced number of matrices to work with.

2. – Getting the matrix A.

We start writing the representatives of the classes of equivalence in a normal form, known as the natural form [6].

It can be shown that every matrix A with elements on a commutative field can be reproduced, on this field, to one and only one natural normal form:

$$A = \begin{bmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_s \end{bmatrix},$$

where each A_i, $i = 1, \ldots, s$, is a square matrix, called a cell, such that it presents the following form:

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix}.$$

Observe that $a_0, a_1, \ldots, a_{n-1}$ are the coefficients of the characteristic polynomial of matrix the A_i:

$$p_i(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 .$$

To characterize the matrix A, we use the matrix $\lambda I - A$; this is a polynomial matrix, since its entries are polynomials in λ. We work the similar relation of the natural normal form of matrices, using the invariant factors of its corresponding polynomial matrix. We work having as basis the following criteria:

Criterion 1. Let P_1 and P_2 be polynomial matrices of order n; they are said equivalent if and only if their respective invariant factors $d_1(\lambda)$, $d_2(\lambda), \ldots, d_n(\lambda)$ are the same.

Criterion 2. Two matrices A and B on a field K are similar if and only if their matrices $\lambda I - A$ and $\lambda I - B$ are equivalent.

For the search of matrices in their natural normal forms that represent the classes, we consider the classification of the matrices A according to the degrees of the invariant polynomials (invariant factors) of the matrix $A - \lambda I$.

Since the invariant factors of $A - \lambda I$ are factors of the characteristic polynomial $p_A(\lambda)$ associated to A, then we can consider the feasible n-decomposition of the polynomial $p_A(\lambda)$.

We give a table of the degrees of the invariant polynomials for the matrices $A - \lambda I$ for $n = 2, 3, 4$. We designate an arbitrary polynomial of degree n by $\boxed{n}$.

It many cases the number 1 is an invariant factor, it is a polynomial of degree zero. In table I, this factor is represented, simply, as 1.

It is interesting to observe that in table I, $d_1(\lambda) = 1$, always. We can prove, in this way, the following theorem.

TABLE I. – *Degrees of the invariant factors of the matrix* $A_i - \lambda I$.

Order n	Invariants			
	$d_1(\lambda)$	$d_2(\lambda)$	$d_3(\lambda)$	$d_3(\lambda)$
2	1	[2]		
3	1	1	[3]	
	1	[1]	[2]	
4	1	1	1	[4]
	1	1	[1]	[3]
	1	1	[2]	[2]
	1	[1]	[1]	[2]

TABLE II. – *Classification of matrices.*

Order n	Cases	Matrices A_i	Invariants
2		$\begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}$	$1,\ \lambda^2 - a$
3	3.1	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & 0 \end{bmatrix}$	$1,\ 1,\ \lambda^3 - \lambda b - a$
	3.2	$\begin{bmatrix} q & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2q^2 & -q \end{bmatrix}$	$1,\ \lambda - q,\ (\lambda + 2q)(\lambda - q)$
4	4.1	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & c & 0 \end{bmatrix}$	$1,\ 1,\ 1,\ \lambda^4 - c\lambda^2 - b\lambda - a$
	4.2	$\begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2q^3 - qb & b & -q \end{bmatrix}$	$1,\ 1,\ \lambda - q,\ \lambda^3 + q\lambda^2 - b\lambda - a$ with $a = q(2q^2 - b)$
	4.3	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a & 0 \end{bmatrix}$	$1,\ 1,\ \lambda^2 - a,\ \lambda^2 - a$
	4.4	$\begin{bmatrix} -3a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix}$	$1,\ \lambda - a,\ \lambda - a,\ (\lambda - a)(\lambda + 3a)$

Theorem 1. Let A be a matrix with conditions (4). The first invariant factor of the matrix $A-\lambda I$ is equal to 1. In other words: $d_1(\lambda)=1$. The proof is followed by the fact that trace $A=0$.

The methodology for finding the matrices A, that we propose in this paper, consists first in building matrices in their natural normal form in correspondence with the dimensions of the cells given in table I. We have found a representative of every class of the partition induced by the properties (4). One among all matrices will be similar to the found matrix.

In this way we are classifying the solutions of eq. (3) and building a representative, that is a general solution to our problem.

In table II are shown the representatives for $n=2,3,4$.

3. – Getting the matrix g.

Let the differential equation (3) for some A be as in (4). Then, we have a system of $s\times n$ linear differential equations. We remember that $p_A(\lambda)$ is the characteristic polynomial of matrix A, $p_{A_i}(\lambda)$ is the characteristic polynomial of each cell of A and $p_e(\lambda)$ is the characteristic polynomial associated to (3).

We enunciate now some of our results without proof.

Theorem 2. If A is a natural normal cell itself, then

$$p_A(\lambda)=p_e(\lambda).$$

Definition. For a square matrix $A=[a_{ij}]$ of order n, an antidiagonal is the set of entries $\alpha_K=\{a_{ij}\,|\,i+j=k\}$; $k=2,\dots,2n$.

For example, the matrix $A=[a_{ij}]$ of order 4, has 7 antidiagonals, they are

$$a_{11};\; a_{12}a_{21};\; a_{13}a_{22}a_{31};\; a_{14}a_{23}a_{32}a_{41};\; a_{24}a_{33}a_{42};\; a_{34}a_{43};\; a_{44}.$$

Theorem 3. Let A be a natural cell. The elements of the matrix Ag, on each α_k; $k=2,3,\dots,n+1$, are the same.

Theorem 4. Let A be a matrix in its natural normal form with s cells. Then $g_{,\lambda}=$ $=Ag$ can be decomposed in a system of $s\times n$ independent homogeneous linear differential equations. Moreover,

$$p_A(\lambda)=\prod_{i=1}^{s}p_{A_i}(\lambda).$$

If the matrix A is a natural cell, we can develop the differential equation associated to $g_{,\lambda}=Ag$ for g_{11}, like in the example below. If we solve such differential equation, then we will get g_{11}. After that, we can compute the elements of the first row and the first column of g, differentiating n times. If we know those elements, then we can know all the elements of the triangular left submatrix of g. To compute the other elements, we need to compute at most $n(n-1)/2$ derivatives.

If A is a matrix in its natural normal form constituted by cells, then all the elements of g can be known, if we solve the differential equations associated to every cell, to make good use of the symmetry of g and Ag. In other words, the equation $g_{,\lambda}=Ag$ can be expressed as a system of $n\times s$ linear homogeneous differential equations, every one of them with some order. If the equation i is of order r_i, then we can get r_i elements of the matrix g after we solve it. If we solve the $n \times s$ equations and we get some derivations, we will get the matrix g. This procedure is shown in the following example.

4. – Example.

We take the matrix 4.3 of table II in order to give an example. The matrix differential equation to solve is

$$g_{,\lambda} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a & 0 \end{bmatrix}, \qquad g = Ag, \tag{5}$$

with $a \in \boldsymbol{R}$. We arrive at

$$g_{,\lambda} = Ag = \begin{bmatrix} g_{21} & g_{22} & g_{23} & g_{24} \\ ag_{11} & ag_{12} & ag_{13} & ag_{14} \\ g_{41} & g_{42} & g_{43} & g_{44} \\ ag_{31} & ag_{32} & ag_{33} & ag_{34} \end{bmatrix}. \tag{6}$$

We begin with the first column. If we compare the two sides of eq. (6), we get

$$g'_{11} = g_{21}, \qquad g'_{31} = g_{41},$$

$$g'_{21} = ag_{11} \qquad g'_{41} = ag_{31}.$$

We observe here two cases: $a \geqslant 0$ and $a < 0$. If $a < 0$, we have that $g_{11} = c \exp[i\lambda\sqrt{-a}] + d \exp[-i\lambda\sqrt{-a}]$, which implies that $g'_{11} = g_{21}$ is complex.

Let then be $a \geqslant 0$. The solution for g_{11} and g_{12} is

$$g_{11} = c_1 \exp[\lambda\sqrt{a}] + d_1 \exp[-\lambda\sqrt{a}] \quad \text{and} \quad g_{12} = c_1\sqrt{a}\exp[\lambda\sqrt{a}] - d_1\sqrt{a}\exp[-\lambda\sqrt{a}];$$

in the same form, we get

$$g_{13} = c_2 \exp[\lambda\sqrt{a}] + d_2 \exp[-\lambda\sqrt{a}] \quad \text{and} \quad g_{14} = c_2\sqrt{a}\exp[\lambda\sqrt{a}] - d_2\sqrt{a}\exp[-\lambda\sqrt{a}].$$

It completes the solution of the first column. In the same way we solve the other columns. Using the symmetry of g, we arrive at

$$g = C\exp[\lambda\sqrt{a}] + D\exp[-\lambda\sqrt{a}],$$

with

$$C = \begin{bmatrix} c_1 & c_1\sqrt{a} & c_2 & c_2\sqrt{a} \\ c_1\sqrt{a} & c_1 a & c_2\sqrt{a} & c_2 a \\ c_2 & c_2\sqrt{a} & c_3 & c_3\sqrt{a} \\ c_2\sqrt{a} & c_2 a & c_3\sqrt{a} & c_3 a \end{bmatrix}$$

and

$$D = \begin{bmatrix} d_1 & -d_1\sqrt{a} & d_2 & -d_2\sqrt{a} \\ -d_1\sqrt{a} & d_1 a & -d_2\sqrt{a} & d_2 a \\ d_2 & -d_2\sqrt{a} & d_3 & -d_3\sqrt{a} \\ -d_2\sqrt{a} & d_2 a & -d_3\sqrt{a} & d_3 a \end{bmatrix},$$

where

$$\begin{vmatrix} d_1 & d_2 \\ d_2 & d_3 \end{vmatrix} \begin{vmatrix} c_1 & c_2 \\ c_2 & c_3 \end{vmatrix} = \frac{1}{16a^2} \qquad (a \neq 0)$$

in order to have $\det g = 1$. Remember that λ fulfills the generalized Laplace's equation:

$$(\rho\lambda_{,z})_{,\bar{z}} + (\rho\lambda_{,\bar{z}})_{,z} = 0 \qquad (x^1 = \rho).$$

One solution of this equation is $\lambda = \ln\rho$. Writing $n = \sqrt{a}$, we get the solution

$$g = C\rho^n + D\rho^{-n},$$

which is an analogous solution of the cylindrically symmetric class in the four-dimensional theory.

Further solutions can be obtained by

1) finding other solutions of the Laplace's equations;

2) using the symmetry of the group $g = Cg_0C^T$, where g_0 is a known solution;

3) using table II.

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