Exact solutions of SL(N,$\mathbb{R}$)-invariant chiral equations one- and two-dimensional subspaces

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A methodology for integrating the chiral equation $(\rho g_{,\bar{z}} g^{-1})_{,\bar{z}}+(\rho g_{,\bar{z}} g^{-1})_{,z}=0$ is developed, when g is a matrix of the SL(N,$\mathbb{R}$) group. In this work the ansätze $g=g(\lambda)$ where λ satisfy the Laplace equation and $g=g(\lambda,\tau)$ are made, where λ and τ are geodesic parameters of an arbitrary Riemannian space. This reduces the chiral equation to an algebraic problem and g can be obtained by integrating a homogeneous linear system of differential equations. As an example of the first ansatz, all the matrices for $N=3$ and one example for $N=8$, which corresponds to exact solutions of the $d=5$ and $d=10$ Kaluza–Klein theory, respectively are given. For the second ansatz the chiral equations are integrated for the subgroups SL(2,$\mathbb{R}$), SO(2,1$\mathbb{R}$), Sp(2,$\mathbb{R}$), and the Abelian subgroups.

I. INTRODUCTION

Chiral fields appear in many problems in physics. The most studied of them are perhaps the SU(N)-invariant chiral fields.[1] Nevertheless, SL(N,$\mathfrak{R}$)-invariant chiral fields are also present in unified theories, as, for example, the Kaluza–Klein theory. Here we are restricted to showing how SL(N,$\mathfrak{R}$), invariant chiral fields appear in n-dimensional Kaluza–Klein theories. The goal of this work is to give a method to obtain exact solutions of these fields.

The n-dimensional Kaluza–Klein theory is one unifing theory of weak–strong, electromagnetic and gravitational interactions.[2] Originally, it was formulated by Kaluza in 1921 and Klein in 1926 in a five-dimensional Riemannian space, where the five-dimensional Ricci tensor vanished. The five-dimensional Kaluza–Klein theory unified electromagnetism and gravitation. The generalization to more than five dimensions has shown to be a mechanism for unifying all the until now well-known interactions in physics. The n-dimensional theory assumes that the whole space V_n has a structure of principal fibre bundle, with a four dimensional Riemannian base space V_4, interpreted as the space-time, and typical fiber, a Lie group G, called the inner space.[3] It is supposed that V_n can be endowed with a Riemannian metric $\hat{g}$, which is invariant under the right action of G on V_n. In such a way, the metric $\hat{g}$, written in local coordinates, reads as

$$\hat{g}=g_{\alpha\beta}\,dx^\alpha\,dx^\beta+\xi_{mn}(\theta^m+A^m_\alpha\,dx^\alpha)(\theta^n+A^n_\beta\,dx^\beta)$$

$$\alpha,\beta=1\cdots 4;\quad m,n=5,\dots,n, \qquad (1)$$

where $g_{\alpha\beta}$, ξ_{mn}, and A^m_α depend only on x^α in order to have right invariance of $\hat{g}$ on G; $\{\theta^m\}m=5\cdots n$, is a basis of right-invariant one-forms on G. In (1) the space-time metric is $g=g_{\alpha\beta}\,dx^\alpha\,dx^\beta$, the G connection is represented by the one-form $A^m=A^m_\alpha\,dx^\alpha$, and the metric on the fibre is $\xi_{mn}\,dy^m\,dy^n$. The field equations are accepted to be the vanishing of the n-dimensional Ricci tensor. If we do so we obtain the four-dimensional Einstein's equations coupled with the Yang–Mills fields and a scalar multiplet.

In this work we are interested in finding exact solutions of these field equations, when the components of the n-dimensional metric $\hat{g}$ depend only on two coordinates, x^1 and x^2. In such a case, and without a loss of generality, we can rewrite (1) in the form

$$\hat{g}=f(d\rho^2+d\zeta^2)+\gamma_{ab}\,dx^a\,dx^b,\quad a,b=3,\dots,n, \qquad (2)$$

where the components of $\hat{g}$, f, and γ_{ab} now depend on ρ and ζ. The field equations $R_{AB}=0$ $A,B=1,\dots,n$ for the metric (2) reduce to[4]

$$\text{(a)}\quad (\ln \rho f)_{,z}=\tfrac{1}{2}\rho\,\mathrm{tr}(g_{,z}g^{-1})^2,$$

$$\text{(b)}\quad (\rho g_{,z}g^{-1})_{,\bar{z}}+(\rho g_{,\bar{z}}g^{-1})_{,z}=0, \qquad (3)$$

$$\det g=-\rho^2,\quad \gamma_{ab}=(g)_{ab},\quad z=\rho+i\zeta.$$

The main goal of this work is to give one method for solving Eq. (3b). Equation (3a) [knowing a solution of (3a)] is a linear differential equation of first order for the function f. Therefore one solution of f always depends on one solution of γ. Now we want to explain the method we are proposing.[5]

II. THE p-DIMENSIONAL SUBSPACES

We suppose now that the matrix g depends on a set of parameters λ^i, $i=1,...,p$, which depend on z and $\bar{z}$, $\lambda^i=\lambda^i(z,\bar{z})$, i.e.,

$$g=g(\lambda^i). \tag{4}$$

In this case Eq. (3b) transforms to

$$[(g_{,i}g^{-1})_{,j}+(g_{,j}g^{-1})_{,i}]\rho\lambda^i_{,z}\lambda^j_{,\bar{z}}+g_{,i}g^{-1}[(\rho\lambda^i_{,z})_{,\bar{z}}+(\rho\lambda^i_{,\bar{z}})_{,z}]=0. \tag{5}$$

Let us now suppose that the parameters λ^i are geodesics of an arbitrary Riemannian space V_p, with Christoffel symbols Γ^k_{ij}, i.e.,

$$(\rho\lambda^i_{,z})_{,\bar{z}}+(\rho\lambda^i_{,\bar{z}})_{,z}+2\rho\Gamma^i_{jk}\lambda^j_{,z}\lambda^k_{,\bar{z}}=0, \quad i,j,k=1,...,p. \tag{6}$$

Defining the matrix

$$A_i(g)=A_i=g_{,i}g^{-1}, \tag{7}$$

and using Eqs. (5) and (6), it is easy to see that the matrices A_i fullfill the Killing equation

$$A_{i;j}+A_{j;i}=0$$

and the relation

$$A_{i;j}-A_{j;i}=[A_i,A_j], \tag{8}$$

in the space V_p, where [ ] means matrix commutator. The matrices A_i, like the matrix g, are $N\times N$ $(N=n-2)$ matrices. The Killing equation then is fullfilled by each component of the matrices A_i. Using the well-known relation $\zeta_{n;b;a}=R^m_{abn}\zeta_m$, where ζ_m are the components of a Killing vector, we find that the covariant derivative of the Riemannian tensor of the space V_p vanishes. That means that the space V_p is symmetric.

Of course, the A matrix is right invariant under the action of the group G_c, G_c being the group of constant matrices in G (i.e., the matrices $A_0\in G$ that do not depend on λ^i). It is easy to show that the relation $A^{g_0}rA$ iff there exist $g_0\in G_c$, such that $A^{g_0}=A\circ L_{g_0}$ (L_{g_0} is the left action of G_c on G) is an equivalence relation. Let us call TB a set of representatives A^j of each class $[A^j]$, such that $\{[A^j]\}=A/\sim$. TB is a set of elements of the $\mathfrak{I}$ algebra of G, because A is the Maurer–Cartan form of G. Each element of TB can be mapped through the exponential map into the G group. Let be $B=\exp TB=\{g\in G\}|g=\exp A, A\in TB\}\subset G$. Then it is possible to show that (G,B,Π,G_0,L_{g_0}) is a principal fiber bundle with projection $\Pi(L(g_0,g))=g, L(g_0,g)=L_{g_0}(g)$.[6] That means that if we can know the basis set B of the bundle, we can construct all the elements of G through the left action L_{g_0} on G. In this work we will give two examples for the symmetric matrices of $G=SL(N,\Re)$. The properties of the matrices A_i can be deduced from the fact that g is a symmetric and real matrix. Furthermore, it is easy to see that the transformation $g\to-\rho^{-2/(n-2)}g$ led Eq. (3b) invariant. That means that we can renormalize g in order to have $\det g=(-1)^{N+1}$. In this case we can summarize the properties of g and A_i as

$$\begin{array}{lcll} g=\bar{g}, & & \text{(a)} & A_i=\bar{A}_i, \\ \det g=(-1)^{N+1} & \Rightarrow & \text{(b)} & T_rA_i=0, \\ g=g^T, & & \text{(c)} & A_ig=gA_i^T. \end{array} \tag{9}$$

In this work we study only the cases when $p=1,2$ for some interesting dimensions. The four-dimensional case was studied in Refs. 4 and 5 and the five-dimensional one in Ref. 7.

III. ONE-DIMENSIONAL SUBSPACES

We start taking the ansatz $g=g(\lambda)$, where λ is a function of z and $\bar{z}$, i.e., $\lambda=\lambda(z,\bar{z})$. Equation (3b) then reduces to

$$g_{,\lambda}=Ag, \tag{10}$$

A being a constant matrix and λ fullfiling the Laplace equation

$$(\rho\lambda_{,z})_{,\bar{z}}+(\rho\lambda_{,\bar{z}})_{,z}=0 \tag{11}$$

[compare (10) and (11) with Eqs. (6) and (8)]. The $N\times N$ matrix A has the properties (9). Now observe that the field equation (3b) is invariant under the transformation $g\to CgC^T$, the left action of $G_c\supset C$ on G, C being a constant matrix of the group $SL(N,\Re)$. Under this transformation the matrix A transforms to $A\to CAC^{-1}$. This last relation separates the set of matrices A in equivalent classes, which lets us to work only with the representatives of each class, because for each member of the class, the corresponding solution will be related with the solution using the representative of the class by the transformation $g\to CgC^T$, C being the matrix that relates A with CAC^{-1}. The next step is to find a convenient representative for each class. The first that one has is the Jordan normal form. But this normal form is diffiicult. In some cases the components of this kind of representatives are complex, although this is not a characteristic of the class. There is a more convenient representatives called the natural normal form. In Appendix A it is shown that all $N\times N$ matrices are similar to one and only one natural normal form, i.e., the representative of each class has the form[8]

$$A=\begin{bmatrix} A_1 & & & & \\ & A_2 & & O & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ O & & & & A_s \end{bmatrix},$$

where A_i, $i=1\cdots s$, is a square matrix called a cell, and is of the form

$$A_i=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & & & & \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \qquad (12)$$

$a_0,\ldots,a_{n-1}$ being the coefficients of its characteristic polynomial

$$p(\lambda)=\lambda^n+a_{n-1}\lambda^{n-1}+\cdots+a_1\lambda+a_0 \qquad (13)$$

[observe that we are using the same λ for Eqs. (11) and (13)].

Next, we will propose an algorithm for finding all the classes. In order to do so, we use criteria 1 and 2 of Appendix A, and separate our algorithm in two steps. First step: we make all the possible partitions of the matrix A in n_i cells, where $n_1+n_2+\cdots+n_s=N$. Taking into account Theorem 4 of Appendix B, we obtain a set of possible representatives. But that is not enough. Second step: We find all the invariant factors of each matrix, comparing them with the corresponding one of the other matrices. When all the invariant factors are different to all the other matrices, we have found a new class, and then a new representative. The result is shown in Table I. In this table we designate an arbitrary polynomial of degree n by (n). When 1 is the corresponding invariant factor, it represents a polynomial of degree zero. The representatives, i.e., the set elements of TB, are shown in Table II.

TABLE I. Degrees of the invariant factors of the matrix $A_i - I$.

Order n	INVARIANTS							
	$d_1(\lambda)$	$d_2(\lambda)$	$d_3(\lambda)$	$d_4(\lambda)$	$d_5(\lambda)$	$d_6(\lambda)$	$d_7(\lambda)$	$d_8(\lambda)$
2	1	(2)						
3	1	1	(3)					
	1	(2)	(2)					
4	1	1	1	(4)				
	1	1	(1)	(3)				
	1	1	(2)	(2)				
	1	(1)	(1)	(2)				
8	1	1	1	1	1	1	1	(8)
	1	1	1	1	1	1	(1)	(7)
	1	1	1	1	1	1	(2)	(6)
	1	1	1	1	1	1	(3)	(5)
	1	1	1	1	1	1	(4)	(4)
	1	1	1	1	1	(1)	(1)	(6)
	1	1	1	1	1	(1)	(2)	(5)
	1	1	1	1	1	(1)	(3)	(4)
	1	1	1	1	1	(2)	(2)	(4)
	1	1	1	1	1	(2)	(3)	(3)
	1	1	1	1	(1)	(1)	(1)	(5)
	1	1	1	1	(1)	(1)	(2)	(4)
	1	1	1	1	(1)	(1)	(3)	(3)
	1	1	1	1	(1)	(2)	(2)	(3)
	1	1	1	1	(2)	(2)	(2)	(2)
	1	1	1	(1)	(1)	(1)	(1)	(4)
	1	1	1	(1)	(1)	(1)	(2)	(3)
	1	1	1	(1)	(1)	(2)	(2)	(2)
	1	1	(1)	(1)	(1)	(1)	(1)	(3)
	1	1	(1)	(1)	(1)	(1)	(2)	(2)
	1	(1)	(1)	(1)	(1)	(1)	(1)	(2)

IV. THE MATRIX g

Now we have to solve the matrix differential equation $g_{,\lambda}=Ag$, in order to find the corresponding B set, A being a representative of a class. Each class will give us a new solution of the field equation (3b) in terms of λ.

For solving this differential equation, we proceed in the following form. First we solve the first column of the equation $g_{,\lambda}=Ag$, which is independent of the others. The first column is divided in blocks, each of them corresponding to one cell A_i, $i=1,\ldots,s$. Solving the blocks in terms of the first component g_{j1} of them, one observes that the solution of it depends only on the roots of the corresponding eigenvalue problem $p_A(\lambda)=0$, $i=1,\ldots,s$ $p_{A_i}(\lambda)$ being the characteristic polynomial of A_i (see Appendix B). The other components can be obtained by derivation. Second, one solves all the columns in the same way. The solution for each column will be the same, but with other integration constants. Third, we relate the in-

TABLE II. Classification of matrices.

Order n	Cases	Matrices A_i	Invariants
2		$\begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}$	$1,\ \lambda^2-a$
3	3.1	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & 0 \end{bmatrix}$	$1,\ 1,\ \lambda^3-\lambda b-a$
		$\begin{bmatrix} q & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2q^2 & -q \end{bmatrix}$	$1, \lambda-q,\ (\lambda+2q)(\lambda-q)$
4	4.1	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & c & 0 \end{bmatrix}$	$1,\ 1,\ 1,\ \lambda^4-c\lambda^2-b\lambda-a$
	4.2	$\begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2q^3-qb & b & -q \end{bmatrix}$	$1,\ 1, \lambda-q,\ \lambda^3+q\lambda^2-b\lambda-a$ con $a=q(2q^2-b)$
	4.3	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a & 0 \end{bmatrix}$	$1,\ 1,\ \lambda^2-a,\ \lambda^2-a$
	4.4	$\begin{bmatrix} -3a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix}$	$1,\ \lambda-a,\ \lambda-a,\ (\lambda-a)(\lambda+3a)$

TABLE III. One-dimensional subspaces.

A	A′	g	$T_r A^2$	Constant Parameters
	$\begin{pmatrix} 0 & 1 \\ d & 0 \end{pmatrix}$	$b\begin{pmatrix} 1 & \sqrt{d} \\ \sqrt{d} & d \end{pmatrix} e^{1/2\sqrt{d}\lambda} + \frac{1}{4db}\begin{pmatrix} -1 & \sqrt{d} \\ \sqrt{d} & -d \end{pmatrix} e^{-1/2\sqrt{d}\lambda}$	$2b$	d, b
	a) $\begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix}$	$g = \begin{pmatrix} ae^{r_1\lambda} & 0 & 0 \\ 0 & be^{r_2\lambda} & 0 \\ 0 & 0 & ce^{r_3\lambda} \end{pmatrix}$	$r_1^2 + r_2^2 + r_3^2$	$r_1^2 + r_2^2 + r_3^2 = 0$ $abc = 1$
$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{pmatrix}$	b) $\begin{pmatrix} -2r_1 & 0 & 0 \\ 0 & r_1 & 1 \\ 0 & 0 & r_1 \end{pmatrix}$	$\begin{pmatrix} ae^{-3r_1\lambda} & 0 & 0 \\ 0 & (b\lambda + c) & b \\ 0 & b & 0 \end{pmatrix} e^{r_1\lambda}$	$6r_1^2$	$ab^2 = -1$ c, r_1
	c) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \beta & 0 \end{pmatrix}$	$\begin{pmatrix} d & -\Lambda c & \beta d \\ -\Lambda c & \beta d & -\beta\Lambda c \\ \beta d & -\beta\Lambda c & \beta^2 d \end{pmatrix} \sin\Lambda\lambda + \begin{pmatrix} c & \Lambda d & \beta c \\ \Lambda d & \beta c & \beta\Lambda d \\ \beta c & \beta\Lambda d & \beta^2 c \end{pmatrix} \cos\Lambda\lambda + \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	2β	$\Lambda^2 = -\beta > 0$ $b\beta^3(c^2 + d^2) = 1$
	c) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha & \beta & 0 \end{pmatrix}$	$a\begin{pmatrix} 1 & -2m & 4m^2 \\ -2m & 4m^2 & -8m^3 \\ 4m^2 & -8m^3 & 16m^4 \end{pmatrix} e^{-2m\lambda} + b\begin{pmatrix} 1 & m+in & (m+in)^2 \\ m+in & (m+in)^2 & (m+in)^3 \\ (m+in)^2 & (m+in)^3 & (m+in)^4 \end{pmatrix} e^{(m+in)\lambda} + \text{c.c.}$	2β	$\alpha = -2m(m^2 + n^2)$ $\beta = n^2 - 3m^2$ $4ab\bar{b} = -\frac{m^4}{n^4(9m^2+n^2)}$
$\begin{pmatrix} q & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2q^2 & -q \end{pmatrix}$	$q \neq 0$ $\begin{pmatrix} -2r_1 & 0 & 0 \\ 0 & r_1 & 0 \\ 0 & 0 & r_1 \end{pmatrix}$	$\begin{pmatrix} ae^{-3r_1\lambda} & 0 & 0 \\ 0 & b & c \\ 0 & c & d \end{pmatrix} e^{r_1\lambda}$	$6r_1^2$	$a(bd - c^2) = 1$ r_1
	$q = 0$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a & b & 0 \\ b & c\lambda + d & c \\ 0 & c & 0 \end{pmatrix}$	0	$ac^2 = -1$ b, d
	$(\rho\lambda_{,\bar{z}})_{,z} + (\rho\lambda_{,z})_{,\bar{z}} = 0$	$(\ln\rho f)_{,z} = \frac{1}{\rho}(\ln\rho)_{,z} + \frac{\rho}{2}(\lambda_{,z})^2\,\mathrm{tr}\,A^2$		

tegration constants, making use of the symmetry relation $g=g^T$, and therefore $g_{,\lambda}=Ag=g^T{}_{,\lambda}$. When A_i is a natural normal cell, we obtain that all the l-antidiagonal, i.e., the components g_{ij} with $i+j=l$, are equal to each other. Finally, one obtains a solution of the field equation (3b) in terms of λ. In order to obtain it in terms of z and $\bar{z}$, we have to write λ in terms of these variables. It can be done solving the Laplace equation (11). For each solution of the Laplace equation we will have a new solution of the field equation (3b) in terms of z and $\bar{z}$.

In the first step of this algorithm, the g_{j1} component of g is determined by a linear differential equation of the order of the corresponding cell A_i with constant coefficients, whose characteristic polynomial is just the characteristic polynomial of A. Therefore the solution of g_{j1} (and of all components of g) will depend on the multiplicity of the roots of the polynomial, i.e., of the roots of the equation $p_A(\lambda)=\Pi_{i=1}^{s}p_i(\lambda)=0$, where $p_i(\lambda)$ is the characteristical polynomial associated with the cell A_i. Therefore it is necessary to do a classification of the eigenvalues of A in order to have an explicit solution of g in terms of λ (see, for example, Ref. 7 for the five-dimensional case).

Example 1: We give now some examples in order to show the method. We take the $N=3$ dimensional case, which corresponds to the five-dimensional Kaluza–Klein theory. As is shown in Table II, this dimension has two representatives. For the first one, the characteristic polynomial is

$$\lambda^3-b\lambda-a=0, \qquad (14)$$

and it is, at the same time, the characteristic polynomial of the differential equation $g_{,\lambda}=Ag$. We have to classify the roots of Eq. (14). There are three possibilities: (a) All the roots are real and different; (b) all of them are real but two are the same; and (c) two are complex and one is real. For case (a) we can diagonalize the matrix A in order to obtain an easier form of the matrix. For case (b) it is better to use the corresponding Jordan form (in both these cases the matrix A remains real). This is a spacial case when (14) has one null eigenvalue. It is shown as case (b) in Table III.

The second representative has the characteristical polynomial

$$(\lambda-q)^2(\lambda+2q)=0. \qquad (15)$$

The roots are well determined, but the explicit solution depends on whether $q\neq0$ or $q=0$. If $q\neq0$ the diagonal form is more convenient. All the corresponding solutions are shown in Table III.

If we want to solve Eq. (3a), we have to take into account the transformation $g\rightarrow-\rho^{-2/N}g$. Under this transformation, Eq. (3a) transforms into

$$(\ln\rho^{1-1/N}f)_{,z}=\tfrac{1}{2}\rho\, tr(g_{,z}g^{-1})^2.$$

The matrix $g_{,z}g^{-1}$ can now be cast into its λ form $g_{,\lambda}g^{-1}\lambda_{,z}=A\lambda_{,z}$, and observing that $\mathrm{tr}\,A^2=2b$ for the first representative of Table III and $\mathrm{tr}\,A^2=6q^2$ for the second representative ($N=3$ in Table III), one arrives at

$$(\ln(\rho^{1-1/N}f))_{,z}=b\rho(\lambda_{,z})^2. \qquad (16)$$

The integrability conditions of Eq. (16) are guaranteed because λ is a solution of the Laplace equation (11). Note that the integration of Eq. (16) only depends on the value of $\mathrm{tr}\,A^2$, which is always a constant; furthermore, it is an invariant number of the class. The $N=2$-*dimensional* case was studied in Ref. 4 and we only give the results in Table III.

Example 2: A very worked dimension is $d=10$, which corresponds to superstrings theory. If we are studing superstrings theory in a curved space-time underground and accept that this underground satisfies the field equations $R^d{}_{ab}=0$, $R^d{}_{ab}$ being the Ricci tensor in d dimensions, the field equations reduce to (3). In Ref. 4 the cylindrically symmetric solutions from the case $N=2$ in Table III were found, with λ:

$$\lambda=(n/\sqrt{a})\ln\rho, \quad n\in\Re. \tag{17}$$

In order to give one example in ten-dimensional relativity, we take the matrix

$$g_{,i}=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \end{bmatrix} g=Ag,$$

which has the set of invariant factors,

$$d_1(\lambda)=d_2(\lambda)=d_3(\lambda)=d_4(\lambda)=1,$$

$$d_5(\lambda)=d_6(\lambda)=d_7(\lambda)=d_8(\lambda)=\lambda^2-a.$$

If we use the methodology given in this paper, we arrive at

$$g=Ce^{\lambda\sqrt{a}}+De^{-\lambda\sqrt{a}},$$

with $a\in\Re$,

$$C=\begin{bmatrix} c_1 & c_1\sqrt{a} & c_3 & c_3\sqrt{a} & c_5 & c_5\sqrt{a} & c_4 & c_4\sqrt{a} \\ c_1\sqrt{a} & c_1a & c_3\sqrt{a} & c_3a & c_5\sqrt{a} & c_5a & c_7\sqrt{a} & c_7a \\ c_3 & c_3\sqrt{a} & c_2 & c_2\sqrt{a} & c_4 & c_4\sqrt{a} & c_6 & c_6\sqrt{a} \\ c_3\sqrt{a} & c_3a & c_2\sqrt{a} & c_2a & c_4\sqrt{a} & c_4a & c_6\sqrt{a} & c_6a \\ c_5 & c_5\sqrt{a} & c_4 & c_4\sqrt{a} & c_8 & c_8\sqrt{a} & c_{10} & c_{10}\sqrt{a} \\ c_5\sqrt{a} & c_5a & c_4\sqrt{a} & c_4a & c_8\sqrt{a} & c_8a & c_{10}\sqrt{a} & c_{10}a \\ c_7a & c_7\sqrt{a} & c_6 & c_6\sqrt{a} & c_{10} & c_{10}\sqrt{a} & c_9 & c_9\sqrt{a} \\ c_7\sqrt{a} & c_7a & c_6\sqrt{a} & c_6a & c_{10}\sqrt{a} & c_{10}a & c_9\sqrt{a} & c_9a \end{bmatrix}$$

and

$$D=\begin{bmatrix} d_1 & -d_1\sqrt{a} & d_3 & -d_3\sqrt{a} & d_5 & -d_5\sqrt{a} & d_7 & -d_7\sqrt{a} \\ -d_1\sqrt{a} & d_1a & -d_3\sqrt{a} & d_3a & -d_5\sqrt{a} & d_5a & -d_7\sqrt{a} & d_7a \\ d_3 & -d_3\sqrt{a} & d_2 & -d_2\sqrt{a} & d_4 & -d_4\sqrt{a} & d_6 & -d_6\sqrt{a} \\ -d_3\sqrt{a} & d_3a & -d_2\sqrt{a} & d_2a & -d_4\sqrt{a} & d_4a & -d_6\sqrt{a} & d_6a \\ d_5 & -d_5\sqrt{a} & d_4 & -d_4\sqrt{a} & d_8 & -d_8\sqrt{a} & d_{10} & -d_{10}\sqrt{a} \\ -d_5\sqrt{a} & d_5a & -d_4\sqrt{a} & d_4a & -d_8\sqrt{a} & d_8a & -d_{10}\sqrt{a} & d_{10}a \\ d_7 & -d_7\sqrt{a} & d_6 & -d_6\sqrt{a} & d_{10} & -d_{10}\sqrt{a} & d_9 & -d_9\sqrt{a} \\ -d_7\sqrt{a} & d_7a & -d_6\sqrt{a} & d_6a & -d_{10}\sqrt{a} & d_{10}a & -d_9\sqrt{a} & d_9a \end{bmatrix}.$$

Using the λ given in (17), one arrives at

$$g=C\rho^n+D\rho^{-n},$$

and the integration from Eq. (16) of the superpotential f is

$$f=\rho^{2n^2-7/8},$$

which integrates the whole ten-dimensional metric.

V. TWO-DIMENSIONAL SUBSPACES

Here we present another method following the mechanics showed in the Introduction. Now we make the ansatz

$$g=g(\lambda,\tau), \quad \lambda=\lambda(z,\bar{z}), \quad \tau=\tau(z,\bar{z}), \tag{18}$$

where g is a symmetric $N\times N$ matrix of the group SL(N,$\mathbb{R}$). We shall suppose that λ and τ are "geodesic" of a certain Riemannian space V_2,

$$(\rho\lambda^i{}_{,z})_{,\bar{z}}+(\rho\lambda^i{}_{,\bar{z}})_{,z}+2\rho\Gamma^i{}_{jk}\lambda^i{}_{,z}\lambda^k{}_{,\bar{z}}=0, \quad i,j,k=1,2, \tag{19}$$

where $(\lambda^1,\lambda^2)=(\lambda,\tau)$. The ansatz (18) and (19) was made first by Neugebauer and Kramer[5] in the case that g is a matrix of the group SU(2,1).

Let us define the two matrices in the corresponding Lie algebra of the group SL(N,$\mathbb{R}$),

$$A_1(g)=A_1=g_{,\lambda}g^{-1}, \quad A_2(g)=A_2=g_{,\tau}g^{-1}. \tag{20}$$

Using the chain rule in (3), we substitute (18) and (19), and obtain that the matrix vector $A=(A_1,A_2)$ satisfies the Killing equation in V_2,

$$A_{1;2}+A_{2;1}=0, \quad \lambda,\tau=1,2. \tag{21}$$

This means that each conponent of A satisfies the Killing equation in V_2. From the definition (20), it is easy to see that the meaning of the covariant derivative in matrix notation becomes the commutator between the matrices A_1 and A_2, i.e.,

$$A_{i;j}=\tfrac{1}{2}[A_j,A_i]. \tag{22}$$

Observe that A_1 and A_2 are traceless and real, because they belong to the Lie algebra sl(N,$\mathbb{R}$). The V_2 space must be symmetric (which means that the curvature of the V_2 space must be constant). Because all space V_2 is conformally flat, we can write the two metric as

$$dS^2=\frac{d\lambda\, d\tau}{(1+K\lambda\tau)^2}, \tag{23}$$

K being the constant curvature of V_2. The metric (23) has three independent Killing vectors. Let be ϕ, ξ, ζ, these three Killing vectors. Then the matrix vector A can be written as

$$A=\sigma_1\phi+\sigma_2\xi+\sigma_3\zeta. \tag{24}$$

Hence the matrices A_1 and A_2 belong to the three-dimensional subalgebras of sl(N,$\mathbb{R}$). Equation (22) is just the connection between the Lie subalgebras and the V_2 spaces. We choose three independent Killing vectors of V_2:

$$\phi=\frac{1}{2V^2}(K\tau^2+1,K\lambda^2+1), \quad \xi=\frac{1}{V^2}(-\tau,\lambda),$$
$$\zeta=\frac{1}{2V^2}(K\tau^2-1,1-K\lambda^2), \quad V=(1+K\lambda\tau), \tag{25}$$

and, using Eq. (22), it is easy to see that the commutation relations of the sl(N,$\mathbb{R}$) subalgebras for the matrices σ_1, σ_2, σ_3 are

$$[\sigma_1\sigma_2]=-4K\sigma_3, \quad [\sigma_2\sigma_3]=4K\sigma_1, \quad [\sigma_3\sigma_1]=-4\sigma_2. \tag{26}$$

VI. CHANGE OF BASE

If we change the base of the Lie algebra σ_1, σ_2, σ_3, for example, as follows:

$$\sigma_1=aX_1+bX_2+cX_3, \quad \sigma_2=dX_1+eX_2+fX_3,$$
$$\sigma_3=gX_1+hX_2+iX_3, \tag{27}$$

where X_1, X_2, and X_3 satisfy the commutator relation (26), then the base of the solutions of the Killing equations changes to

$$\phi'=a\phi+d\xi+g\zeta, \; \xi'=b\phi+e\xi+h\zeta, \; \zeta'=c\phi+f\xi+i\zeta. \tag{28}$$

There exists a one-to-one correspondence between the group $GL(3,\mathbb{R})$ and the set of transformations

$$MX=\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}=\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}=\sigma, \tag{29}$$

which we can use in order to obtain a new representation of the matrix A. This new representation does not satisfy the same commutation relations (26), in general. But there is a subset of $GL(3, \mathbb{R})$, that makes the representation (26) invariant. The elements of this subset have the inverses

$$M^{-1}=\frac{1}{a^2-Kb^2-c^2}\begin{pmatrix} a & -d/K & -g \\ -Kb & e & Kh \\ -c & f/K & i \end{pmatrix}. \tag{30}$$

The equation $MM^{-1}=I$ gives five independent algebraic equations for the nine components of M. Thus we can change the Killing vectors base (28), making the commutation relation (26) invariant using the transformation (30). If $K=0$, the components of M satisfy $af=cd$ and $di=fg$, $e=0$, b and h remain free, but one of them must be different from zero in order to conserve the dimensionality.

VII. THE SL(N R) INVARIANCE

Equation (3) is invariant under transformations of the group SL(N,ℝ). But if we also note that g is a symmetric matrix of this group, we have to take the invariance transformation (i.e., the left action L_c on G)

$$g\to CgC^T=L_c(g), \tag{31}$$

in order to conserve symmetry (g^T denotes the matrix transposition of g). In the Lie algebra sl(N ℝ) this transformation is translated as the equivalence relation

$$A_1\to CA_1C^{-1}, \quad A_2\to CA_2C^{-1}. \tag{32}$$

We have already used this last relation for finding classes of solutions in the one-dimensional case. Relation (32) makes clear, again, that it is enough to work with the representatives of the classes. We will use the classification of Table II now. Let us solve Eq. (3). In order to do so, we have to give a base of the Killing vector space in V_2. Using Eq. (22), one finds the corresponding commutation relations for the three-dimensional base of the subalgebra of sl(N, ℝ). Having the explicit form of the Lie algebra one uses the exponential map for finding the group elements, or, equivalently, one integrates the first-order differential equations system (20).

There are not too much subalgebras of dimension three of the Lie algebra sl(N,ℝ). Among the classical groups there are well-known isomorphisms of dimension three. Here we shall study the group isomorphisms SO(2, 1;ℝ) = Sp(2, ℝ) = SL(2,ℝ) (see, for example, Ref. 9).

VIII. THE FLAT SUBSPACE $k=0$

A very interesting case is if $K=0$ in (26). Then the space V_2 is flat and the λ and τ parameters fulfill the Laplace equation separately. To this case belongs the algebras of dimension one and two, with $\sigma_2=\sigma_3=0$ and $\sigma_2=0$, respectively. We start by supposing $K=0$ and $\sigma_2=0$, in order to study the two-dimensional subalgebras. They are Abelian algebras with

$$[\sigma_1,\sigma_3]=0, \quad \sigma_1,\sigma_3\in \mathrm{sl}(N,\mathbb{R}). \tag{33}$$

For $N=2$ there is no representation. For $N=3$ we first classify the matrices σ_1, because they are traceless and real matrices, and therefore they must be of one of the forms shown in Table II. It is convenient to change the base vectors in the killing space. We choose

$$\phi\to\phi-\zeta=(1,0), \quad \zeta\to\phi+\zeta=(0,1) \tag{34}$$

(remember that we need only two vectors, because $\sigma_2=0$). Hence the base of the Abelian Lie algebra is transformed in such a form that $A_1=\sigma_1$ and $A_2=\sigma_3$:

$$\sigma\to\sigma_1+\sigma_3=A_1, \quad \sigma_3\to-\sigma_1+\sigma_3=A_2. \tag{35}$$

For $N=3$ we have only two normal forms for the matrix A_1:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha & \beta & 0 \end{pmatrix} \text{ and } \begin{pmatrix} q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2q^2 & -q \end{pmatrix}. \tag{36}$$

For the first matrix of (36), one can show that the matrix commutating with it is

$$\sigma_3=\begin{pmatrix} -2\beta/3 & 0 & 1 \\ \alpha & \beta/3 & 0 \\ 0 & \alpha & \beta/3 \end{pmatrix}. \tag{37}$$

We now have to solve the system of differential equations of first order

$$g_{,\lambda}=\sigma_1 g \quad \text{and} \quad g_{,\tau}=\sigma_3 g, \tag{38}$$

the characteristic polynomial of σ_1 is $r^3-\beta r-\alpha=0$ and of σ_3:

$$-\left(\frac{2\beta}{3}+\tau\right)\left(\frac{\beta}{3}-\tau\right)^2+\alpha^2=0. \tag{39}$$

Because of the symmetry of g, $g_{,\lambda}$ and $g_{,\tau}$ must be also symmetric matrices. For the first equation in (38), it is enough to take $g_{22}=g_{13}$, $g_{23}=\alpha g_{11}+\beta g_{12}$, and $g_{33}=\alpha g_{12}$

TABLE IV. Two-dimensional subspaces.

N = 3	σ_1	σ_3	g		Constant Parameters
$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha & 30 & \end{pmatrix}$	$\begin{pmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{pmatrix}$	$\begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix}$	$\begin{pmatrix} ae^{\gamma_1\lambda+t_1\tau} & 0 & 0 \\ 0 & be^{\gamma_2\lambda+t_2\tau} & 0 \\ 0 & 0 & ce^{\gamma_3\lambda+t_3\tau} \end{pmatrix}$		$\gamma_1+\gamma_2+\gamma_3 = 0$ $t_1+t_2+t_3 = 0$ $\gamma_1 t_2 - \gamma_2 t_1 \neq 0$ $abc = 1$
	$\begin{pmatrix} -2\gamma_1 & 0 & 0 \\ 0 & \gamma_1 & 0 \\ 0 & 0 & \gamma_1 \end{pmatrix}$	$\begin{pmatrix} -2\beta & 0 & 0 \\ 0 & \beta & \alpha \\ 0 & 0 & \beta \end{pmatrix}$	$\begin{pmatrix} ae^{-3\gamma_1\lambda-3\beta\tau} & 0 & 0 \\ 0 & (\lambda+\alpha\tau)c+b & c \\ 0 & c & 0 \end{pmatrix} e^{\gamma_1\lambda+\beta\tau}$		$\gamma_1\cdot\alpha-\beta\neq 0$ $ac^2=-1, b$
	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha & \beta & 0 \end{pmatrix}$	$\begin{pmatrix} -2\beta/3 & 0 & 1 \\ \alpha & \beta/3 & 0 \\ 0 & \alpha & \beta/3 \end{pmatrix}$	$a\begin{pmatrix} 1 & -2m & 4m^2 \\ -2m & 4m^2 & -8m^3 \\ 4m^2 & -8m^3 & 16m^2 \end{pmatrix} e^{-2m\lambda+\left(4m^2\frac{2\beta}{3}\right)\tau} +$ $b\begin{pmatrix} 1 & m+in & (m+in)^2 \\ m+in & (m+in)^2 & (m+in)^3 \\ (m+in)^2 & (m+in)^3 & (m+in)^4 \end{pmatrix} e^{(m+in)\lambda+\left((m+in)^2-\frac{2\beta}{3}\right)\tau} + \text{c.c.}$		a, b, m, n
$\begin{pmatrix} q & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 2q^2 & -q \end{pmatrix}$	$r_1\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -(\alpha+\beta) & 0 & 0 \\ 0 & \alpha & \delta \\ 0 & \gamma & \beta \end{pmatrix}$	$\begin{pmatrix} ae^{-3r_1\lambda-(\alpha+\beta)\tau} & 0 & 0 \\ 0 & (be^{q_1\tau}+ce^{q_2\tau}) & \left(\frac{\gamma}{q_1-\beta}be^{q_1\tau}+\frac{\gamma}{q_2-\beta}ce^{q_2\tau}\right) \\ 0 & g_{32}=g_{23} & \left[\left(\frac{\gamma}{q_1-\beta}\right)^2 be^{q_1\tau}+\left(\frac{\gamma}{q_2-\beta}\right)^2 ce^{q_2\tau}\right] \end{pmatrix} e^{r_1\lambda}$	$q_i^2-(\beta+\alpha)q_i$ $+\alpha\beta-\delta\gamma=0$ $i=1,2$	$abc\gamma^2\frac{q_2-q_1}{(q_1-\beta)(q_2-\beta)}=1$ $q_1, q_2\in\mathbf{R}$ $q_1-q_2\neq 0$ a, b, c
			$\begin{pmatrix} ae^{-3r_1\lambda-\frac{3(\alpha+\beta)}{2}\tau} & 0 & 0 \\ 0 & (b\tau+c) & \left[\frac{\beta-\alpha}{2\delta}b\tau+\frac{1}{\delta}\left(b-\frac{\alpha-\beta}{2}\right)c\right] \\ 0 & g_{23} & \left[\left(\frac{\beta-\alpha}{2\delta}\right)^2 b\tau-\frac{\gamma}{\delta}c-\frac{\alpha-\beta}{\delta^2}b\right] \end{pmatrix} e^{\frac{\alpha+\beta}{2}\tau+r_1\lambda}$		$q_1, q_2\in\mathbf{R}$ $\delta\neq 0\neq b$ $a=-\left(\frac{\delta}{b}\right)^2$ $q_1=q_2=(\alpha+\beta)/2$ a, b, c
			$\begin{pmatrix} ae^{-3r_1\lambda-(\alpha+\beta)\tau} & 0 & 0 \\ 0 & (be^{q\tau}+\bar{b}e^{\bar{q}\tau}) & \left(\frac{q-\alpha}{\delta}be^{q\tau}+\frac{\bar{q}-\alpha}{\delta}\bar{b}e^{\bar{q}\tau}\right) \\ 0 & g_{23} & \left(\frac{q-\alpha}{\delta}\right)^2 be^{q\tau}+\left(\frac{\bar{q}-\alpha}{\delta}\right)^2\bar{b}e^{\bar{q}\tau} \end{pmatrix} e^{r_1\lambda}$		$(4ab\bar{b}/\delta^2)\times$ $(4\gamma\delta+(\alpha-\beta)^2)=1$ $a, b, q=m+in$
	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -2\alpha & 0 & \gamma \\ \beta & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$	$\begin{pmatrix} ae^{-2\alpha\tau} & -\beta(a/3\alpha)(e^{\alpha\tau}+e^{-2\alpha\tau}) & 0 \\ g_{12}=g_{21} & \left[\left(-\frac{\beta}{\gamma}\lambda-\frac{\beta^2}{3\alpha}\tau\right)a+b\right]+e^{-2\alpha\tau}a\left(\frac{\beta}{3\alpha}\right)^2 & -\frac{\beta}{\gamma}ae^{\sigma\tau} \\ 0 & -\alpha\frac{\beta}{\gamma}e^{\alpha\tau} & 0 \end{pmatrix}$		$a=-(\gamma/\beta)^{2/3}$ $\alpha\neq 0\neq\gamma$ b, β
			$\begin{pmatrix} a & a\beta\tau+b & 0 \\ \beta a\tau+b & \frac{\beta^2}{2}a\tau^2+b\tau+c & \frac{\beta}{\gamma}a \\ 0 & \frac{\beta}{\gamma}a & 0 \end{pmatrix}$		$a=-(\gamma/\beta)^{2/3}$ $\gamma\neq 0 \quad \alpha=0$ b, c, β

$+\beta g_{13}$. For the second one we get just the same relations, thus the set of matrices $\sigma_1=A_1$, $\sigma_3=A_2$, which we can use, is restricted to (36). The solution of the differential equations (4) depends on the kind of roots, which are obtained from the respective characteristic polynomial. It is easy to see that this condition depends on whether $D=27\alpha^3-32$ is greater than, less than, or equal to zero for both matrices, σ_1 and σ_3. Hence both matrices have the same kind of eigenvalues. Let be $D>0$. We look for solutions of the form

$$g_j=G_je^{r\lambda+t\tau}, \tag{40}$$

where g_j is the jth column of g. We solve the equation column by column. The substitution of (40) in (38) shows that the two linear algebraic systems,

$$(A_1-rI)G_j=0, \quad (A_2-tI)G_j=0, \tag{41}$$

must be consistent. This condition is satisfied if we take $t=r^2-2\beta/3$ or $r=-\alpha(\beta/3-t)$. From the eigenvalue equation of σ_1 follows the eigenvalue equation for σ_3. The other two cases $D<0$ and $D=0$ do not cause any difficulty if we take the corresponding Jordan form for σ_1.

The second matrix in (36) can be divided into two cases, $q\neq 0$ and $q=0$. If $q\neq 0$, we can transform the matrix to

$$\sigma_1=\begin{pmatrix} -2\gamma_1 & 0 & 0 \\ 0 & \gamma_1 & 0 \\ 0 & 0 & \gamma_1 \end{pmatrix}, \quad \sigma_3=\begin{pmatrix} a & 0 & 0 \\ 0 & e & f \\ 0 & h & i \end{pmatrix}, \tag{42}$$

σ_3 being the matrix with which σ_1 commutates. The first equation (38) for (41) is easy to solve. The second one gives some restrictions to g. The results are shown in Table IV.

The generalization for arbitrary N can be done as follows. One starts from the normal form of σ_1 and determines the most general matrix that commutates with it calling this matrix σ_3. One calculates the characteristic polynomial of both matrices, and with each eigenvalue one looks for solutions of the form (40), g_j being the jth column of g. One arrives at an algebraic equation of type (41) and looks for the conditions with which they are compatible. If the compatibility does exist, one finds the solutions for the matrix column G_j, fulfilling both algebraic equations (41). If the eigenvalues of σ_1 are real, it is sometimes convenient to transform the natural normal form to the Jordan form. An alternative algorithm for passing over from the Lie algebra to the corresponding Lie group is via the exponential map. The problem of this alternative method is that sometimes one obtains an infinite series of coefficients and matrices that are not appropiate for analytic calculations.

For the three-dimensional subalgebras with $K=0$ we did not have success in $N=3$. We find two algebras:

$$\sigma_1=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&0&0\\\delta&0&\rho\\0&0&0\end{pmatrix},$$

$$\sigma_3=4\begin{pmatrix}-\rho&p&-\delta^{2/\delta}\\d&0&f\\\delta&0&\rho\end{pmatrix},\quad \delta\neq 0$$

and

$$\sigma_1=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&0&\beta\\\delta&0&\rho\\0&0&0\end{pmatrix},$$

$$\sigma_3=4\begin{pmatrix}\rho&-\beta&c\\\rho^{2/\rho}&-\rho&f\\\delta&0&0\end{pmatrix},\quad \beta\neq 0,$$

but both are not compatible with the symmetry of g, $g=g^T$. Hence we did not find any solution. Perhaps it is possible to find such a solution after some transformation of type (1).

IX. THE CLASSICAL GROUPS

We deal now with the case $K\neq 0$ in (25). We have to look for Lie algebras of dimension three, which are, at the same time, subalgebras of $\mathrm{sl}(N,\mathbb{R})$. The first idea is the algebra $\mathrm{sl}(2,\mathbb{R})$. This algebra has an irreducible representation in the set of traceless real 2×2 matrices (a very good study of these algebras is given in Ref. 10):

$$\begin{pmatrix}a&b\\c&-a\end{pmatrix}.$$

Since $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{Sp}(2,\mathbb{R})$ are isomorphic, the corresponding Lie algebra $\mathrm{sp}(2,\mathbb{R})$ must have a representation with the commutation relations (26) as well. If we want to use the base of the killing vector space (25), the corresponding representation $\mathrm{sl}(2,\mathbb{R})=\mathrm{sp}(2,\mathbb{R})$ is

$$\sigma_1=2\sqrt{-K}\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \sigma_2=\pm 2\sqrt{-K}\begin{pmatrix}0&b\\a&0\end{pmatrix},\quad \sigma_3=2\begin{pmatrix}0&-b\\a&0\end{pmatrix},\quad ab=1, \tag{43}$$

being $K<0$. We now want to come back to the Lie group. We have to solve the differential equation system (32), where

$$g_{,\lambda}g^{-1}=A_1=\frac{1}{(1+K\lambda\tau)^2}\times\begin{pmatrix}\sqrt{-K}(k\tau^2+1)&b(\sqrt{-K}\tau-1)^2\\-a(\sqrt{-K}\tau+1)^2&-\sqrt{-K}(k\tau^2+1)\end{pmatrix},$$
$$g_{,\tau}g^{-1}=A_2=\frac{1}{(1+K\lambda\tau)^2}\times\begin{pmatrix}\sqrt{-K}(k\lambda^2+1)&-b(\sqrt{-K}\lambda-1)^2\\-a(\sqrt{-K}\lambda+1)^2&-\sqrt{-K}(K\lambda^2+1)\end{pmatrix}. \tag{44}$$

For the integration of (44) it is helpful to take into account the symmetry of the matrix g. If we do so, we obtain

$$g=\frac{1}{1+K\lambda\tau}\begin{pmatrix}c(1-\sqrt{-K}\lambda)(1-\sqrt{-K}\tau)&e\sqrt{-K}(\tau-\lambda)\\e\sqrt{-K}(\tau-\lambda)&d(1+\sqrt{-K}\lambda)(1+\sqrt{-K}\tau)\end{pmatrix}\quad cd<0,\quad e^2=-cd,\quad a=\frac{e}{c},\quad b=-\frac{e}{d}, \tag{45}$$

where λ and τ satisfy the "geodesic" equations (19), Γ^i_{jk} being the Christofel symbols of the metric (23).

The last Lie algebra we want to deal with, is $\mathrm{so}(2,1,\mathbf{R})$. This is the set of matrices of the form

$$\begin{pmatrix}0&b&a\\-b&0&c\\a&c&0\end{pmatrix}. \tag{46}$$

We can calculate the generators of this Lie algebra directly from the matrix (46), and arrive at

$$X_1=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix},\quad X_2=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},\quad X_3=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}. \tag{47}$$

Again, if we use the base of the Killing space of V_2 (25), we can see that one set of generators, which are compatible with (25), is

$$\sigma_1=4\sqrt{K}X_1,\quad \sigma_2=4KX_2,\quad \sigma_3=4\sqrt{K}X_3. \tag{48}$$

In other words, the three matrices (48) satisfy the commutator relations (26). With these matrices and Killing vectors, Eqs. (20) obtain the form

$$g_{,\lambda}{}^{-1}=\frac{4\sqrt{K}}{2(1+K\lambda\tau)^2}\begin{pmatrix}0 & K\tau^2+1 & -2\sqrt{K}\tau\\ -K\tau^2-1 & 0 & K\tau^2-1\\ -2\sqrt{K}\tau & K\tau^2-1 & 0\end{pmatrix}, \tag{49}$$

$$g_{,\tau}{}^{-1}=\frac{4\sqrt{K}}{2(1+K\lambda\tau)^2}\begin{pmatrix}0 & K\lambda^2+1 & -2\sqrt{K}\lambda\\ -K\lambda^2-1 & 0 & 1-K\lambda^2\\ -2\sqrt{K}\lambda & 1-K\lambda^2 & 0\end{pmatrix}.$$

Unfortunately, we were not able to integrate Eqs. (49). In order to obtain a solution from the algebra so(2,1,$\mathbb{R}$), we propose a transformation of the form (29) with the property (30). Let be M such a transformation:

$$M=\begin{pmatrix}0 & b & 0\\ d & 0 & f\\ g & 0 & i\end{pmatrix},\quad M^{-1}=\frac{-1}{Kb^2}\begin{pmatrix}0 & -d/K & -g\\ -Kb & 0 & 0\\ 0 & f/K & i\end{pmatrix};$$

$$\sigma=MX \tag{50}$$

after that, we transform (48) into the following manner:

$$\sigma_i\to c\sigma_i c^{-1},\quad \text{with } c=\begin{pmatrix}1 & 1 & 0\\ 0 & 0 & \frac{1}{2}\\ 1 & -1 & 0\end{pmatrix}.$$

For this last transformation let the commutation relations invariant. We obtain the representation

$$\sigma_1=4\sqrt{-K}\begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix},\quad \sigma_2=\sqrt{-K}\begin{pmatrix}0 & 0 & d\\ 0 & 0 & c\\ b & a & 0\end{pmatrix}$$

$$\sigma_3=\begin{pmatrix}0 & 0 & -d\\ 0 & 0 & c\\ b & -a & 0\end{pmatrix},\quad db=ac=\tfrac{1}{2}. \tag{51}$$

Using (51) and (25), the differential equations system (20) gets the form

$$g_{,\lambda}g^{-1}=\frac{1}{2(1+K\lambda\tau)^2}\begin{pmatrix}-4\sqrt{-K}(K\tau^2+1) & 0 & d(\sqrt{-K}\tau-1)^2\\ 0 & 4\sqrt{-K}(K\tau^2+1) & -c(\sqrt{-K}\tau+1)^2\\ -b(\sqrt{-K}\tau+1)^2 & a(\sqrt{-K}\tau-1)^2 & 0\end{pmatrix}, \tag{52}$$

$$g_{,\tau}g^{-1}=\frac{1}{2(1+K\lambda\tau)^2}\begin{pmatrix}-4\sqrt{-K}(K\lambda^2+1) & 0 & -d(\sqrt{-K}\lambda-1)^2\\ 0 & 4\sqrt{-K}(k\lambda^2+1) & -c(\sqrt{-K}\lambda+1)^2\\ -b(\sqrt{-K}\lambda+1)^2 & -a(\sqrt{-K}\lambda-1)^2 & 0\end{pmatrix}.$$

In order to solve it, we first take into account the symmetry relations of $g_{,\lambda}$ and $g_{,\tau}$ and find some algebraic equations. These equations help us to solve column by column the system of differential equations. We arrive at

$$g=\frac{1}{(4-\lambda\tau)^2}\begin{pmatrix}a'(\lambda-2)^2(\tau-2)^2 & b'(\lambda-\tau) & c'(\lambda-2)(\tau-2)(\lambda-\tau)\\ b'(\lambda-\tau) & d'(\tau+2)^2(\lambda+2)^2 & e'(\lambda+2)(\tau+2)(\lambda-\tau)\\ c'(\lambda-2)(\tau-2)(\lambda-\tau) & e'(\lambda+2)(\tau+2)(\lambda-\tau) & f'[(4-\lambda\tau)^2-8(\tau-\lambda)^2]\end{pmatrix},$$

$$a'=\frac{ad}{bc},\quad b'=8ad,\quad c'=-4\frac{ad}{c},\quad d'=1,\quad e'=4a,\quad f'=-\frac{a}{c},\quad K=-\frac{1}{4}. \tag{53}$$

The matrix g is an exact solution of the chiral equations (3) if λ and τ are solutions of the "geodesic" equations (19) with the metric (23).

It is clear that solution (53) cannot be used for two-dimensional chiral fields. But either solution (45) or solution (53), or a combination of both, can be used for chiral fields of dimension $N>3$. For example, for $N=5$, one can take

$$\sigma_1=2\sqrt{-k}\begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\sigma_2=\sqrt{-k}\begin{pmatrix} 0 & 0 & d & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ b & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b' \\ 0 & 0 & 0 & a' & 0 \end{pmatrix},$$

$$\sigma_3=\begin{pmatrix} 0 & 0 & -d & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ b & -a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -b' \\ 0 & 0 & 0 & a' & 0 \end{pmatrix},$$

$$db=ac=\tfrac{1}{2}, \quad a'b'=4. \tag{54}$$

The set of matrices (54) satisfies the commutation relations (26). Thus the final solution will be a symmetric 5×5 matrix with the solution (53) in the upper diagonal and the solution (44) in the lower diagonal. Higher dimensionality will give us more possibilities to combine matrices and so more possibilities to obtain chiral fields.

X. CONCLUSIONS

We have developed a method for finding exact solutions of the chiral equations (2) if the matrix g is a matrix of the group $SL(N,\mathbb{R})$. It consists of reducing the chiral equations to a Killing equation of certain Riemannian space V^d for each component of the corresponding Lie algebra matrices. After that, one looks for representations of the p-dimensional subalgebras of $\mathrm{sl}(N,\mathbb{R})$, and for a base of the p-dimensional Killing vector space of V^d. Of course, this method can be applied to any Lie group G. Having the explicit Lie algebra general matrix in terms of the Killing vectors, one comes back to the Lie group, either with the exponential map or by integration. We also gave an algorithm for integrating the matrix g, using the one-dimensional subalgebras, and other algorithm for integrating the two- and three-dimensional subalgebras in order to get an analytic expression of the elements of the Lie group.

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APPENDIX A

In this appendix, we present the definitions and theorems that we use in the present work, known in the mathematical literature. Some of them can be found in the text of linear algebra, although the theorem T is presented with its proof, because it is less popular. The theorem T can be read in Ref. 8. The matrices used in this appendix are of order n and its entries belong to a commutative field K. Here, we are interested in two equivalence relations: equivalence and similarity.

Definition 1: A matrix B is said to be equivalent to a matrix A if there exist nonsingular matrices P and Q such that $B=PAQ$.

Definition 2: Two square matrices A and B are said to be similar if there exists a nonsingular matrix P of the same order, such that $B=P^{-1}AP$.

We denote $A\underset{E}{\sim}B$ for equivalence and $A\underset{S}{\sim}B$ for similarity.

Let A be a matrix with constant entries; we denote as $p_A(\lambda)$ the characteristic polynomial of A. The matrix $\lambda I-A$ is a polynomial matrix in the variable λ. In the definition that follows, we present the invariant factors of $\lambda I-A$ that correspond to equivalence relation $\underset{\sim}{E}$.

Definition 3: Let P be a polynomial matrix of order n:

$$P=\begin{bmatrix} p_{11}(\lambda) & p_{12}(\lambda) & \cdots & p_{1n}(\lambda) \\ p_{21}(\lambda) & p_{22}(\lambda) & \cdots & p_{2n}(\lambda) \\ & \vdots & & \\ p_{n1}(\lambda) & p_{n2}(\lambda) & \cdots & p_{nn}(\lambda) \end{bmatrix},$$

and $D_k(\lambda)$ the greatest common divisor of all minors of order k, in P ($k=1,\ldots,n$).

The invariant factors of P are defined as the following:

$$d_1(\lambda)=D_1(\lambda),$$

$$d_2(\lambda)=\frac{D_2(\lambda)}{D_1(\lambda)},$$

$$\vdots$$

$$d_r(\lambda)=\frac{D_r(\lambda)}{D_{r-1}(\lambda)},$$

$$d_{r+1}(\lambda)=0,$$

$$\vdots$$

$$d_n(\lambda)=0.$$

If all minors of order k are equal to zero, then $D_k(\lambda)=0$.

Notice that $D_1(\lambda)$ is the greatest common divisor of the elements of matrix P and $D_n(\lambda)$ is the determinant of P.

Below, we present two criteria that join the relations of definitions 1 and 2.

Criterion 1: Let P_1 and P_2 be polynomial matrices of order n; they are said to be equivalent if and only if their respective factors, $d_1(\lambda)$, $d_2(\lambda),...,d_n(\lambda)$, are the same.

The $d_j(\lambda)$ are known as the invariant factors.

Criterion 2: Two matrices A and B on a field K are similar if and only if their characteristic matrices $\lambda I-A$ and $\lambda I-B$ are equivalent.

In other words, the invariant factors $d_j(\lambda)$ ($j=1,...,n$) characterize to a polynomial matrix and to all equivalent matrices to it.

Now, we want to construct a convenient procedure for finding an appropriate representative element of every class of equivalence. Given a matrix A, we need to become A in a simpler form similar to A. These forms are called normal. A classic example is the Jordan normal form. In the complex field, all matrices can be represented in its Jordan normal form; but in the field K, the reduction on K cannot be possible. If the entries of matrix A belong to a commutative field, A can be reduced to a normal form, known as natural. This is studied in the following rows.

Definition 4: Let

$$f(\lambda)=\lambda^n+a_{n-1}\lambda^{n-1}+\cdots+a_1\lambda+a_0\,,$$

be a polynomial with principal coefficient 1 and $a_j \in K$, $j=0,1,...,n-1$. The associate matrix A_f to the polynomial $f(\lambda)$, is defined as the following:

$$A_f=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & \cdots & & \cdots & & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}. \tag{A1}$$

Lemma 1: Let $f(\lambda)$ be a polynomial with principal coefficient 1 and $n \geqslant 1$. If A_f is its associated matrix, then the invariant factors of the matrix $\lambda I-A_f$ are equal to $1,...,1,f(\lambda)$.

The proof of this lemma can be established easily using the following ideas: $D_k(\lambda)=1$, for $k=1,...,n-1$. Then $d_1(\lambda)=d_2(\lambda)=\cdots=d_{n-1}(\lambda)=1$, and we can calculate the determinant of $\lambda I-A_f$ by the row n for getting

$$D_n(\lambda)=a_0+a_1\lambda+\cdots+a_{n-1}\lambda^{n-1}+\lambda^n=d_n(\lambda).$$

Definition 5.1: A matrix in the form (A1), with 1's in the diagonal to the right of the principal diagonal, and with coefficients in the field K in the last row, is called a natural normal cell.

Definition 5.2: A matrix with natural normal cells in its diagonal is called the natural normal matrix, if

$$A_F=\begin{bmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_s \end{bmatrix},$$

where A_j is an associated matrix to a polynomial $f_j(\lambda)$, $j=1,...,s$. If for every $j=1,...,s$, $f_j(\lambda)$ is a divisor of $f_{j+1}(\lambda)$, it is said that A_F is a canonical natural matrix.

For the matrix A_F, we have the following lemma.

Lemma 2: Let $\operatorname{gr} f(\lambda)$ be the degree of the $f(\lambda)$.

The invariant factors of the matrix A_F are equal to 1, $1,...,1, f_1(\lambda), f_2(\lambda),...,f_s(\lambda)$, where the number of the 1's is given by

$$\sum_{j=1}^{s} \operatorname{gr} f_j(\lambda)-s.$$

To establish Lemma 2, we can use the following ideas: to every cell A_j ($j=1,2,...,s$) corresponds its characteristic polynomial $f_j(\lambda)$. Due to the fact that polynomials $f_j(\lambda)$ are factors of $p_{A_F}(\lambda)(Aj\ =\ 1,...,S)$, we can interchange the rows for getting the invariant factors, as in the enunciated lemma.

It is important to observe that $f(\lambda)$ coincide with the characteristic polynomial of the cell A_f, and A_f is the associated matrix to $f(\lambda)$.

Then, if the characteristic polynomial of a matrix has not a factorization on K, the associated matrix of such a polynomial is a natural cell. If there is a factorization of the polynomial, then its associated matrix is constituted by as many cells as factors have the characteristic polynomial, and they are in order, such as enunciated in Definition 5.

Last, we enunciate the most important theorem of this appendix; to do it we give a formal proof, because this theorem is a basis for the choosing of the natural normal form for the classification of representatives of equivalence classes on similarity.

Theorem: Every matrix A with elements on K can be reduced on this field to one natural normal form, and only one.

Proof: Let $\lambda I-A$ be the polynomial matrix and its set of invariant factors: $I=\{1,...,1,f_1(\lambda),...,f_t(\lambda)\}$, without loss of generality, we can suppose that the invariant factors are ordered, such that $f_i(\lambda)$ is divided by $f_{i-1}(\lambda)$ for $i=2,3,...,t$. If n is the order of the matrix A, we have that

$$\sum_{i=1}^{t} \mathrm{gr}[f_i(\lambda)]=n,$$

and the number of the 1's is $n-s$.

For every $f_i(\lambda)$ we can build its associated natural normal cell N_i, after we build the matrix N in the following way:

$$N=\begin{bmatrix} N_1 & & & \\ & N_2 & 0 & \\ & 0 & \ddots & \\ & & & N_s \end{bmatrix}.$$

N is a matrix in its natural normal form because every polynomial $f_i(\lambda)$ is divided by the previous. Hence, the invariant factors of the matrix $\lambda I-N$ are $1,1,...,1,f_1(\lambda),...,f_t(\lambda)$, and they coincide with the invariant factors of $\lambda I-A$. Then the matrices A and N are similar.

We presented a method for reducing the matrix A to its natural normal form. The unity of N follows from the fact that their invariant factors are built in only one way from the matrix A, if we employ the method suggested by Definition 3. However, the matrix N is built in only one way by the invariant factors.

APPENDIX B

1. FIRST PART

Consider the linear system of differential equations:

$$g_{,\lambda}=Ag, \tag{B1}$$

as defined in (3).

Remember that $p_A(\lambda)$ is the characteristic polynomial of the matrix A and $p_e(\lambda)$ is the characteristic polynomial associated to (B1).

Theorem 1: If A is a natural normal cell itself, then $p_A(\lambda)=p_e(\lambda)$.

Proof: Since A consists of a single natural normal cell, then

$$A=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ b_1 & b_2 & b_3 & \cdots & b_n \end{bmatrix}.$$

We know, by Appendix A, that the entries of the last row of A are the coefficients of the characteristic polynomial of A. Thus,

$$p_A(\lambda)=\lambda^n-b_n\lambda^{n-1}-b_{n-1}\lambda^{n-2}-\cdots-b_2\lambda-b_1. \tag{B2}$$

Let g be a real and symmetric matrix of order n. If we represent $g_{,\lambda}$ as $[g'_{ij}]$, then Eq. (B1) can be written as follows:

$$\begin{bmatrix} g'_{11} & g'_{12} & \cdots & g'_{1n} \\ g'_{21} & g'_{22} & \cdots & g'_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ g'_{n1} & g'_{n2} & \cdots & g'_{nn} \end{bmatrix}$$

$$=\begin{bmatrix} g_{12} & g_{22} & \cdots & g_{n2} \\ g_{13} & g_{23} & \cdots & g_{n3} \\ \cdot & \cdot & \cdots & \cdot \\ g_{1n} & g_{2n} & \cdots & g_{nn} \\ \Sigma_{i=1}^n bg_{1i} & \Sigma_{i=1}^n bg_{2i} & \cdots & \Sigma_{i=1}^n bg_{ni} \end{bmatrix}. \tag{B3}$$

For each column j ($j=1,...,n$), we get, from (B3),

$$g'_{ij}=g_{i+1,j}, \quad i=1,...,n-1, \tag{B4}$$

$$g'_{nj}=\sum_{i=1}^{n} bg_{ij}. \tag{B5}$$

Developing the last equation using Eqs. (B4), we obtain the following differential equation, in terms of g_{1j}:

$$g_{1j}^{(n)}-b_n g_{1j}^{(n-1)}-b_{(n-1)}g_{1j}^{(n-2)}-\cdots-b_2 g'_{1j}-b_1 g_{1j}=0. \tag{B6}$$

The characteristic polynomial associated to (B6) is

$$p_e(\lambda)=\lambda^n-b_n\lambda^{n-1}-b_{n-1}\lambda^{n-2}-\cdots-b_3\lambda^2-b_2\lambda-b_1. \tag{B7}$$

From (B2) and (B7) it follows that $p_A(\lambda)=p_e(\lambda)$.

Notice that when we use a natural normal matrix such that its trace vanishes, then $p_A(\lambda)$ does not present the term that corresponds to the derivative of order $n-1$, i.e.,

$$p_A(\lambda)=\lambda^n-b_{n-1}\lambda^{n-2}-\cdots-b_2\lambda-b_1.$$

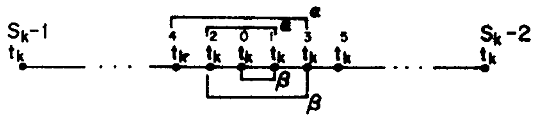
Definition 1: For a square matrix $A=[a_{ij}]$ of order n, an antidiagonal is the set of entries $\alpha_k=\{a_{ij}|i+j=k\}$; $k=2,3,...,2n$.

For example, the matrix $A=[a_{ij}]$ of order 4, have seven antidiagonals; they are a_{11}; $a_{12}a_{21}$; $a_{13}a_{22}a_{31}$; $a_{14}a_{23}a_{32}a_{41}$; $a_{24}a_{33}a_{42}$; $a_{34}a_{43}$; a_{44}.

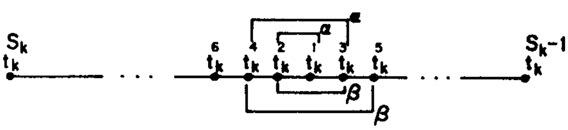
Theorem 2: Let A be a natural cell. The elements of the matrix Ag, on each α_k, are the same.

Proof: g is a symmetric matrix, and from this it follows that $g_{,\lambda}$ is also symmetric, and hence Ag must be symmetric as well. Let Ag be like the right member of equality (B3), and let S_k be the number of elements α_k corresponding to Ag, $k=3,4,...,2n$ and n order of Ag. Here S_k can be odd or even.

If S_k is odd: for some k, we assign t_k^0 to the element $g_{(k-1)/2,(k-1)/2+1}$ of the antidiagonal α_k and numerate the other elements of α_k, as is shown in the following diagram:



If S_k is even for some k, we asign t_k^1 to the element $g_{k/2,k/2}$ of the α_k and numerate the elements of α_k as follows:



If we say that $t_k^i \gamma t_k^j \Leftrightarrow t_k^i = t_k^j$ because of some matrix symmetry γ, then, since α is the symmetry of matrix Ag and β is the symmetry of matrix g, the theorem follows. It is interesting to observe that, for $k=n+2,...,2n$; $t_k^{S_k-1}$ (odd case) or $t_k^{S_k}$ (even case), such t_k^m corresponds to one sum, then this element t_k^m only is relationed with the symmetry α. But we have the equality of all elements of α_k following a spiral of symmetries.

Let A be a matrix in its natural normal form, with s cells,

$$A=\begin{bmatrix} A_1 & & & \\ & A_2 & 0 & \\ & 0 & \ddots & \\ & & & A_s \end{bmatrix},$$

let n_j be the order of the cell Aj, $j=1,...,s$.

We represent by g_j the jth column of the matrix g,

$$g_j=[g_{ij}], \quad i=1,...,n.$$

We make a partition of each column g_j, of g according to the cells of A. Let us denote by

$$\gamma_{ij}=\begin{bmatrix} g_{m_{i-1}+1,j} \\ \vdots \\ g_{m_i+1,j} \end{bmatrix},$$

each block of g_j, where $m_i=n_1+\cdots+n_i$, $i=1,...,s$.

Now we can write Ag as follows:

$$Ag=\begin{bmatrix} A_1\gamma_{11} & A_1\gamma_{12} & \cdots & A_1\gamma_{1n} \\ A_2\gamma_{21} & A_2\gamma_{22} & \cdots & A_2\gamma_{2n} \\ & \cdot & \cdots & \\ A_s\gamma_{s1} & A_s\gamma_{s2} & \cdots & A_s\gamma_{sn} \end{bmatrix}. \tag{B8}$$

In this way, Ag consists of $n\times s$ block, each of which defines a linear system of differential equations, namely,

$$\gamma_{kj,\lambda}=A_k\gamma_{kj}, \tag{B9}$$

with $k=1,...,s$ and $j=1,...,n$.

Notice that these systems are independent. If we represent by $p_e^{(i)}(\lambda)$ the characteristic polynomial associated to the matrix A_i, then we can enunciate the following theorem.

Theorem 3: Let A be a matrix in its natural normal form with s cells. Then $g_{,\lambda}=Ag$ can be decomposed in a system of $s\times n$ independent homogeneous linear differential equations. Moreover,

$$p_A(\lambda)=\prod_{i=1}^{s} p_{A_i}(\lambda), \tag{B10}$$

where $p_A(\lambda)$ is the characteristic polynomial of the matrix A and $p_{A_i}(\lambda)$ are the characteristic polynomials of the cells of A.

Proof: The first part has been proved already. Expression (B10) is known for matrices descomposed in diagonal cells.

To solve equation $g_{,\lambda}=Ag$, we have to solve each system in (B9). This amounts to solving only s equations since for each k, all system in (B9) with different j having the same form.

2. SECOND PART

To generate the numbers into the little squares of Table I, we use the following definition.

Definition 2: For some $n\in\mathfrak{N}$, consider all the possible sets of natural numbers, such that, if we add the numbers in each set, we get the number n.

Suppose that there are t such sets:

$$S_i=\{n_{i1},n_{i2},...,n_{ik_i}\}; \quad i=1,...,t.$$

We call the arrangement

$$\begin{bmatrix} n_{11} & n_{12} & \cdots & n_{1k_1} \\ n_{21} & n_{22} & \cdots & n_{2k_2} \\ \vdots & & & \\ n_{t1} & n_{t2} & \cdots & n_{tk_t} \end{bmatrix},$$

an ordered table of sums, if it satisfies the following conditions:

(1) $n_{ij} \leqslant n_{ik}$, for $j<k$ and $i=1,...,t$;

(2) $k_1 \leqslant k_2 \cdots \leqslant k_t$.

In Definition 3 of Appendix A, the concept of invariant elements of a polynomial matrix was presented to identify a class of matrices. If the invariant elements are ordered, as explained in Appendix A, we have the following result.

Theorem 4: Let A be a non-null matrix of order n in its natural normal form, such that trace $A=0$. The first invariant element of the matrix $\lambda I - A$ is 1, i.e.,

$$d_1(\lambda)=1.$$

Proof: A is a real matrix; then $D_1(\lambda)=1$ or $D_1(\lambda)$ is a polynomial of first degree. If $D_1(\lambda)=1$, then $d_1(\lambda)=1$, as the theorem asserts. In the other case, we can suppose $D_1(\lambda)=d_1(\lambda)=\lambda-\alpha$. If $d_1(\lambda)=\lambda-\alpha$, A should have one α in each place of the principal diagonal and zeros in the other places. If this happens, we have that $d_1(\lambda) =d_2(\lambda)=\cdots=d_n(\lambda)=\lambda-\alpha$ and then $p(\lambda)=(\lambda-\alpha)^n$ and trace $A=n\alpha$, but trace $A=0$ by hypothesis. Hence $\alpha=0$, but this is a contradiction to the condition that A is a non-null matrix.

ACKNOWLEDGMENT

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