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The linear problem for the five-dimensional projective field theory

T. Matos, Jena

Friedrich-Schiller-Universität Jena

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A linear problem is given for the five-dimensional projective theory when the metric depends only on two coordinates. The symmetries of this problem are investigated.

Ein lineares Problem wird für die fünfdimensionale projective Relativitätstheorie angegeben, wenn die Metrik nur von zwei Koordinaten abhängt. Die Symmetrien dieses Problems werden untersucht.

1. Introduction

The five-dimensional projective field theory unifies the electromagnetic and gravitational fields (Schmutzer 1968). It is postulated that the five-dimensional Ricci tensor in the electro-scalar-vacuum case is equal to zero and it is assumed that there is a non-null Killing vector field X^μ. The projective metric is related to the electromagnetic 4-potential A_i, the scalar field I, and the space-time metric g_{jk}. In the adapted coordinate system with $X^\mu = \delta^\mu_5$, these quantities are identified with the five-dimensional metric $\gamma_{\mu\nu}$ according to

$$g_{ik} = \gamma_{ik} - I^2 A_i A_k\,, \qquad A_k = I^2 \gamma_{5k}\,, \qquad I^2 = \gamma_{55}\,, \qquad i, k = 1 \dots 4\,. \tag{1.1}$$

We are interested in the case when the metric depends only on two coordinates x^1 and x^2. For such a metric the respective field equations (see for example Belinsky 1980) are

$$\begin{aligned}
&(\alpha A)_{,\bar z} + (\alpha B)_{,z} = 0\,,\\
&(\ln f)_{,z} = \frac{(\ln\alpha)_{,zz}}{(\ln\alpha)_{,z}} + \frac{1}{4\alpha\alpha_{,z}} T_r(\alpha A)^2\,, \qquad A = \gamma_{,z}\gamma^{-1}; \qquad B = \gamma_{,\bar z}\gamma^{-1}; \qquad \gamma = \begin{pmatrix} g_{33} & \cdots \\ \cdots & g_{55}\end{pmatrix}.\\
&(\ln f)_{,\bar z} = \frac{(\ln\alpha)_{,\bar z\bar z}}{(\ln\alpha)_{,\bar z}} + \frac{1}{4\alpha\alpha_{,\bar z}} T_r(\alpha B)^2\,, \qquad \alpha^2 = \det\gamma\,, \qquad z = x^1 + x^2
\end{aligned} \tag{1.2}$$

(a bar denotes complex conjugation). In the present paper it is given an associated linear problem with the purpose to generate new solutions to (1.2)

2. The field equations

We will calculate the field equations in the potential space. It is necessary to write down explicitly the matrices A and B in (1.2),

$$\begin{aligned}
&A_{11} = \frac{I^2}{\alpha^2}\begin{vmatrix} g_{33,z} + I^2 A_3 A_{3,z} & g_{34} \\ g_{34,z} + I^2 A_3 A_{4,z} & g_{44}\end{vmatrix}, \qquad A_{12} = \frac{I^2}{\alpha^2}\begin{vmatrix} g_{33,z} + I^2 A_3 A_{3,z} & g_{33} \\ g_{34,z} + I^2 A_3 A_{4,z} & g_{34}\end{vmatrix},\\
&A_{12} = \frac{I^2}{\alpha^2}\begin{vmatrix} g_{34} & g_{34,z} + I^2 A_4 A_{3,z} \\ g_{44} & g_{44,z} + I^2 A_4 A_{4,z}\end{vmatrix}, \qquad A_{22} = \frac{I^2}{\alpha^2}\begin{vmatrix} g_{33} & g_{34,z} + I^2 A_4 A_{3,z} \\ g_{34} & g_{44,z} + I^2 A_4 A_{4,z}\end{vmatrix},\\
&A_{13} = -A_4 A_{12} - A_3 A_{11} + \frac{(I^2 A_3)_{,z}}{I^2}, \qquad A_{23} = -A_3 A_{21} - A_4 A_{22} + \frac{(I^2 A_4)_{,z}}{I^2},\\
&A_{31} = \frac{I^2}{\alpha}\begin{vmatrix} I^2 A_{3,z} & g_{34} \\ I^2 A_{4,z} & g_{44}\end{vmatrix}, \qquad A_{32} = \frac{I^2}{\alpha^2}\begin{vmatrix} g_{33} & I^2 A_{3,z} \\ g_{34} & I^2 A_{4,z}\end{vmatrix}, \qquad T_r A = (\ln\alpha^2)_{,z}
\end{aligned} \tag{2.1}$$

and the components of B with z replaced by $\bar z$.

In our case, the metric g_{ik} admits two Killing vectors (for example, stationarity and axial symmetry). There are then three Killing vectors in the five-dimensional space: X^α, Y^α, and Z^α. In terms of these three Killing vectors we can define the matrix

$$M^{\alpha\beta\gamma\delta} := \begin{pmatrix} X^\alpha Y^\beta Z^{\gamma;\delta} & Z^\alpha X^\beta Z^{\gamma;\delta} & Y^\alpha Z^\beta Z^{\gamma;\delta} \\ X^\alpha Y^\beta Y^{\gamma;\delta} & Z^\alpha X^\beta Y^{\gamma;\delta} & Y^\alpha Z^\beta Y^{\gamma;\delta} \\ X^\alpha Y^\beta X^{\gamma;\delta} & Z^\alpha X^\beta X^{\gamma;\delta} & Y^\alpha Z^\beta X^{\gamma;\delta} \end{pmatrix} \tag{2.2}$$

(a colon denotes covariant differentiation). In the special coordinate system where the Killing vectors are given by

$$X^\mu = \delta^\mu_5\,, \qquad Y^\mu = \delta^\mu_4\,, \qquad Z^\mu = \delta^\mu_3 \tag{2.3}$$

we can rewrite the matrices A and B in terms of the matrix $M^{\alpha\beta\mu\delta}$

$$A = \frac{2}{\alpha}\,\varepsilon_{\alpha\beta\gamma\delta 1} M^{\alpha\beta\gamma\delta}\,, \qquad B = -\frac{2}{\alpha}\,\varepsilon_{\alpha\beta\gamma\delta 2} M^{\alpha\beta\gamma\delta}\,,$$

where $\varepsilon_{\alpha\beta\gamma\delta\mu}$ is the five-dimensional Levi-Civita tensor.

The five real scalar potentials I, ν, ψ, χ and ε are defined in terms of two Killing vectors (see NEUGEBAUER 1969)

$$\begin{aligned} &I^2 = X^\mu X_\mu\,, \qquad e^\nu = -I^{-2}[I^2 Y^\mu Y_\mu - (X^\mu Y_\mu)^2]\,, \qquad \Psi = -I^2 X^\mu Y_\mu\,, \\ &\chi_{,\mu} = -2\varepsilon_{\alpha\beta\gamma\delta\mu} X^\alpha Y^\beta X^{\gamma;\delta}\,, \qquad \varepsilon_{,\mu} = -2\varepsilon_{\alpha\beta\gamma\delta\mu} X^\alpha Y^\beta Y^{\gamma;\delta}\,. \end{aligned} \tag{2.4}$$

The potentials introduced in (2.4) are the gravitational potential $\nu/2c^2$, the electrostatic potential Ψ, the magnetostatic potential χ, the rotational potential ε, and the scalar potential I. Substituting the definition (2.4) into the field equations (1.2) we find, with (2.1) and (2.3), the following differential equations

$$\begin{aligned} &\left(\frac{\alpha I_{,z}}{I^2}\right)_{,\bar z} + \left(\frac{\alpha I_{,\bar z}}{I^2}\right)_{,z} - \frac{I\alpha}{h}\Psi_{,z}\Psi_{,\bar z} + \frac{\alpha}{hI^5}\chi_{,z}\chi_{,\bar z} + \frac{4\alpha I_{,z} I_{,\bar z}}{I^3} = 0\,, \\ &\left(\frac{\alpha}{hI^4}\chi_{,z} + \frac{\alpha\Psi}{h^2I^2}(\varepsilon_{,z} + \Psi\chi_{,z})\right)_{,\bar z} + \left(\frac{\alpha}{hI^4}\chi_{,\bar z} + \frac{\alpha\Psi}{h^2I^2}(\varepsilon_{,\bar z} + \Psi\chi_{,\bar z})\right)_{,z} = 0\,, \\ &\left(\frac{\alpha I^2}{h}\Psi_{,z}\right)_{,\bar z} + \left(\frac{\alpha I^2}{h}\Psi_{,\bar z}\right)_{,z} - \frac{\alpha}{h^2I^2}[(\varepsilon_{,z} + \Psi\chi_{,z})\chi_{,\bar z} + (\varepsilon_{,\bar z} + \Psi\chi_{,\bar z})\chi_{,z}] = 0\,, \\ &\left(\frac{\alpha}{h^2I^2}(\varepsilon_{,z} + \Psi\chi_{,z})\right)_{,\bar z} + \left(\frac{\alpha}{h^2I^2}(\varepsilon_{,\bar z} + \Psi\chi_{,\bar z})\right)_{,z} = 0\,, \\ &\left(\frac{\alpha h_{,z}}{h^2}\right)_{,\bar z} + \left(\frac{\alpha h_{,\bar z}}{h^2}\right)_{,z} + \frac{2\alpha}{h^3}\left(h_{,z}h_{,\bar z} + \frac{1}{I^2}(\varepsilon_{,z} + \Psi\chi_{,z})(\varepsilon_{,\bar z} + \Psi\chi_{,z})\right) + \frac{2\alpha I^2}{h^2}\Psi_{,z}\Psi_{,\bar z} = 0\,, \end{aligned} \tag{2.5}$$

with $l/ \equiv h$. Our objective is to solve these equations by means of the inverse scattering method.

The equations (2.5) can be derived from the Lagrange function (an equivalent Lagrangian is given in NEUGEBAUER (1969)).

$$\mathfrak{L} = \frac{\alpha}{2h^2I^2}[(Ih)^{,i}(Ih)_{,i} + (\varepsilon^{,i} + \Psi\chi^{,i})(\varepsilon_{,i} + \Psi\chi_{,i})] + \frac{\alpha}{2h}\left(I^2\Psi_{,i}\Psi^{,i} + \frac{1}{I^4}\chi^{,i}\chi_{,i}\right) + \frac{3}{2}\frac{\alpha}{I^2}I_{,i}I^{,i}\,. \tag{2.6}$$

3. The first-order form of the field equations

In order to simplify the equations (2.5), we define the following quantities:

$$\begin{aligned} A_1 &= \frac{1}{2}\left[(\ln Ih)_{,z} - \frac{i}{Ih}(\varepsilon_{,z} + \Psi\chi_{,z})\right], \\ B_1 &= \frac{1}{2}\left[(\ln Ih)_{,z} + \frac{i}{Ih}(\varepsilon_{,z} + \Psi\chi_{,z})\right], \\ E_1 &= -\frac{1}{2\sqrt{2}}h^{-1/2}\left[\frac{\chi_{,z}}{I^2} - iI\Psi_{,z}\right], \\ F_1 &= -\frac{1}{2\sqrt{2}}h^{-1/2}\left[\frac{\chi_{,z}}{I^2} + iI\Psi_{,z}\right], \\ D_1 &= \frac{1}{2}(\ln I)_{,z}\,, \\ C_1 &= (\ln\alpha)_{,z} \end{aligned} \tag{3.1}$$

and corresponding expressions for A_2, B_2, ... are obtained by replacing z by $\bar{z}$. With the new quantities, the equations (2.5) are transformed to ten non-linear first-order differential equations:

$$
\begin{aligned}
A_{1,\bar{z}} &= A_1A_2 - A_1B_2 - \tfrac{1}{2}\,C_2A_1 - 2E_1F_2 - \tfrac{1}{2}\,C_1A_2\,,\\
A_{2,z} &= A_1A_2 - A_2B_1 - \tfrac{1}{2}\,C_2A_1 - \tfrac{1}{2}\,C_1A_2 - 2E_2F_1\,,\\
B_{1,\bar{z}} &= B_1B_2 - A_2B_1 - \tfrac{1}{2}\,C_2B_1 - \tfrac{1}{2}\,C_1B_2 - 2E_2F_1\,,\\
B_{2,z} &= B_1B_2 - A_1B_2 - \tfrac{1}{2}\,C_2B_1 - \tfrac{1}{2}\,C_1B_2 - 2E_1F_2\,,\\
E_{1,\bar{z}} &= A_1E_2 + \tfrac{1}{2}\,A_2E_1 - \tfrac{1}{2}\,B_2E_1 - \tfrac{1}{2}\,C_1E_2 - \tfrac{1}{2}\,C_2E_1 + 3D_1F_2\,,\\
E_{2,z} &= A_2E_1 + \tfrac{1}{2}\,A_1E_2 - \tfrac{1}{2}\,B_1E_2 - \tfrac{1}{2}\,C_1E_2 - \tfrac{1}{2}\,C_2E_1 + 3D_2F_1\,,\\
F_{1,\bar{z}} &= B_1F_2 + \tfrac{1}{2}\,B_2F_1 - \tfrac{1}{2}\,A_2F_1 - \tfrac{1}{2}\,C_1F_2 - \tfrac{1}{2}\,C_2F_1 + 3D_1E_2\,,\\
F_{2,z} &= B_2F_1 + \tfrac{1}{2}\,B_1F_2 - \tfrac{1}{2}\,A_1F_2 - \tfrac{1}{2}\,C_1F_2 - \tfrac{1}{2}\,C_2F_1 + 3D_2E_1\,,\\
D_{1,\bar{z}} &= -(E_1E_2 + F_1F_2) - \tfrac{1}{2}\,C_1D_2 - \tfrac{1}{2}\,C_2D_1\,,\\
D_{2,z} &= -(E_1E_2 + F_1F_2) - \tfrac{1}{2}\,C_1D_2 - \tfrac{1}{2}\,C_2D_1\,.
\end{aligned}
\tag{3.2}
$$

4. The associated linear problem

The equations (3.2) are equivalent to the integrability conditions of the linear problem

$$
\begin{aligned}
\Omega_{,z} &= \left[\begin{pmatrix} B_1 & 0 & E_1 \\ 0 & A_1 & -F_1 \\ -F_1 & E_1 & \tfrac{1}{2}(A_1+B_1) \end{pmatrix} + \lambda \begin{pmatrix} -D_1 & B_1 & F_1 \\ A_1 & -D_1 & -E_1 \\ E_1 & -F_1 & 2D_1 \end{pmatrix}\right]\Omega\,,\\
\Omega_{,\bar{z}} &= \left[\begin{pmatrix} B_2 & 0 & E_2 \\ 0 & A_2 & -F_2 \\ -F_2 & E_2 & \tfrac{1}{2}(A_2+B_2) \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} -D_2 & B_2 & F_2 \\ A_2 & -D_2 & -E_2 \\ E_2 & -F_2 & 2D_2 \end{pmatrix}\right]\Omega\,,
\end{aligned}
\tag{4.1}
$$

for the 3×3 matrix $\Omega = \Omega(\lambda, z, \bar{z})$ (the corresponding linear problem for Einstein-Maxwell fields is, e.g. given in (NEUGEBAUER, KRAMER (1983)). The quantity λ defined by

$$
\lambda = \lambda(K) = \left[\frac{K - i\bar{z}}{K + iz}\right]^{1/2}
\tag{4.2}
$$

contains the constant spectral parameter K, and the 3×3 pseudopotential matrix Ω is normalized according to

$$
\Omega(1) = \frac{1}{I^{1/2}} \begin{bmatrix} -ihI + \varepsilon & 1 & \dfrac{1}{2\sqrt{2}}\chi \\ -ihI - \varepsilon & -1 & -\dfrac{1}{2\sqrt{2}}\chi \\ \sqrt{2}\, I^2 h^{1/2}\psi & 0 & -\dfrac{1}{2}\, I^2 h^{1/2} \end{bmatrix},
\tag{4.3}
$$

at $\lambda = 1$ (we have used the notation $\Omega \equiv \Omega(\lambda, z, \bar{z}) \equiv \Omega(\lambda)$). The linear problem (4.1) and the normalization (4.3) imply the definitions (3.1) and the properties

$$
\begin{aligned}
&\text{a)}\quad \Omega(\lambda) = \sigma\Omega^*(\lambda)^T\, K(k)\,,\\
&\text{b)}\quad \Omega^*(\lambda)\,\Omega(-\lambda) = \left(\det \overline{\Omega^*(\lambda)} \det \Omega(-\lambda)\right)^{1/3} I\,,\\
&\text{c)}\quad \det \Omega^*((\lambda) = \det \Omega(\lambda) = \det \Omega(')\,.
\end{aligned}
\tag{4.4}
$$

$\left(T\right.$ denotes matrix transposition and $\Omega^*(\lambda)$ is defines by $\left.\Omega^*(\lambda) = \overline{\Omega\left(\frac{1}{\lambda}\right)^T}\right)$.

It is possible to check that all of these relations (4.4) are consistent with the linear eigenvalue equations (4.1) and the definitions (3.1). These properties will lead us to solutions of the equations (3.2). The polynomial method i.e., the generalized Bäacklund transformations provide us with a means to solve these complicated differential equations. In a later work we will give a solution of (1.2) using the linear problem (4.1), the normalization (4.3) and the properties (4.4).

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Address of the author:

T. MATOS
Friedrich-Schiller-Universität Jena, Sektion Physik
Max-Wien-Platz 1
DDR-6900 Jena
German Democratic Republic