Five-dimensional Schwarzschild-like spacetimes with an arbitrary magnetic field

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We find a new class of exact solutions of the five-dimensional Einstein equations whose corresponding four-dimensional spacetime possesses a Schwarzschild-like behavior. The electromagnetic potential depends on a harmonic function and can be chosen to be of a monopole, dipole, etc., field. The solutions are asymptotically flat and for a vanishing magnetic field the four-metrics are of the Schwarzschild solution. The spacetime is singular in $r=2m$ for higher multipole moments, but regular for monopoles or vanishing magnetic fields in this point. The scalar field possesses a singular behavior.

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In recent years there has been great interest in the study of exact solutions of actions of the type

$$S=\int d^4x\sqrt{-g}\,[-R+2(\nabla\Phi)^2+e^{-2\alpha\phi}F^2]\ , \tag{1}$$

because they reduce to the 4D low-energy Lagrangian for string theory for $\alpha=1$, to the Einstein-Maxwell-Scalar theory for $\alpha=0$, and they also reduce to 5D gravity for $\alpha=\sqrt{3}$, after dimensional reduction. Some exact solutions of the field equations for charged bodies of this action are known [1]. It seems that the properties of electrically charged solutions depend on the value of α, but they are only different for the extreme case $\alpha=0$ [2]. In this Brief Report we want to show that magnetic fields do not alter the properties of the spacetime for static bodies, for $\alpha=\sqrt{3}$. We present a set of exact static solutions of this Lagrangian where the magnetic field depends on a harmonic map, and can be chosen to be of a monopole, dipole, quadrupole, etc.

Einstein theory is a good model for describing gravitational interactions in the Universe. Nevertheless, there are some phenomena in the cosmos where gravitation is interacting with electromagnetism. Such is the case, for example, in planets and stars possessing a magnetic field such as Earth or the Sun. Our Galaxy also possesses a magnetic field and there is not yet a convincing explanation for it. One would expect that the Einstein-Maxwell theory should give such an explanation by means of a simple exact solution possessing a magnetic field such as the celestial bodies. There is an exact solution of the Einstein-Maxwell equations containing a magnetic dipole moment satisfying the required stationary and static limits [3]. But it is not simple at all. Five-dimensional (5D) theory is an alternative model for understanding gravitational and electromagnetical interactions together. In this work we want to show that there exists a class of very simple exact solutions of the 5D Einstein equations possessing magnetic fields, the four-dimensional (4D) metric of which behaves like the Schwarzschild solution. In a past work [4] we developed a method for generating exact solutions for the 5D Einstein equations with a G_3 group of motion, putting the solutions in terms of two harmonic maps λ and τ. These solutions can also be interpreted as solutions of Lagrangian (1) for the case $\alpha=\sqrt{3}$. We separated the solutions in five tables (Tables III–VII) for the one- and two-dimensional (Abelian and non-Abelian) subgroups of SL $(2,R)$, in the spacetime and the potential space, and demonstrated that many of the well-known solutions are contained in these tables. In this Brief Report we want to present a set of new solutions which belong to the class of solutions i, j, and k of Table VI in Ref. [4], specializing the harmonic maps, because it represents a class of very well-behaved solutions, if we choose the harmonic maps λ and τ conveniently. In terms of the five potentials [5] the solutions are

$$\text{(i)}\ \ \chi=\frac{a_1e^{q\tau}+a_2e^{-q\tau}}{g_{22}}\ ,\ \ g_{22}=be^{q\tau}+ce^{-q\tau},\ \ b+c=\frac{1}{I_0}\ ,$$

$$\text{(j)}\ \ \chi=\frac{a_1\tau+a_2}{g_{22}}\ ,\ \ g_{22}=b\tau+\frac{1}{I_0}\ ,$$

$$\text{(k)}\ \ \chi=\frac{a_1e^{iq\tau}+\bar{a}_1e^{-iq\tau}}{g_{22}}\ ,\ \ g_{22}=be^{iq\tau}+\bar{b}e^{-iq\tau}\ ,\ \ b+\bar{b}=\frac{1}{I_0}$$

(here we have set $\alpha+\beta=0$ in Table VI of Ref. [4]). The gravitational potential and the scalar potential are the same for all cases given by

$$f=\frac{e^{\Lambda\lambda}}{\sqrt{I_0g_{22}}}\ ,\ \ I^2=\frac{I_0e^{-2\Lambda\lambda/3}}{g_{22}}\ ,$$

where a_1, a_2, q, b, c, I_0, and Λ are constants restricted by $bcq^2=I_0\delta\neq0$, while the electrostatic and rotational potentials vanish, i.e., $\psi=\epsilon=0$. Now it is easy to write the spacetime metric. Let us write it in Boyer-Lindquist coordinates:

$$\rho=\sqrt{r^2+2mr}\,\sin\theta,\ \ z=(r-m)\cos\theta\ .$$

In these coordinates the five-metric reads

$$dS^2=\frac{1}{I}\left\{\frac{1}{f}e^{2k}\left[1-\frac{2m}{r}+\frac{m^2\sin^2\theta}{r^2}\right]\left[\frac{dr^2}{1-\frac{2m}{r}}+r^2d\theta^2\right]+\frac{1}{f}\left[1-\frac{2m}{r}\right]r^2\sin^2\theta d\varphi^2-fdt^2\right\}+I^2(A_3d\varphi+dx^5)^2 .$$

The expression in curly brackets is interpreted as the 4D metric in the 5D theory and corresponds to the spacetime metric of Lagrangian (1). The functions k and A_3 are completely determined by the potentials f, χ, and $\kappa^2=I^3$:

$$k_{,\zeta}=\rho\left[\frac{(f_{,\zeta})^2}{2f^2}+\frac{1}{2f}\frac{(\chi_{,\zeta})^2}{\kappa^2}+\frac{2}{3}\frac{(\kappa_{,\zeta})^2}{\kappa^2}\right]=\frac{\rho}{2}\left[\frac{4}{3}\Lambda^2(\lambda_{,\zeta})^2+q^2(\tau_{,\zeta})^2\right],$$

$$A_{3,\zeta}=-\frac{\rho}{f\kappa^2}\chi_{,\zeta}=-\rho\tau_{,\zeta}\,,$$

$$A_{3,\bar\zeta}=\rho\tau_{,\bar\zeta},\quad \zeta=\rho+iz\ .$$

Observe that the function A_3 is integrable because τ satisfies the Laplace equation $(\rho\tau_{,\zeta})_{,\bar\zeta}+(\rho\tau_{,\bar\zeta})_{,\zeta}=0$. In Ref. [6] a set of solutions of the Laplace equation and their corresponding magnetic potential A_3 is listed. Two examples are

$$\text{(a)}\quad \tau=\tau_0\ln\left[1-\frac{2m}{r}\right],\quad A_3=2\tau_0m(1-\cos\theta)$$

and

$$\text{(b)}\quad \tau=\frac{\tau_0m^2\cos\theta}{(r-m)^2-m^2\cos^2\theta}\,,$$

$$A_3=\frac{m^2\tau_0(r-m)\sin^2\theta}{(r-m)^2-m^2\cos^2\theta}$$

written in Boyer-Lindquist coordinates. The magnetic potentials (a) and (b) represent a magnetic dipole, respectively. In general the harmonic function τ determines the magnetic field in the solution and can be chosen to obtain monopole, dipole, quadrupole, etc., fields. The harmonic function λ determines the gravitational potential f. Let us choose $\lambda=\lambda_0\ln(1-2m/r)$. The five-metric transforms to

$$dS^2=\left[\frac{g_{22}}{I_0}\right]^{1/2}\left[1-\frac{2m}{r}\right]^{\Lambda\lambda_0/3}\left\{\frac{\left[1-\frac{2m}{r}+\frac{m^2\sin^2\theta}{r^2}\right]^{1-4\Lambda^2\lambda_0^2/3}}{\left[1-\frac{2m}{r}\right]^{\Lambda\lambda_0-4\Lambda^2\lambda_0^2/3}}\sqrt{I_0g_{22}}\,e^{2k_1}\left[\frac{dr^2}{1-\frac{2m}{r}}-r^2d\theta^2\right]+\left[1-\frac{2m}{r}\right]^{1-\Lambda\lambda_0}\sqrt{I_0g_{22}}\,r^2\sin^2\theta d\phi^2-\frac{1}{\sqrt{I_0g_{22}}}\left[1-\frac{2m}{r}\right]^{\Lambda\lambda_0}dt^2\right\}+I^2(A_3d\phi+dx^5)^2\,, \tag{2}$$

where $k_{1,\zeta}=\frac{1}{2}q^2\tau_0(\tau_{,\zeta})^2$, g_{22}, and A_3 are determined only by the harmonic function τ. If we choose τ to vanish for some limit $r\gg m$ [the two examples (a) and (b) satisfy this condition], then the metric (2) is asymptotically flat. If τ and m vanish, metric (2) is flat.

If we put $\Lambda=-2,\lambda_0=-\frac{1}{2}$ in (2) we can interpret m as the mass parameter and $\sqrt{I_0g_{22}}$ as the contribution of the magnetic field to the metric. In this case, metric (2) reads

$$dS^2=\frac{1}{I}\left\{\sqrt{I_0g_{22}}\,e^{2k_2}\left[\frac{dr^2}{1-2m/r}+r^2d\theta^2\right]+\sqrt{I_0g_{22}}\,r^2\sin^2\theta d\phi^2-\frac{1}{\sqrt{I_0g_{22}}}\left[1-\frac{2m}{r}\right]dt^2\right\}+I^2(A_3d\phi+dx^5)^2\,,$$
$$I^2=\frac{I_0(1-2m/r)^{-2/3}}{g_{22}}\ . \tag{3}$$

This metric can be interpreted as a magnetized Schwarzschild solution in 5D gravity. The difference to a previous one [7] is that in metric (3) the magnetic potential can be chosen in many ways. If the magnetic field A_3 in (3) vanishes, the expression in curly brackets is just the Schwarzchild metric. Therefore we can interpret $r=2m$ as the horizon of the four-metric. Observe that the presence of the magnetic field does not alter the horizon of the metric, conserving the main feature of its topology. Nevertheless the scalar field does. We can see that the scalar potential tends very rapidly to I_0 for $r\gg 2m$ and is singular for $r=2m$. If we interpret the expression in curly brackets as the spacetime metric, we find that its Riemannian invariant $R^{abcd}R_{abcd}$ and its Ricci invariant $R^{ab}R_{ab}$ are singular for $r=2m$ (but not its scalar curvature R), when τ depends on θ. This is so for the case when A_3 represents the magnetic field of a dipole, but when A_3 represents a monopole, all invariants remain regular on $r=2m$. So, one expects that $r=2m$ is a coordinate's singularity when A_3 is a monopole field, but the spacetime is really singular for higher multipole moments at this point, and is not a black hole. However, for geodesical trajectories around the surface $r>2m$, the effective potential is regular for $r=2m$ even for magnetic dipole fields, but the scalar field increases without bound for all these cases when r approaches $2m$. The scalar field I is topologically the radius of the fifth dimension, which is a circle. This circle has a constant radius for $r \gg 2m$, but tends to a line when r approaches $2m$. That means that the scalar potential is really important only near the horizon, but disappears very rapidly far away from it. One would suspect that the properties of the geometry change near the horizon with respect to Schwarzschild's geometry due to the interaction of the scalar field. That means that the relevant modifications of Schwarzschild's geometry is not due to the magnetic field, but due to the scalar interaction. The geodesic motion in this spacetime will be published elsewhere [8].

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