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Generalized Gross–Perry–Sorkin-like solitons

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Abstract. We present two new classes of solutions for the effective theory of the Einstein–Maxwell dilaton, low-energy string and Kaluza–Klein theories, which contains among other solutions the well known Kaluza–Klein solitonic monopole of Gross–Perry–Sorkin as a special case. We also explicitly show the magnetic dipole solutions contained in the general one.

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1. Introduction

One of the exact solutions of the vacuum Einstein field equations in five-dimensional gravity is the Gross–Perry–Sorkin (GPS) spacetime [1], which is stationary, everywhere regular and without an event horizon. Actually, it represents a monopole, and there is no reason why the charge carried by the monopole should be labelled 'magnetic', it might equally well be deemed 'electric' and A_μ treated as a potential for the dual field ${}^*F_{\mu\nu}$.

As is well known [1], the monopoles, in addition to charge, are characterized by the topology of their spatial solutions. They carry one unit of Euler character, and therefore one can construct stationary dipole solutions.

Moreover, 5D gravity is one example of the unified theories of electromagnetism and general relativity. Einstein–Maxwell dilaton, Kaluza–Klein, and low-energy string theories are also examples of this kind of unified theory [2]. Mathematically their effective actions in four dimensions are very similar, they differ in the value of the scalar dilatonic field coupling constant. Thus we can write the four-dimensional effective action for all of the above mentioned theories in the form [2]

$$S = \int \mathrm{d}^4x\, \sqrt{-g}\, \big[- R + 2(\nabla\Phi)^2 + \mathrm{e}^{-2\alpha\Phi} F_{\mu\nu} F^{\mu\nu} \big] \tag{1.1}$$

where R is the Ricci scalar, Φ is the scalar dilaton field, $F_{\mu\nu}$ is the Faraday electromagnetic tensor, and α is the dilaton coupling constant. For $\alpha = 0$, (1.1) is the effective action of the Einstein–Maxwell dilaton theory, where the scalar dilaton field appears minimaly coupled to the electromagnetic one. For $\alpha = 1$, (1.1) represents the low-energy string theory, where only the $U(1)$ vector gauge field has not dropped out, and $\alpha = \sqrt{3}$ reduces the action (1.1) to that of the 5D Kaluza–Klein theory. As is well known, all of these theories unify gravity

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and electromagnetism. It is interesting to note that for the string and Kaluza–Klein theories, the electromagnetic field cannot be decoupled from the scalar dilaton field.

The GPS solution has already been studied in [1], and recently the harmonic map ansatz [3] has been applied to the action (1.1) in order to find exact solutions of this kind to its corresponding field equations [4].

In this work we present two new classes of exact solutions to the field equations coming from (1.1), for arbitrary values of the α coupling constant. The solutions are written in terms of a harmonic map, in such a way that for spacial values of this harmonic map, the solution represents electromagnetic monopoles, dipoles, quadrupoles etc. If we choose the harmonic map in order to have monopoles, for the particular case $\alpha = \sqrt{3}$ it reduces just to the GPS solution. This new solitonic solution is the spacetime of a monopole with arbitrary α. In general, this class represents a solitonic spacetime (with only a few exceptions pointed out below), it does not contain horizons and it is regular everywhere for $r \neq 0$. Nevertheless, the influence of the dilaton field becomes important only in regions very close to the soliton.

The plan of the paper is as follows. In section 2 we review the harmonic map ansatz. In section 3 we review very briefly the GPS solution. In section 4 we present the new classes of solutions and their corresponding spacetimes in each of the above mentioned theories. In section 5 we discuss the results and present the conclusions.

2. Harmonic map ansatz

We begin by considering the Papapetrou metric in the following parametrization

$$\mathrm{d}S^2 = \frac{1}{f}[\mathrm{e}^{2k}(\mathrm{d}\rho^2 + \mathrm{d}\zeta^2) + \rho^2\,\mathrm{d}\varphi^2] - f\,\mathrm{d}t^2 \,. \tag{2.1}$$

The harmonic map ansatz [5] supposes that all terms of the metric depend on a set of functions λ^i $(i = 1, \ldots, p)$, such that these functions λ^i satisfy the Laplace equation [6, 7]

$$\Delta\lambda^i = (\rho\lambda^i_{,z})_{,\bar{z}} + (\rho\lambda^i_{,\bar{z}})_{,z} = 0 \tag{2.2}$$

where

$$z = \rho + \mathrm{i}\zeta \,. \tag{2.3}$$

Thus the field equations derived from the Lagrangian (1.1) reduce to equations in terms of the functions λ^i. In general these equations are easier to solve than the original ones. One of the advantages of this method is based on the fact that it is possible to generate exact spacetimes for each solution of the Laplace equation.

Fortunately, the harmonic map determines the gravitational and electromagnetic potentials in such a way that we can choose them to have electromagnetic monopoles, dipoles, quadrupoles, etc. [4].

In the Papapetrou parametrization, the field equations reduce to the set of equations

$$\Delta \ln f = \mathrm{e}^{-2\alpha\Phi}\frac{1}{\rho} f \;\; A_{\varphi,z} A_{\varphi,\bar{z}} \tag{2.4}$$

for the function f

$$2k_{,z} = 4\rho(\Phi_{,z})^2 - \mathrm{e}^{-2\alpha\Phi}\frac{f}{\rho}(A_{\varphi,z})^2 + \rho\;(\ln f_{,z})^2 \tag{2.5}$$

$$2k_{,\bar{z}} = 4\rho(\Phi_{,\bar{z}})^2 - \mathrm{e}^{-2\alpha\Phi}\frac{f}{\rho}(A_{\varphi,\bar{z}})^2 + \rho\;(\ln f_{,\bar{z}})^2 \tag{2.6}$$

for the function k. Let us suppose that the components of the Papapetrou metric depend only on one harmonic map λ. Here we present two solutions to these field equations (2.4)–(2.6). By solving the general field equations coming from the metric (2.1) in terms of one harmonic map λ with no electromagnetic field at all, we arrive at a solution given by [4, 8]

$$f = e^{\lambda} \qquad k_{,z} = \frac{\rho}{2}(4\alpha^2 a^2 + 1)(\lambda_{,z})^2 \tag{2.7}$$

with the following form for the scalar dilaton field

$$e^{2\alpha\Phi} = \kappa_0^2 e^{2\alpha^2 a\lambda} \qquad a = \text{constant} \tag{2.8}$$

where λ is a harmonic map, i.e. a solution of the Laplace equation (2.2).

The field equation for the function k is always integrable if λ is a solution of the Laplace equation. (For more details of the method see [3, 4].)

The second solution we want to deal with here contains an electromagnetic field. It is given by

$$f = \frac{1}{(1-\lambda)^{\frac{2}{1+\alpha^2}}} \qquad k = 0 \qquad A_{\varphi,z} = Q\rho\lambda_{,z} \qquad A_{\varphi,\bar{z}} = -Q\rho\lambda_{,\bar{z}} \tag{2.9}$$

and the corresponding form for the scalar dilaton field is given by

$$e^{-2\alpha\Phi} = \frac{e^{-2\alpha\Phi_0}}{(1-\lambda)^{\frac{2\alpha^2}{1+\alpha^2}}} \tag{2.10}$$

where the magnetic charge is related to the scalar one by $Q^2 = 4e^{2\alpha\Phi_0}/(1+\alpha^2)$. This class contains, among others, the GPS solitonic solution as a particular case. In the next section we briefly review the Kaluza–Klein monopole.

3. Gross–Perry–Sorkin monopole

The Kaluza–Klein monopole, known as the GPS solution, represents the simplest and basic soliton, actually it is a generalization of the self-dual Euclidean Taub–Nut solution [9], and is described by the metric

$$\begin{aligned} ds^2 = -dt^2 &+ \left(1 + \frac{4m}{r}\right)^{-1}(dx^5 + 4m[1-\cos\theta]\,d\varphi)^2 \\ &+ \left(1 + \frac{4m}{r}\right)(dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2) \end{aligned} \tag{3.1}$$

where (r, θ, ϕ) are polar coordinates. For $dt = 0$ the Taub–Nut instanton is obtained. As is well known, the coordinate singularity at $r = 0$ is absent if x^5 is periodic with period $16\pi m = 2\pi R$ [10], with R the radius of the fifth dimension. Thus

$$m = \frac{\sqrt{\pi G}}{2e} \tag{3.2}$$

and the electromagnetic potential A_μ is that of a monopole

$$A_\varphi = 4m(1 - \cos\theta) \tag{3.3}$$

and

$$\boldsymbol{B} = \frac{4m\boldsymbol{r}}{r}\,. \tag{3.4}$$

The magnetic charge of the monopole is fixed by the radius of the Kaluza–Klein circle

$$g = \frac{4m}{\sqrt{16\pi G}} = \frac{R}{2\sqrt{16\pi G}} = \frac{1}{2e}\,. \tag{3.5}$$

Moreover, the mass of the soliton is given by

$$M = \frac{m}{G}\,. \tag{3.6}$$

The GPS solitonic solutions are also solitonic solutions to the effective four-dimensional theory, for the 4-metric $g_{\mu\nu}$ and for a massless scalar dilaton field Φ, as well, with

$$\mathrm{d}s_4^2 = -\frac{\mathrm{d}t^2}{\sqrt{1+4m/r}} + \sqrt{1+\frac{4m}{r}}(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\varphi^2) \tag{3.7}$$

and

$$\Phi = \frac{\sqrt{3}}{4}\ln\left(1+\frac{4m}{r}\right). \tag{3.8}$$

Although this is a singular solution of the effective four-dimensional theory, it is a perfectly sensible soliton. A singularity arises because the conformal factor $\mathrm{e}^{2\alpha\Phi}$ is singular at $r = 0$.

4. Generalized Gross–Perry–Sorkin solution

In the Boyer–Lindquist coordinates

$$\rho = \sqrt{r^2 - 2mr}\,\sin\theta \qquad \zeta = (r-m)\cos\theta \tag{4.1}$$

the group of metrics we want to deal with can be written as

$$\mathrm{d}s_4^2 = (1-\lambda)^{\frac{2}{(1+\alpha^2)}}\left\{\left[1-\frac{2m}{r}+\frac{m^2\sin^2\theta}{r^2}\right]\left[\frac{\mathrm{d}r^2}{1-2m/r}+r^2\,\mathrm{d}\theta^2\right] + \left(1-\frac{2m}{r}\right)r^2\sin^2\theta\,\mathrm{d}\varphi^2\right\} - \frac{\mathrm{d}t^2}{(1-\lambda)^{\frac{2}{(1+\alpha^2)}}} \tag{4.2}$$

with the scalar dilaton field given by

$$\mathrm{e}^{2\Phi} = \frac{\mathrm{e}^{2\Phi_0}}{(1-\lambda)^{\frac{2\alpha}{1+\alpha^2}}}\,. \tag{4.3}$$

The metric (4.2) is an exact solution of the field equations derived from (1.1), where λ is a harmonic map. In what follows, we study the two subclasses namely $m = 0$ and $m \neq 0$.

4.1. Subclass $m = 0$

This subclass is interesting because it corresponds to conformally spherically symmetric spacetimes. Choosing $m = 0$ in (4.2), the metric reduces to

$$\mathrm{d}s^2 = (1-\lambda)^{\frac{2}{1+\alpha^2}}\{\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2\} - \frac{\mathrm{d}t^2}{(1-\lambda)^{\frac{2}{1+\alpha^2}}} \tag{4.4}$$

with λ a solution of the Laplace equation in spherical coordinates

$$\Delta\lambda = (r^2\lambda_{,r})_{,r} + \frac{1}{\sin\theta}(\sin\theta\,\lambda_{,\theta})_{,\theta} = 0 \tag{4.5}$$

and as is well known, the spherical harmonics are solutions of (4.5). Moreover, it is easy to show that it is possible to construct arbitrary multipoles, by performing the identification

$$A_{3,z} = Q\rho\lambda_{,z} \tag{4.6}$$

$$A_{3,\bar{z}} = -Q\rho\lambda_{\bar{z}} \tag{4.7}$$

with z, ρ and ζ given by (2.3) and (4.1), respectively. The GPS solution is thus obtained by choosing the electromagnetic potential $A_3 = Q(1-\cos\theta)$ and $\alpha = \sqrt{3}$ in (4.4), consequently

$$\begin{aligned} \mathrm{d}s_4^2 &= \left(1+\frac{4M}{r}\right)^{\frac{1}{2}}(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\varphi^2) - \left(1+\frac{4M}{r}\right)^{-\frac{1}{2}}\mathrm{d}t^2 \\ \mathrm{e}^{2\Phi} &= \frac{\mathrm{e}^{2\Phi_0}}{(1+4M/r)^{\frac{3}{2}}}\,. \end{aligned} \tag{4.8}$$

As mentioned in the introduction, for $\alpha = 0$ (4.4) is a solution in the framework of the Einstein–Maxwell theory, with the same magnetic field A_3

$$\begin{aligned} \mathrm{d}s_4^2 &= \left(1+\frac{4M}{r}\right)^{2}(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\varphi^2) - \left(1+\frac{4M}{r}\right)^{-2}\mathrm{d}t^2 \\ \mathrm{e}^{2\Phi} &= \mathrm{e}^{2\Phi_0}\,. \end{aligned} \tag{4.9}$$

For $\alpha = 1$, (4.4) reduces to a low-energy string theory solution

$$\begin{aligned} \mathrm{d}s_4^2 &= \left(1+\frac{4M}{r}\right)(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\varphi^2) - \left(1+\frac{4M}{r}\right)^{-1}\mathrm{d}t^2 \\ \mathrm{e}^{2\Phi} &= \frac{\mathrm{e}^{2\Phi_0}}{(1+4M/r)}\,. \end{aligned} \tag{4.10}$$

Finally, for $\alpha = \sqrt{3}$ a 5D Kaluza–Klein solution is obtained in the sense that for this particular case

$$\mathrm{d}S_5^2 = \frac{1}{I}\,\mathrm{d}S_4^2 + I^2(A_\mu\,\mathrm{d}x^\mu + \mathrm{d}x^5)^2 \tag{4.11}$$

with

$$I^3 = \mathrm{e}^{2\alpha\Phi}\,. \tag{4.12}$$

This last fact suggests that in higher-dimensional theories, the four-dimensional part should be multiplied by a conformal factor related to the dilaton field in order to be physically meaningful.

Further generalizations can be carried out by taking $\lambda = \cos\theta/r$, $m = 0$

$$\mathrm{d}s^2 = \left(1-\frac{\cos\theta}{r}\right)^{\frac{2}{1+\alpha^2}}\{\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2\} - \frac{\mathrm{d}t^2}{(1-\cos\theta/r)^{\frac{2}{1+\alpha^2}}} \tag{4.13}$$

and the scalar dilaton field takes the form

$$\mathrm{e}^{-2\Phi} = \frac{\mathrm{e}^{-2\Phi_0}}{(1-\cos\theta/r)^{\frac{2\alpha}{1+\alpha^2}}} \tag{4.14}$$

with $\alpha = 0, 1, \sqrt{3}$. This metric contains a magnetic dipole moment whose magnetic 4-potential is

$$A_3 = -\frac{Q}{2\sqrt{2}}\,\frac{\sin^2\theta}{r}.$$

After a dual transformation

$$F_{ij} \to {}^*F_{ij} = \mathrm{e}^{-2\alpha\Phi}\epsilon_{ijkl}F^{kl} \qquad \Phi \to -\Phi$$

these solutions correspond to electrically charged point particles, where ϵ_{ijkl} is the complete antisymmetric Levi–Civita tensor.

4.2. Subclass $m \neq 0$

The same can be done for $m \neq 0$. The monopole solutions are obtained by choosing $\lambda = \ln(1 - 2m/r)$ (but now with gravitational mass $2m/(1+\alpha^2)$) and $A_3 = -\frac{1}{\sqrt{2}}Q(1-\cos\theta)$, thus the corresponding metric is given by (see also [11])

$$\mathrm{d}s_4^2 = \left[1 - \ln\left(1 - \frac{2m}{r}\right)\right]^{\frac{2}{1+\alpha^2}} \left\{\left[1 - \frac{2m}{r} + \frac{m^2\sin^2\theta}{r^2}\right]\left[\frac{\mathrm{d}r^2}{1-2m/r} + r^2\,\mathrm{d}\theta^2\right] + \left(1 - \frac{2m}{r}\right)r^2\sin^2\theta\,\mathrm{d}\varphi^2\right\} - \frac{\mathrm{d}t^2}{[1 - \ln(1-2m/r)]^{\frac{2}{1+\alpha^2}}} \tag{4.15}$$

with the scalar dilaton field given by

$$\mathrm{e}^{-2\Phi} = \frac{\mathrm{e}^{-2\Phi_0}}{[1 - \ln(1-2m/r)]^{\frac{2\alpha}{1+\alpha^2}}}\,. \tag{4.16}$$

Moreover, in order to achieve a magnetic dipole solution, we identify

$$\lambda = \frac{\cos\theta}{(r-m)^2 - m^2\cos^2\theta}$$

with the electromagnetic potential

$$A_3 = \frac{Q}{2\sqrt{2}}\frac{(r-m)\sin^2\theta}{(r-m)^2 - m^2\cos^2\theta}$$

which corresponds to a magnetic dipole field. The corresponding metric is given by

$$\mathrm{d}s_4^2 = \left(1 - \frac{\cos\theta}{(r-m)^2 - m^2\cos^2\theta}\right)^{\frac{2}{1+\alpha^2}} \left\{\left[1 - \frac{2m}{r} + \frac{m^2\sin^2\theta}{r^2}\right]\left[\frac{\mathrm{d}r^2}{1-2m/r} + r^2\mathrm{d}\theta^2\right] + \left(1 - \frac{2m}{r}\right)r^2\sin^2\theta\,\mathrm{d}\varphi^2\right\} - \frac{\mathrm{d}t^2}{\left(1 - \frac{\cos\theta}{(r-m)^2 - m^2\cos^2\theta}\right)^{\frac{2}{1+\alpha^2}}} \tag{4.17}$$

with the scalar dilaton field given by

$$\mathrm{e}^{-2\Phi} = \frac{\mathrm{e}^{-2\Phi_0}}{\left(1 - \frac{\cos\theta}{(r-m)^2 - m^2\cos^2\theta}\right)^{\frac{2\alpha}{1+\alpha^2}}}\,. \tag{4.18}$$

All metrics are asymptotically flat and they are also flat for $Q = m = 0$.

5. Discussion

We have presented two new classes of solutions to the Einstein–Maxwell dilaton field equations with arbitrary scalar coupling constant. These classes contain a family of solutions in all of the three most important theories which unify gravitation and electromagnetism. The first class does not contain the electromagnetic field at all, it represents a gravitational object with a dilatonic interaction and its spherical symmetry is conformal. The second class contains two subclasses, i.e. $m = 0$ and $m \neq 0$. Both classes are written in terms of a harmonic map in spherical coordinates, resulting in the spherical harmonics corresponding to the electromagnetic multipoles in the electromagnetic potential of the corresponding class. Moreover, we present a magnetic molopole solution, which for the Kaluza–Klein case ($\alpha = \sqrt{3}$) reduces to the GPS soliton. The $m = 0$ subclass is spherically symmetric for monopoles, due to the symmetry of its electromagnetic field, but for dipoles this subclass is only conformally spherically symmetric. However, the second subclass ($m \neq 0$) does not present this symmetry. By performing a dual transformation, the magnetic solutions correspond to electrically charged solutions. From the astrophysical point of view, dipole solutions are more interesting. There exist some conclusions about Kaluza–Klein theories, derived using the GPS soliton [12, 13]. Nevertheless, this solution corresponds to a magnetic monopole, which has not yet been observed in nature and such conclusions could only be valid for electromagnetic monopoles. Moreover, magnetic dipoles are well known objects in nature, many astrophysical objects including our own planet, are gravitational bodies endowed with magnetic dipoles, therefore these kind of solutions could be more realistic for testing the Kaluza–Klein, as well as the low-energy string and the Einstein–Maxwell theories from the classical point of view.

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