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Stationary dilatons with arbitrary electromagnetic field

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Abstract. We present two new classes of axisymmetric stationary solutions of the Einstein–Maxwell–dilaton equations with coupling constant $\alpha^2 = 3$. Both classes are written in terms of two harmonic maps λ and τ. λ determines the gravitational potential and τ the electromagnetic one in such a form that we can have an arbitrary electromagnetic field. As examples we generate two solutions with mass (M), rotation (s) and scalar (δ) parameters, one with electric charge (q) and the other with magnetic dipole (Q) parameter. The first solution contains the Kerr metric for $q = \delta = 0$.

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1. Introduction

There exists in the universe a great number of astrophysical objects possessing both gravitational and electromagnetic fields. Such is the case for some planets and stars possessing a magnetic dipole field like the Earth or the Sun. Einstein–Maxwell (EM) theory predicts the existence of gravitational objects endowed with magnetic dipoles by means of complicated exact solutions [1, 2]. Furthermore, EM theory is actually not a unification theory; here the electromagnetic field appears as an energy–momentum tensor, there is in fact no explanation for its existence, it is put in by hand and the electromagnetic field appears like a model. On the other hand, five-dimensional (5D) gravity is an alternative theory for understanding combined gravitational and electromagnetic interactions. In this theory the electromagnetic field is a consequence of a more general unified field, it is not a model. Nevertheless, it contains an extra field not observed in nature, a scalar field called the dilaton. Scalar fields are not new in physics, they appear naturally in all the most important unified theories, like Kaluza–Klein theory [3] and superstring theory, and are put in by hand in unified models like inflation or the standard model, but nobody has ever seen one. In [4] it is shown that there exists a class of very simple static exact solutions of the Einstein–Maxwell–dilaton field equations, where the coupling constant between the scalar field and electromagnetism α remains arbitrary. These solutions possess a gravitational and a magnetic dipole field; the four-dimensional (4D) metric behaves very similarly to that in the Schwarzschild solution, coupled with a magnetic dipole. They are very simple solutions, but they possess a scalar field interaction coming from the compactification in

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5D, not observed in astrophysical objects. Nevertheless, in [4] it is shown that for these solutions, the dilaton interaction cannot be measured in weak gravitational fields like that of the Sun, even if the Sun possessed one, but it would be perhaps possible to measure it in stronger gravitational fields like that of a pulsar. A star like the Sun is essentially static, but a pulsar is essentially not, therefore it is worth generalizing this solution to a rotating body. One expects that the scalar field interaction will not modify the interaction of test particles for non-compact bodies like the Sun, even if the rotation is taken into account. But one expect that a rotating solution will give us more information about the behaviour of rotating compact bodies.

In [5], Matos developed a method for generating exact static solutions of the 5D Einstein equations with a G_3 group of motion, putting the solutions in terms of two harmonic maps, λ and τ. The harmonic map λ determined the gravitational field and the harmonic map τ the electromagnetic one. Therefore one can choose the electromagnetic field of a monopole, dipole, quadrupole, etc. In this work we generalize a set of static solutions for $\alpha^2 = 3$. The first class represents an electrically charged static body with mass and scalar field parameters, the second one represents a magnetic dipole with mass and scalar field parameters [4]. We obtain their corresponding rotating solution using invariant transformations in the potential space. Unfortunately this method can be used only when $\alpha^2 = 3$ or 0, only in these two cases is the potential space symmetric (see [5, 6]). This work is organized as follows. In section 2 we introduce the potential space formalism for 5D gravity. In section 3 we write the field equations in chiral form. In section 4 we generalize the first class of solutions and in section 5 we generalize the second class. We write the solutions in terms of two harmonic maps and give some explicit solutions in section 6. In section 7 we give some conclusions and remarks.

2. Potential space field equations

In this section we introduce the potential formalism for 5D gravity. This theory is characterized by the existence of a Killing vector field $\boldsymbol{X}$, which generates the $U(1)$ gauge electromagnetic group. Introducing an extra timelike Killing vector field, Neugebauer [6] and later Maison [7] introduced the potential formalism in 5D gravity (see also [9], where the procedure of [6, 7] and [5, 10, 18] was repeated and a Bonnor-like solution was derived [8]). The field equations are then those for stationary fields. This formalism consists of covariantly defining five potentials in terms of the two commuting Killing vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$; $\boldsymbol{X}$ being related to the $U(1)$ isometry and $\boldsymbol{Y}$ being related to stationarity. The five potentials are (see also [5])

$$\begin{aligned} &I^2 = \kappa^{4/3} = X^\mu X_\mu \qquad f = -IY^\mu Y_\mu + I^{-1}(X^\mu Y_\mu)^2 \\ &\psi = -I^{-2} X^\mu Y_\mu \qquad \epsilon_{,\mu} = \epsilon_{\alpha\beta\gamma\delta\mu} X^\alpha Y^\beta Y^{\gamma;\delta} \\ &\chi_{,\mu} = -\epsilon_{\alpha\beta\gamma\delta\mu} X^\alpha Y^\beta X^{\gamma;\delta} \end{aligned} \tag{1}$$

where $\epsilon_{\alpha\beta\gamma\delta\mu}$ is the Levi-Civita pseudotensor. In the adapted coordinate system, where $X = X^A\partial/\partial x^A = \partial/\partial x^5$, $Y = Y^A\partial/\partial x^A = \partial/\partial x^4$, one finds that $\Psi^A = (f, \epsilon, \psi, \chi, \kappa)$, $A = 1, \ldots, 5$ are the gravitational, rotational, electrostatic, magnetostatic and scalar potentials, respectively. The five-dimensional field equations, in terms of the potentials (1), can be derived from the Lagrangian [5, 6]

$$L = \frac{\rho}{2f^2}\left[f_{,i} f^{,i} + (\epsilon_{,i} + \psi\chi_{,i})(\epsilon^{,i} + \psi\chi^{,i})\right] + \frac{\rho}{2f}\left(\kappa^2\psi_{,i}\psi^{,i} + \frac{1}{\kappa^2}\chi_{,i}\chi^{,i}\right) + \frac{2}{3}\frac{\rho}{\kappa^2}\kappa_{,i}\kappa^{,i} \tag{2}$$

(variation is with respect to Ψ^A). Now we can define a 5D (abstract) Riemannian space V_5, called the potential space, inspired in the Lagrangian (2) with metric

$$\mathrm{d}S^2 = \frac{1}{2f^2}\left[\mathrm{d}f^2 + (\mathrm{d}\epsilon + \psi\,\mathrm{d}\chi)^2\right] + \frac{1}{2f}\left(\kappa^2\,\mathrm{d}\psi^2 + \frac{1}{\kappa^2}\,\mathrm{d}\chi^2\right) + \frac{2}{3}\,\frac{\mathrm{d}\kappa^2}{\kappa^2}. \tag{3}$$

On V_5, the five potentials Ψ^A are the local coordinates which define a symmetric Riemmanian space, i.e. the covariant derivative of the curvature tensor of V_5 with respect to each coordinate vanishes.

3. The chiral form of the field equations

Axisymmetry is represented by the existence of a third spacelike Killing vector field $\boldsymbol{Z}$. One can then choose a coordinate system in which the components of the 5-metric depend only on two coordinates. In this case the matrix representation of the 5D Einstein field equations in the potential space V_5 are [10]

$$(\rho g_{,z}\, g^{-1})_{,\bar{z}} + (\rho g_{,\bar{z}}\, g^{-1})_{,z} = 0 \tag{4}$$

where Weyl's canonical coordinates ρ and ζ are given by $z = \rho + \mathrm{i}\zeta$ and its complex conjugate $\bar{z}$. The matrix g in (4) is a symmetric matrix, and is an element of the group $SL(3,\mathbb{R})$, i.e.

$$g = g^T, \qquad g = \bar{g}, \qquad \det g = 1 \tag{5}$$

(T denotes matrix transposition). Using the invariant transformations of the Lagrangian (2), we generate new solutions from a seed one. This means that if we have a solution of the field equation Ψ^A, an invariant transformation of (2) $\Psi^A \to \Psi'^A(\Psi^B)$ will give us a new solution (see [11]). All the invariant transformations of (2) were found in [10]. The group of motion of the metric (3) is $SL(3,\mathbb{R})$. This group has eight parameters. The invariant transformations of the Lagrangian (2) can be cast into a very simple form as

$$g = C g_0 C^T \tag{6}$$

where the constant matrix C is also an element of $SL(3,\mathbb{R})$. The matrix g can be parametrized in terms of (1) as [10]

$$g = \frac{-2}{f\kappa^{2/3}} \begin{pmatrix} f^2 + \epsilon^2 - f\kappa^2\psi^2 & -\epsilon & -\frac{1}{2\sqrt{2}}(\epsilon\chi + f\kappa^2\psi) \\ -\epsilon & 1 & \frac{1}{2\sqrt{2}}\chi \\ -\frac{1}{2\sqrt{2}}(\epsilon\chi + f\kappa^2\psi) & \frac{1}{2\sqrt{2}}\chi & \frac{1}{8}(\chi^2 - \kappa^2 f) \end{pmatrix}. \tag{7}$$

The correct choice of the C matrix in (6) will generate new solutions. We use this method for generating the Belinsky–Ruffini solution [13].

4. The first class of solutions: seed solution $\Psi_0^A = (f_0, \epsilon_0, 0, 0, \kappa_0)$

In order to obtain the new solution, we write the potentials Ψ^A in terms of the components of the matrix g and its inverse

$$g^{-1} = -\frac{1}{2}\frac{\kappa^{2/3}}{f} \begin{pmatrix} 1 & \epsilon + \chi\psi & -2\sqrt{2}\psi \\ \epsilon + \chi\psi & f^2 + (\epsilon + \chi\psi)^2 - f\chi^2\kappa^{-2} & 2\sqrt{2}[f\chi\kappa^{-2} - \psi(\epsilon + \chi\psi)] \\ -2\sqrt{2}\psi & 2\sqrt{2}[f\chi\kappa^{-2} - \psi(\epsilon + \chi\psi)] & -8(f\kappa^{-2} - \psi^2) \end{pmatrix} \tag{8}$$

thus the potentials Ψ^A can be written as

$$\kappa^{4/3} = \frac{4g_{11}^{-1}}{g_{22}}, \qquad f^2 = \frac{1}{g_{11}^{-1} g_{22}}, \qquad \chi = 2\sqrt{2}\frac{g_{23}}{g_{22}},$$
$$\psi = \frac{1}{2\sqrt{2}} \frac{g_{13}^{-1}}{g_{11}^{-1}}, \qquad \epsilon = -\frac{g_{12}}{g_{22}}, \tag{9}$$

where g_{ij} are the components of the matrix g, and g_{ij}^{-1} are the components of the matrix g^{-1}. As the seed solution we take f_0, ϵ_0 and κ_0 arbitrary and $\psi_0 = \chi_0 = 0$ in the matrix g_0 in (6). Using the invariant transformations (6) with the C matrix as

$$C = \begin{pmatrix} a & b & c \\ d & e & j \\ i & h & k \end{pmatrix} = \begin{pmatrix} q & p & t \\ u & v & w \\ s & y & z \end{pmatrix}^{-1}, \tag{10}$$

we evaluate matrix g using relations (9), and obtain

$$\kappa^{4/3} = \frac{V}{W}\kappa_0^{4/3}, \qquad f^2 = \frac{f_0^2}{VW},$$
$$\psi = \frac{1}{2\sqrt{2}V}\left[(u\epsilon_0 + q)(w\epsilon_0 + t) + f_0(wuf_0 - 8sz\kappa_0^{-2})\right],$$
$$\chi = \frac{2\sqrt{2}}{W}\left[(d\epsilon_0 - e)(i\epsilon_0 - h) + f_0\left(dif_0 - \tfrac{1}{8}kj\kappa_0^2\right)\right], \tag{11}$$
$$\epsilon = -\frac{1}{W}\left[(a\epsilon_0 - b)(d\epsilon_0 - e) + f_0\left(adf_0 - \tfrac{1}{8}cj\kappa_0^2\right)\right],$$

with

$$V = (q + u\epsilon_0)^2 + f_0(u^2 f_0 - 8s^2\kappa_0^{-2})$$

and

$$W = (d\epsilon_0 - e)^2 + f_0\left(d^2 f_0 - \frac{j^2\kappa_0^2}{8}\right).$$

Solution (11) is endowed with eight new free parameters. The transformation (11) yields a solution for the potentials Ψ^A from an arbitrary seed solution without electromagnetism. If we want to obtain an asymptotically flat solution, we must fix some of the eight parameters (see [10]). In general, one should give an explicit seed solution in (11) and integrate (1) in order to obtain the corresponding metric in the spacetime; we gave an example in [10]. Here we give a general integration for a special case of the matrix C. There exists a class of solutions which can be integrated for any seed solution. This class is derived from the matrix

$$C_M = \begin{pmatrix} q & 0 & -s \\ 0 & 1 & 0 \\ -s & 0 & q \end{pmatrix}, \tag{12}$$

with $\det C_M = q^2 - s^2 = 1$. Substituting C_M into (10), the five potentials Ψ^A read

$$f^2 = \frac{f_0^2}{q^2 - 8s^2 f_0 \kappa_0^{-2}}, \qquad \epsilon = q\epsilon_0, \qquad \chi = 2\sqrt{2}s\epsilon_0,$$
$$\kappa^{4/3} = \kappa_0^{4/3}(q^2 - 8s^2 f_0 \kappa_0^{-2}), \qquad \psi = \frac{1}{2\sqrt{2}} \frac{qs(1 - 8 f_0 \kappa_0^{-2})}{q^2 - 8s^2 f_0 \kappa_0^{-2}}. \tag{13}$$

Using definitions (1) we can integrate (13) to obtain the spacetime metric, we arrive at

$$\mathrm{d}S^2 = \frac{1}{I_0 f_0} \mathrm{e}^{2k}\, \mathrm{d}z\, \mathrm{d}\bar{z} + \left(g_{033} - \frac{8s^2 g_{034}^2}{T\, I_0^2} \right) \mathrm{d}\phi^2 + 2\frac{q g_{034}}{T}\, \mathrm{d}\phi\, \mathrm{d}t - \frac{f_0}{I_0 T}\, \mathrm{d}t^2$$
$$+ I^2 \left[- \frac{2\sqrt{2}\, s g_{034} I_0}{T}\, \mathrm{d}\phi - \frac{qs}{2\sqrt{2}} \frac{1 - 8 f_0 \kappa_0^{-2}}{T}\, \mathrm{d}t + \mathrm{d}x^5 \right]^2. \tag{14}$$

where $T = q^2 - 8s^2 f_0 \kappa_0^{-2}$ (a subscript 0 denotes the seed solution).

Seed stationary solutions of the Einstein equations in terms of harmonic maps are well known (see [11, 12]), they are written in terms of the Ernst potential $\mathcal{E} = f + \mathrm{i}\epsilon$. These seed solutions can be substituted into (14) in order to obtain the metric in terms of harmonic maps. Let us give an example. We start from the Kerr–NUT solution together with a κ_0 potential as a seed solution given by

$$f_0 = \frac{\omega - 2mr - 2lL_+}{\omega}, \qquad \epsilon_0 = \frac{2(mL_+ - lr)}{\omega}, \qquad \kappa_0 = \left(\frac{r - m + \sigma}{r - m - \sigma} \right)^{\delta},$$
$$L_+ = a\cos\theta + l, \qquad L_- = a\cos\theta - l, \qquad \omega = r^2 + (a\cos\theta + l)^2, \tag{15}$$

where r and θ are the Boyer–Lindquist coordinates, the constants a, m and l are, respectively, the rotation, mass and NUT parameters. Solution (14) with seed solution (15) is an axisymmetric stationary exact solution of 5D gravity. The resulting metric is

$$\mathrm{d}S^2 = \frac{\omega}{\omega - 2mr - 2lL_+} \left(\frac{r - m - \sigma}{r - m + \sigma} \right)^{2\delta/3} (r^2 - 2mr + L_+ L_-) \mathrm{e}^{2k_s} \left(\frac{\mathrm{d}r^2}{\Delta} + \mathrm{d}\theta^2 \right)$$
$$+ \frac{1}{D} \left(\frac{r - m + \sigma}{r - m - \sigma} \right)^{4\delta/3} \Bigg\{ -(\omega - 2mr + 2lL_+)\, \mathrm{d}t^2$$
$$+ (4qa(mr + l)\sin^2\theta + 2l\cos\theta\Delta) \frac{\omega - 2mr + 2lL_+}{r^2 - 2mr + L_+ L_-}\, \mathrm{d}t\, \mathrm{d}\phi$$
$$+ \Bigg[\frac{\omega}{\omega - 2mr - 2lL_+} \Delta \sin^2\theta\, D \left(\frac{r - m - \sigma}{r - m + \sigma} \right)^{2\delta}$$
$$- q^2 (\omega - 2mr - 2lL_+) \left(\frac{4a\sin^2\theta\, (mr + l) + 2l\cos\theta\Delta}{r^2 - 2mr + L_+ L_-} \right)^2 \Bigg] \mathrm{d}\phi^2 \Bigg\}$$
$$+ \left(\frac{r - m + \sigma}{r - m - \sigma} \right)^{2\delta/3} \frac{D}{\omega} (A_3\, \mathrm{d}\phi + A_4\, \mathrm{d}t + \mathrm{d}x^5)^2 \tag{16}$$

where

$$A_3 = -2\sqrt{2}\left(\frac{r-m+\sigma}{r-m-\sigma}\right)^{2\delta}\left(\frac{4a\sin^2\theta\,(mr+l)+2l\cos\theta\,\Delta}{r^2-2mr+L_+L_-}\right)\frac{\omega-2mr+2lL_+}{D},$$
$$A_4 = -\frac{qs}{2\sqrt{2}}\,\frac{\omega\big[(r-m+\sigma)/(r-m-\sigma)\big]^{2\delta}-8(\omega-2mr+2lL_+)}{D},$$
$$D = \omega\left(\frac{r-m+\sigma}{r-m-\sigma}\right)^{2\delta}q^2-8s^2(\omega-2mr+2lL_+), \tag{17}$$
$$\Delta = r^2-2mr+a^2-l^2$$
$$e^{2k_s} = 6\frac{\left(\sqrt{\rho^2+(\zeta-m)^2}+\sqrt{\rho^2+(\zeta+m)^2}\right)^2-4m^2}{4\sqrt{[\rho^2+(\zeta-m)^2][\rho^2+(\zeta+m)^2]}}$$

with

$$\rho = \sqrt{r^2-2mr+a^2-l^2}\,\sin\theta \qquad \zeta = (r-m)\cos\theta. \tag{18}$$

Here m, a, q and σ are constants related by the restrictions $\sigma^2 = m^2+l^2-a^2$, $q^2-s^2=1$. The metric (16) reduces to the Kramer metric [14] for $C = \mathrm{diag}(1,1,1)$, $l=0$, to the Kerr metric for $C = \mathrm{diag}(1,1,1)$, $l=0$, $\delta=0$, and to the Belinsky–Ruffini [13] solution by setting $\delta=\frac{1}{2}$, $l=0$. A study of the singularities of the metric (16) will be given elsewhere [15].

5. The second class of solutions: seed solution $\Psi_0^A = (f_0, 0, 0, \chi_0, \kappa_0)$

From the astrophysical point of view, magnetic dipoles are more interesting. Now we start from a magnetized static seed solution, with f, χ and κ arbitrary and $\epsilon=\psi=0$. In order to obtain the new solution, we proceed in the same way as in the previous case. Thus, we find the following expressions for the potentials:

$$\kappa^{4/3} = \frac{4B}{\kappa_0^2 A}\kappa_0^{4/3} \qquad f^2 = \frac{\kappa_0^2}{AB}f_0^2$$
$$\chi = -\frac{1}{\sqrt{2}A}\left\{8idf_0^2+8e\left(h+\frac{1}{2\sqrt{2}}k\chi_0\right)+j[2\sqrt{2}h\chi_0+k(\chi_0^2-f_0\kappa_0^2)]\right\} \tag{19}$$
$$\epsilon = \frac{1}{4A}\left\{8adf_0^2+8be+2\sqrt{2}\chi_0(bj+ce)+cj(\chi_0^2-f_0\kappa_0^2)\right\}$$
$$\psi = -\frac{1}{4\sqrt{2}B}\left\{tq\kappa_0^2+uw(f_0^2\kappa_0^2-f_0\chi_0^2)+2\sqrt{2}f_0\chi_0(ws+uz)-8zsf_0\right\}$$

where

$$A = -\tfrac{1}{4}\left\{8d^2f_0^2+e(8e+4\sqrt{2}j\chi_0)+j^2(\chi_0^2-f_0\kappa_0^2)\right\}$$
$$B = -\tfrac{1}{2}\left\{q^2\kappa_0^2+u^2(f_0^2\kappa_0^2-f_0\chi_0^2)+4sf_0(\sqrt{2}u\chi_0-2s)\right\} \tag{20}$$

and again we can integrate solution (20) for any seed solution if we use matrix (12), so the solution in the potential space takes the form

$$\kappa^{4/3} = \kappa_0^{4/3}(q^2 - 8s^2 f_0 \kappa_0^{-2}), \qquad \chi = q\chi_0, \qquad \epsilon = \frac{1}{2\sqrt{2}} s\chi_0$$
$$f = \frac{f_0}{\sqrt{q^2 - 8s^2 f_0 \kappa_0^{-2}}}, \qquad \psi = \frac{1}{2\sqrt{2}} \frac{qs(1 - 8 f_0 \kappa_0^{-2})}{q^2 - 8s^2 f_0 \kappa_0^{-2}}. \tag{21}$$

In the spacetime, solution (21) can be integrated to obtain

$$\mathrm{d}S^2 = \frac{1}{I}\left\{T^{1/2}\frac{\mathrm{e}^{k_0}}{f_0}\,\mathrm{d}z\,\mathrm{d}\bar{z} + \left[T^{1/2}\frac{\rho^2}{f_0} - \frac{8s^2 A_{03}^2 f_0}{T^{1/2}}\right]\mathrm{d}\phi^2 - \frac{2\sqrt{2}\,s A_{03} f_0}{T^{1/2}}\,\mathrm{d}\phi\,\mathrm{d}t - \frac{f_0}{T^{1/2}}\,\mathrm{d}t^2\right\}$$
$$+ I^2\left(\frac{qA_{03}}{T}\,\mathrm{d}\phi - \frac{qs}{2\sqrt{2}}\frac{1 - 8 f_0 \kappa_0^{-2}}{T}\,\mathrm{d}t + \mathrm{d}x^5\right)^2 \tag{22}$$

where $T = q^2 - 8s^2 f_0 \kappa_0^{-2}$, the electromagnetic 4-potential is given by

$$A_3 = \frac{qA_{03}}{q^2 - 8s^2 f_0 \kappa_0^{-2}} \qquad A_4 = \frac{-qs(1 - 8s^2 f_0 \kappa_0^{-2})}{2\sqrt{2}(q^2 - 8s^2 f_0 \kappa_0^{-2})}, \tag{23}$$

with $A_1 = A_2 = 0$, and the scalar field fulfils the identity $I^3 = \kappa^2$. We can combine solution (22) with the seed solutions in terms of harmonic maps, in order to have rotating solutions with arbitrary electromagnetic field. This is done in the next section.

6. Explicit solutions

In this section we use the results given in [16], where static solutions of 5D gravity were written in terms of two harmonics maps λ and τ. Many classes of solutions in terms of one and two harmonic maps were found in [16], here we give only one example. There exist two subclasses of static solutions of five-dimensional gravity which are very similar to Schwarzschild spacetime (see [17]). These subclasses have been generalized for any dilaton theory in [18]. Here we use their five-dimensional version. The static metric reads [17, 18]

$$\mathrm{d}s^2 = \frac{1}{I}\left\{\mathrm{e}^{2(k_s+k_e)} g_{22}^{\gamma}\frac{\mathrm{d}r^2}{1 - 2m/r} + g_{22}^{\gamma} r^2(\mathrm{e}^{2(k_s+k_e)}\,\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) - \frac{1 - 2m/r}{g_{22}^{\gamma}}\,\mathrm{d}t^2\right\}$$
$$+ I^2(A_3\,\mathrm{d}\phi + \mathrm{d}x^5)^2 \tag{24}$$
$$A_{3,\zeta} = Q\rho\tau_{,\zeta}, \qquad A_{3,\bar{\zeta}} = -Q\rho\tau_{,\bar{\zeta}}, \qquad \kappa^2 = I^3 = \frac{h^3 \mathrm{e}^{\tau_0 \tau}}{(1 - 2m/r) g_{22}^{\beta}}.$$

The functions g_{22}, k_s and k_e for the subclass (a) are

$$g_{22} = a_1 \mathrm{e}^{q_1 \tau} + a_2 \mathrm{e}^{q_2 \tau},$$
$$k_{s,\zeta} = \frac{\rho}{2\alpha^2}(\lambda_{,\zeta} - \tau_0 \tau_{,\zeta})^2, \qquad k_{e,\zeta} = -\rho\gamma q_1 q_2 (\tau_\zeta)^2, \qquad \tau_0 = q_1 + q_2,$$

and for the subclass (b) are

$$g_{22} = a_1 \tau + 1, \qquad k_{s,\zeta} = \frac{\rho}{2\alpha^2}(\lambda_{,\zeta})^2, \qquad k_e = 0, \qquad \tau_0 = 0,$$

where $\zeta = \rho + \mathrm{i}z = \sqrt{r^2 - 2mr}\sin\theta + \mathrm{i}(r-m)\cos\theta$. $\boldsymbol{A} = A_i\,\mathrm{d}x^i$, $i = 1, \ldots, 4$ is the electromagnetic 4-potential, m is the mass parameter, $\gamma = \frac{1}{2}$, $\beta = \frac{3}{2}$; Q, $a_1 + a_2 = 1$, q_1 and q_2 are constants, subjected to the restrictions

$$2\gamma a_1 a_2 (q_1 - q_2)^2 + \kappa_0^2 Q^2 = 0$$

for subclass (a), and

$$2\gamma a_1^2 - \kappa_0^2 Q^2 = 0$$

for subclass (b). This metric is convenient because we can interpret m as the mass parameter, g_{22} as the magnetic field contribution to the metric and the expression between braces $\{\cdots\}$ as the four-dimensional spacetime metric. Metric (24) can be interpreted as a 5D magnetized Schwarzschild solution. Reference [19] contains a list of a set of solutions of the harmonic map equation $(\rho\tau_{,\zeta})_{,\bar\zeta} + (\rho\tau_{,\bar\zeta})_{,\zeta} = 0$ and their corresponding magnetic potential A_3. The harmonic map λ determines the gravitational potential, while the harmonic map τ determines the electromagnetic one. For a magnetic dipole, the harmonic map τ and its corresponding magnetic field read

$$\tau = \frac{\cos\theta}{(r-m)^2 - m^2\cos^2\theta} \qquad A_3 = \frac{Q(r-m)\sin^2\theta}{(r-m)^2 - m^2\cos^2\theta}. \tag{25}$$

In general, τ can be chosen in such a way as to obtain monopoles, dipoles, quadrupoles, etc.

6.1. Mass and angular momentum

Now we substitute the values of g_{033}, g_{044} and A_{03} from metric (24) for an arbitrary electromagnetic field, i.e. solution (24) into the metric (22), to obtain

$$\begin{aligned} \mathrm{d}S^2 = \frac{1}{I}\bigg\{ & T^{1/2}\mathrm{e}^{2(k_s+k_e)} g_{22}^{\gamma}\left(\frac{\mathrm{d}r^2}{1-2m/r} + r^2\,\mathrm{d}\theta^2\right) \\ & + \left[T^{1/2} g_{22}^{\gamma} r^2 \mathrm{e}^{2(k_s+k_e)}\sin^2\theta - \frac{8s^2}{T^{1/2}} A_{03}^2 \frac{1-2m/r}{g_{22}^{\gamma}}\right]\mathrm{d}\phi^2 \\ & - \frac{2\sqrt{2}}{T^{1/2}} s A_{03} \frac{1-2m/r}{g_{22}^{\gamma}}\,\mathrm{d}\phi\,\mathrm{d}t - \frac{1}{T^{1/2}}\frac{1-2m/r}{g_{22}^{\gamma}}\,\mathrm{d}t^2\bigg\} \\ & + I^2\left(\frac{q}{T}A_{03}\,\mathrm{d}\phi - \frac{qs}{T}\left(1 - 8\frac{1}{I_0^3}\mathrm{e}^{-\tau_0\tau}\left(1-\frac{2m}{r}\right)^2 g_{22}\right)\mathrm{d}t + \mathrm{d}x^5\right)^2. \end{aligned} \tag{26}$$

where $T = q^2 - 8s^2(1/I_0^3)\mathrm{e}^{-\tau_0\tau}(1-2m/r)^2 g_{22}$ and $q^2 - s^2 = 1$. Metric (26) is an exact solution of 5D gravity. If we substitute $s = 0$ we recover the seed metric (24). For the magnetic field (25), we can obtain the mass and the electromagnetic parameters. For both cases (a) and (b), metric (26) contains the mass parameter

$$M = m\left[1 + \frac{8s^2}{h^3}\right] = qm \tag{27}$$

and an angular momentum per unit of mass given by

$$a = \frac{sQ}{4M}, \tag{28}$$

provided that $h = 2$.

6.2. Electromagnetic field components

For the magnetic dipole, the electromagnetic 4-potential reads

$$A_3 = \frac{qQ(r-m)\sin^2\theta}{T[(r-m)^2 - m^2\cos^2\theta]}, \qquad A_4 = -\frac{qs(1-8(1/I_0^3)\mathrm{e}^{-\tau_0\tau}(1-2m/r)^2 g_{22})}{2\sqrt{2}\,T} \tag{29}$$

which corresponds to a dipole magnetic field with magnetic charge

$$Q_M = qQ, \tag{30}$$

for both cases (a) and (b).

7. Final remarks

We have found two classes of solutions of five-dimensional gravity. The first one can be generated from any vacuum solution of the Einstein equations, plus a scalar field, the result is a charged solution of five-dimensional gravity. In fact, if we write the seed metric in Papapetrou form

$$\mathrm{d}S^2 = \frac{1}{I_0}\left\{\frac{1}{f_0}[\mathrm{e}^{2k_0}\,\mathrm{d}z\,\mathrm{d}\bar{z} + \rho^2\,\mathrm{d}\phi^2] - f_0(\mathrm{d}t + A_0\,\mathrm{d}\phi)^2\right\} + I_0^2\,\mathrm{d}x^{5^2}, \tag{31}$$

the resulting metric reads

$$\mathrm{d}S^2 = \frac{1}{I}\left\{\frac{1}{f}[\mathrm{e}^{2k_0}\,\mathrm{d}z\,\mathrm{d}\bar{z} + \rho^2\,\mathrm{d}\phi^2] - f(\mathrm{d}t + qA_0\,\mathrm{d}\phi)^2\right\} + I^2(A_3\,\mathrm{d}\phi + A_4\,\mathrm{d}t + \mathrm{d}x^5)^2, \tag{32}$$

with the functions f, I, A_3 and A_4 given by

$$f = \frac{f_0}{T^{1/2}}, \qquad I = I_0 T^{1/2},$$

$$A_3 = -\frac{2\sqrt{2}\,s f_0 A_0}{T} \qquad A_4 = -\frac{qs}{2\sqrt{2}}\frac{1-8f_0 I_0^{-3}}{T}.$$

The second class can be generated from a seed static solution of five-dimensional gravity, which possesses a magnetic field, i.e.

$$\mathrm{d}S^2 = \frac{1}{I_0}\left\{\frac{1}{f_0}[\mathrm{e}^{2k_0}\,\mathrm{d}z\,\mathrm{d}\bar{z} + \rho^2\,\mathrm{d}\phi^2] - f_0\,\mathrm{d}t^2\right\} + I_0^2(A_{03}\,\mathrm{d}\phi + \mathrm{d}x^5)^2, \tag{33}$$

the result is a rotating solution

$$\mathrm{d}S^2 = \frac{1}{I}\left\{\frac{1}{f}[\mathrm{e}^{2k_0}\,\mathrm{d}z\,\mathrm{d}\bar{z} + \rho^2\,\mathrm{d}\phi^2] - f(\mathrm{d}t - 2\sqrt{2}\,sA_{03}\,\mathrm{d}\phi)^2\right\} + I^2(A_3\,\mathrm{d}\phi + A_4\,\mathrm{d}t + \mathrm{d}x^5)^2, \tag{34}$$

with the functions f, I, A_3 and A_4 given by

$$f = \frac{f_0}{T^{1/2}}, \qquad I = I_0 T^{1/2},$$

$$A_3 = \frac{qA_{03}}{T} \qquad A_4 = -\frac{qs}{2\sqrt{2}}\frac{1-8f_0 I_0^{-3}}{T}.$$

With the first one we generated some well known solutions, such as the Belinsky–Ruffini spacetime. This solution is not a black hole, the scalar field forms a naked singularity because the scalar field parameter δ is fixed. Solution (16) contains black holes for different values of δ. At least the Kerr black hole is contained here for $l = \delta = 0$, but we do not know at the moment if other black holes are contained in the metric (16). The behaviour of this metric is in some sense similar to the Kerr–Newman black hole. It contains a magnetic dipole moment, but it vanishes if the rotation parameter $a = 0$. This means that the magnetic dipole is provoked by induction from the electric field. This metric represents a rotating electrically charged mass, with an induced magnetic dipole. The second class of solutions is quite different. If the rotation parameter s vanishes, the solution becomes static and the magnetic field is just that of a magnetically charged sphere, but the electric charge vanishes as well. This means that the electric charge is here induced by the rotation of the magnetic dipole. This solution represents a rotating mass with a magnetic dipole charge. It is quite surprising that the magnetic field does not alter during the rotation, as can be seen from (21). Nevertheless, there is no way to conserve the rotation when the magnetic field is zero; if $Q = 0$, the rotation parameter a vanishes as well.

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