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AXIALLY SYMMETRIC SOLUTIONS IN DILATONIC THEORY AND THE SOLAR SYSTEM TESTS

TONATIUH MATOS* and HUGO VILLEGAS†

Depto. de Física, CINVESTAV IPN, A.P. 14-740, 07000 México D.F., México

OCTAVIO OBREGÓN‡

Instituto de Física, Universidad de Guanajuato, A.P. E-143, León, Gto., México

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An analysis of the three classical solar system tests for dilatonic gravity coupled to electromagnetism is discussed using an exact, axially symmetric solution. At the post-Newtonian order there is no difference with general relativity, but a constraint on the coupling constant α is obtained at the next order.

1. Introduction

Einstein's theory of gravity has surmounted every confrontation with the experiments. Other theories have been proposed and the dynamics they predict varies from Einstein's laws of gravity. Some have been regarded as viable, but they have been proven to disagree with experiments.[1] One is thus led to consider only theories satisfying three criteria for viability: self-consistency, completeness,[2] and agreement with past observations.[3] These "worthy" theories provide proposals against which general relativity (GR) can be tested.

Once the first two criteria are fulfilled by an alternative theory of gravity, the first step is to compare its weak field, slow motion limit predictions with the observations already made in the solar system. Usually, some reasonable assumptions are made, like perfect fluid structure equations and/or spherically symmetric matter distributions. Starting with the equations of motion, the post-Newtonian approximation of the theory is obtained, and from this, analytical expressions for the standard tests are derived.[4] These tests, namely, the electromagnetic-wave deflection, the radar echo delay, and the perihelion shift of Mercury, have been able to rule out several alternative theories to general relativity, and to impose more or less stringent constraints on free parameters in other viable theories. To deal with these calculations

*E-mail: tmatos@fis.cinvestav.mx; also at Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Ciudad Universitaria, 58260, Morelia, Mich., México.
†E-mail: brena@fis.cinvestav.mx
‡E-mail: octavio@ifug3.ugto.mx

for different theories in a systematic way, the so-called *parametrized post-Newtonian* (PPN) formalism was developed in the early '70s.[1,5,6] Nevertheless, one can assume that certain exact solutions could represent astronomical objects of astrophysical relevance. Some conditions would need to be imposed in these kinds of solutions to be sure that they represent a particular physical configuration of interest. One could, e.g., require the specification of the interior solution, and not only the exterior one; the corresponding matching between them should also be determined, and they should be regular.

It is not always easy for an exact solution to exhibit all these requirements. However, some have a form very close to already well-known solutions, so that one is tempted to extend the physical significance from the old to the new ones. Thus, the first question to be answered is whether its post-Newtonian limit coincides with the observations, and if, at different orders, this limit coincides in its turn with the formal expansions obtained from the equations of motion, or at least, if the free parameters in the theory (if any) are actually constrained in the expected way. Moreover, by knowing the exact solution we do not need to care about any particular structure of the expansion of Einstein's equations, and we can carry out approximations to any order; in particular, we can still use the full exact solution even for a very strong gravitational field.

In this letter we shall study one such solution, namely, a metric in dilatonic gravity, confronting its predictions with the solar system tests. This letter is organized as follows: in Sec. 2, the metric is introduced; then, in Sec. 3, expressions for the three classical tests are obtained. At the post-Newtonian order there is no difference with GR, but at the next order the coupling parameter α must be constrained; finally, in Sec. 4, a brief discussion about the results is presented.

2. The Metric

Here we shall be concerned with a solution for the dilaton coupled to electromagnetism action:

$$S = \int d^4x\sqrt{-g}(R - 2(\nabla\Phi)^2 - e^{-2\alpha\Phi}F^2)\,, \tag{1}$$

which has recently been studied in the context of superstring theories (see, e.g., Refs. 7, 8 and references therein). From the above action, the following field equations are obtained after variations with respect to the metric $g_{\mu\nu}$ and the scalar, dilatonic field Φ:

$$\begin{gathered}
(e^{-2\alpha\Phi}F^{\mu\nu})_{;\mu} = 0\,,\\
\Phi^{;\mu}{}_{;\mu} + \frac{\alpha}{2}e^{-2\alpha\Phi}F_{\mu\nu}F^{\mu\nu} = 0\,,\\
R_{\mu\nu} = 2\Phi_{,\mu}\Phi_{,\nu} + 2e^{-2\alpha\Phi}\left(F_{\mu\lambda}F_{\nu}{}^{\lambda} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right).
\end{gathered} \tag{2}$$

In particular, we shall focus on the metric[9]:

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + e^{2k_s} \frac{dr^2}{1 - \frac{2m}{r}} + r^2(e^{2k_s} d\theta^2 + \sin^2\theta \, d\phi^2)\,, \tag{3}$$

with

$$e^{2k_s} = \left(1 + \frac{m^2 \sin^2\theta}{r^2(1 - \frac{2m}{r})}\right)^{-1/\alpha^2} \qquad \text{for} \quad \alpha \neq 0\,,$$

$$\Phi = \frac{1}{2\alpha} \ln\left(1 - \frac{2m}{r}\right),$$

and where $F_{\mu\nu} = 0$. Note that although in this particular case there is no electromagnetic field, the coupling constant α still appears in the metric elements. It can also be noted that, in the limit when $\alpha \to \infty$, we recover the Schwarzschild solution; however, because of the form of the action, this would imply an infinite coupling strength between gravity and the scalar field, whereas in GR there is no scalar field at all. So the usual coupling with matter would correspond to $\alpha = 0$. Here and throughout, units in which $c = G = 1$ are used.

Due to the similarity with the Schwarzschild solution, one is tempted to assume that the metric above could represent an axially symmetric body of mass m in dilatonic gravity. It might then be used as a simplified model for the sun, for example, and it can be found whether it satisfies the three standard tests. This is discussed in Sec. 3.

When the formal expansion in powers of $1/r$ for Eq. (3) is obtained, the first-order post-Newtonian (1PN) limit is found to coincide with that of GR. However, in the second-order post-Newtonian (2PN) approximation, some terms involving the quantity $1/\alpha^2$ appear, thus providing the possibility to restrict the value of the coupling constant α.

On the other hand, it might be interesting to know what this particular solution looks like in other *conformal frames*, which could be useful in the context of string theories.[10] We therefore use the following general result: Let $\mathcal{L}^*$ be a Lagrangian for a particular tensor–scalar theory (e.g., (1)), and let $\tilde{\mathcal{L}}$ be a Lagrangian obtained from $\mathcal{L}^*$ by a conformal transformation, say $d\tilde{s}^2 = \Omega\, ds_*^2$. It can then be easily proved that the equations of motion derived from $\tilde{\mathcal{L}}$ are identical to those obtained by applying the conformal transformation directly to the equations of motion derived from $\mathcal{L}^*$. In particular, we shall consider conformal transformations which have $\Omega \equiv \Omega(\Phi)$, where Φ is the dilaton introduced in (1).

3. Geodesics Equation and the Three Classical Tests

We begin with the action (1), from which the geodesic equations are written down; after fixing $\theta = \pi/2$ for simplicity, one gets:

$$\left(\frac{dr}{ds}\right)^2 + e^{-2k_s}\left(\frac{B^2}{r^2} + \varepsilon\right)\left(1 - \frac{2m}{r}\right) = e^{-2k_s} A^2\,, \tag{4}$$

where A and B are constants of motion, and ε takes the value 1 or 0 for material particle or photon, respectively (the case $\varepsilon = -1$ is not considered). The constants A and B are then expressed in terms of several observable parameters (see Ref. 4 and below), and an expansion for the relevant quantities up to 2PN order is obtained.

3.1. *Light deflection*

Here and in the next paragraph, an external photon travelling towards the sun (or whatever we want metric (3) to represent) will be considered. Let $r_0 = r(\phi_0)$ be the minimum distance from the photon to the sun. Assuming that $dr/d\phi = 0$ for $r = r_0$, we get:

$$B = r_0 \frac{A}{\sqrt{1 - 2m/r_0}}\,,$$

and the geodesic equation (4) leads to:

$$\phi(r) - \phi(\infty) = \int_r^\infty \frac{dx}{x\left(\frac{(1-\frac{m}{x})^2}{1-\frac{2m}{x}}\right)^{1/2\alpha^2}\left[\frac{1}{r_0^2}\left(1-\frac{2m}{r_0}\right)x^2 - \left(1-\frac{2m}{x}\right)\right]^{1/2}}\,.$$

The total deflected angle $\Delta\phi_{ST}$ is then, up to 2PN order:

$$\Delta\phi_{ST} = 2|\phi(r) - \phi(\infty)|_{r=r_0} - \pi \approx \frac{4m}{r_0} + \frac{m^2}{r_0^2}\left(\frac{15\pi}{4} - 2 - \frac{\pi}{4\alpha^2}\right).$$

Denoting by $\Delta\phi_{GR}$ the classical result obtained from GR, and making m and r_0 equal to the sun mass and radius, respectively, we arrive to:

$$\varepsilon_\phi \equiv \left|\frac{\Delta\phi_{ST} - \Delta\phi_{GR}}{\Delta\phi_{GR}}\right| \approx \frac{m\pi}{16 r_0 \alpha^2}\,. \tag{5}$$

Now, for metric (3) to represent approximately the sun, the result (5) must be within the observational error, which in turn limits the value of α. Taking an error of $\varepsilon_\phi \leq 1.0\%$,[1] one has $\alpha > 6.5 \times 10^{-3}$.

3.2. *Radar echo delay*

In this case we assume $\frac{dr}{dt}\big|_{r=r_0} = 0$, with $r_0 = r(t_0)$ the minimum distance from the photon to the sun. The relation between A and B is the same as in Sec. 3.1 and the geodesic equation (4) gives:

$$t(r, r_0) = \int_{r_0}^{r} \frac{dx}{\left(\frac{(1-\frac{m}{x})^2}{1-\frac{2m}{x}}\right)^{1/2\alpha^2}\left[1 - \frac{(1-\frac{2m}{x})}{(1-\frac{2m}{r_0})}\left(\frac{r_0}{x}\right)^2\right]^{1/2}\left(1-\frac{2m}{x}\right)}\,.$$

For a radar signal travelling to Mercury at superior conjunction, we put $r_0 =$ the sun radius, and introducing the quantities $r_{\mathrm{M}} =$ Mercury distance to the sun,

$r_{\rm E}$ = Earth distance to the sun and m = sun mass, the total delay Δt_{ST} due to the travelling through curved space–time is:

$$\Delta t_{ST} = 2\left[t(r_E, r_0) + t(r_0, r_M) - \sqrt{r_E^2 - r_0^2} - \sqrt{r_M^2 - r_0^2}\right].$$

Expanding up to 2PN order there results:

$$\Delta t_{ST} \approx 4m\left(1 + \ln\left(\frac{4r_M r_E}{r_0^2}\right)\right) + \frac{m^2}{r_0}\left[-4 + \left(\frac{15}{2} - \frac{1}{2\alpha^2}\right)\left(\pi - \frac{r_0}{r_E} - \frac{r_0}{r_M}\right)\right].$$

Denoting again by Δt_{GR} the result from GR, we obtain:

$$\varepsilon_t \equiv \left|\frac{\Delta t_{ST} - \Delta t_{GR}}{\Delta t_{GR}}\right| \approx \frac{\left(\pi - \frac{r_0}{r_E} - \frac{r_0}{r_M}\right)}{1 + \ln\left(\frac{4r_M r_E}{r_0^2}\right)}\frac{m}{8r_0\alpha^2}. \qquad (6)$$

Here we take an error of $\varepsilon_t \leq 0.1\%$,[1] and a limit $\alpha > 8.3 \times 10^{-3}$ is obtained.

3.3. *Perihelion shift of Mercury*

Here we put $\frac{dr}{d\phi}\big|_{r=r_\pm} = 0$, where r_+ and r_- are the distances from Mercury to the sun at perihelion and aphelion, respectively. From this, the following expressions are obtained:

$$B^2 = \frac{2m\left(\frac{1}{r_-} - \frac{1}{r_+}\right)}{\frac{1}{r_-^2}\left(1 - \frac{2m}{r_-}\right) - \frac{1}{r_+^2}\left(1 - \frac{2m}{r_+}\right)},$$

$$\frac{A^2}{B^2} = \frac{1}{2m}\left(1 - \frac{2m}{r_+}\right)\left(1 - \frac{2m}{r_-}\right)\left(\frac{1}{r_-} + \frac{1}{r_+}\right).$$

The geodesic equation (4) allows us to write:

$$\phi(r_+) - \phi(r_-) = \int_{r_-}^{r} \frac{dx}{x^2\left(\frac{\left(1-\frac{m}{x}\right)^2}{1-\frac{2m}{x}}\right)^{1/2\alpha^2}\left[\frac{A^2}{B^2} - \left(\frac{1}{B^2} + \frac{1}{x^2}\right)\left(1 - \frac{2m}{x}\right)\right]^{1/2}},$$

which, when expanded to 2PN order, leads to:

$$\phi(r_+) - \phi(r_-) \approx \left[1 + m\left(\frac{1}{r_+} + \frac{1}{r_-}\right) + \frac{3}{2}m^2\left(\frac{1}{r_+} + \frac{1}{r_-}\right)^2\right]$$

$$\times \int_{r_-}^{r} \frac{dx}{x^2\left[\left(\frac{1}{r_-} - \frac{1}{x}\right)\left(\frac{1}{x} - \frac{1}{r_+}\right)\right]^{1/2}}$$

$$\times \left[1 + \frac{m}{x} + \left(\frac{3}{2} - \frac{1}{2\alpha^2}\right)\frac{m^2}{x^2}\right].$$

This can be easily integrated with the aid of the standard transformation:

$$\frac{1}{x} = \frac{1}{2}\left(\frac{1}{r_+} + \frac{1}{r_-}\right) + \frac{1}{2}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)\sin\psi\,,$$

and, noting that $\psi(r_\pm) = \pm\pi/2$, the following expression is obtained for the perihelion shift, $\Delta\omega_{ST}$:

$$\begin{aligned}\Delta\omega_{ST} &= 2|\phi(r_+) - \phi(r_-)| - 2\pi \\ &\approx \frac{6\pi m}{a(1-e^2)} + \left(19 - \frac{1}{\alpha^2}\right)\frac{m^2\pi}{a^2(1-e^2)^2} + \left(3 - \frac{1}{\alpha^2}\right)\frac{m^2\pi e^2}{2a^2(1-e^2)^2}\,,\end{aligned}$$

where e and a are the eccentricity and semi major axis of the orbit of Mercury, respectively, and m is the sun mass. Using again the result from GR, $\Delta\omega_{GR}$, and putting $L \equiv (1-e^2)a$, one is led to:

$$\varepsilon_\omega \equiv \left|\frac{\Delta\omega_{ST} - \Delta\omega_{GR}}{\Delta\omega_{GR}}\right| \approx \frac{(2+e^2)\,m}{12L\alpha^2}\,. \tag{7}$$

When the error of 0.5% in the observations is taken into account,[1] result (7) imposes the limit $\alpha > 1.2 \times 10^{-3}$, which is consistent with (5) and (6).

4. Discussion

As the preceding results show, all predictions from metric (3) are consistent with the three classical solar system tests, provided that we can adjust the coupling constant α, whose lower bounds were obtained. These results differ clearly from the Schwarzschild metric, which is not a limit case of the theory here considered.

We have assumed that the axially symmetric solution we have used represents an astrophysical object of interest, in particular the sun. Having the exact solution we have the advantage of being able to calculate at any order of approximation without using the PPN formalism.[1,5] Some conditions would be desirable in these kinds of solutions to identify them with a real physical configuration of interest. In particular, the existence of an interior solution and its matching with the exterior one. These aspects are the matter of future work.

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