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The Harmonic Map Ansatz in Gravitational Theories

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We study the action $S = \int \sqrt{-g}\{-R + 2(\nabla\Phi)^2 + e^{-2\alpha\Phi}F_{\mu\nu}F^{\mu\nu}\}d^4x$, this action represents a gravitational field coupled with electromagnetism and a dilaton field Φ. This action reduces to Einstein–Maxwell-dilaton theory for $\alpha = 0$, Kaluza–Klein theory for $\alpha = \sqrt{3}$ and to a part of low energy super strings theory for $\alpha = 1$. We suppose α arbitrary. Using the harmonic map ansatz, we reduce the field equations with two Killing vectors to five non-linear ordinary differential equations. We find a class of solutions representing static gravitational fields coupled to electrical and magnetic monopoles, dipoles, quadripoles etc., and to a dilaton field which can model the exterior field of a star.

KEY WORDS : Dilaton ; electrodynamic

1. INTRODUCTION

A great effort has been made to find exact solutions of the Einstein equations representing the field of a star, even to find a reasonable model of a compact star. Most of these efforts are made supposing that those stars are spherically symmetric, that the field equations are ordinary differential equations depending only on the radial coordinate and they seem to be not too hard to manipulate. In this work we will suppose axial symmetry

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and let the metric functions with the most freedom as we can. Compact objects are big stars without any nuclear fuel, stars where there is nothing for avoiding the gravitational collapse. Compact objects like pulsars and black holes contain a strong gravitational field, many results about them has been found using perturbative methods. In our opinion this methods work very good in week fields, but they could not give reliable information close to a strong field or in the interior of such an object, therefore exact solutions could be the only reliable way to extract information from one theory about strong gravitational fields. A star is a gravitational system coupled to a electromagnetic field. In some theories of gravity, like Kaluza–Klein theory and superstrings, there exist a scalar field coupled to electromagnetism. We suppose the existence of a scalar field [1] which can be coupled to the electromagnetic one, letting the interactions constant arbitrary. In this work we investigate what the Einstein–Maxwell-dilaton theory can say about the exterior field of a star.

Let us start with the following action:

$$\mathcal{L} = \sqrt{-g}\,[-R + 2(\Delta\Phi)^2 + e^{-2\alpha\Phi}F^2], \tag{1}$$

where R is the scalar curvature, $F_{\mu\nu}$ is the Faraday tensor and Φ is the dilaton. The main interest in the Lagrangian (1) is that it contains the most important theories unifying gravitation with electromagnetism. The constant α determines special sub-theories: with $\alpha = \sqrt{3}$, we derive from (1) the Kaluza–Klein field equations, for $\alpha = 1$, (1) is the Lagrangian of the low energy limit of string theory and with $\alpha = 0$, (1) represents the Einstein–Maxwell theory minimally coupled to Φ. The field equations derived from (1) become

$$(e^{-2\alpha\Phi}F^{\mu\nu})_{;\mu} = 0,$$

$$\Phi^{;\mu}{}_{;\mu} + \frac{\alpha}{2}e^{-2\alpha\Phi}F_{\mu\nu}F^{\mu\nu} = 0,$$

$$R_{\mu\nu} = 2\Phi_{,\mu}\Phi_{,\nu} + 2e^{-2\alpha\Phi}(F_{\mu\lambda}F_{\nu}{}^{\lambda} - \tfrac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}). \tag{2}$$

2. FIELD EQUATIONS

We assume the existence of two Killing vector fields in the space-time, stationarity $X = \partial/\partial t$ and axisimmetry $Y = \partial/\partial\varphi$. The line element can be written as

$$ds^2 = f(dt - \omega d\varphi)^2 - f^{-1}[e^{2k}(d\rho^2 + d\zeta^2) + \rho^2 d\varphi^2], \tag{3}$$

where f, ω and k are functions of ρ and ζ only.

Let us define an abstract potential space V_p (see Refs. 2 and 3) given by

$$\mathcal{L} = \frac{\rho}{2f^2}[Df^2 + (D\varepsilon - \psi D\chi)^2] - \frac{\rho\kappa^2}{2f}\left(D\psi^2 + \frac{1}{\kappa^4}D\chi^2\right) + \frac{2\rho}{\alpha^2\kappa^2}D\kappa^2, \quad (4)$$

where we have used the definitions $D = (\partial/\partial\rho, \partial/\partial\zeta)$, and $\tilde{D} = (\partial/\partial\zeta, -\partial/\partial\rho)$ such that $D\tilde{D}G(\rho,\zeta) = 0$. The "coordinates" of the Lagrangian (4) are defined by

$$\psi = 2A_t, \qquad \kappa^2 = e^{-2\alpha\Phi}, \qquad \tilde{D}\chi = 2\frac{f\kappa^2}{\rho}(\omega DA_t + DA_\phi),$$

$$\tilde{D}\varepsilon = \frac{f^2}{\rho}D\omega + \psi\tilde{D}\chi,$$

where $A_\mu = (A_t, 0, 0, A_\varphi)$ is the electromagnetic potential one-form. The functions f, ε, ψ, χ and κ may be interpreted as the gravitational, rotational, electrostatic, magnetostatic, and scalar potentials, respectively. Variation with respect to the potentials leads to an equivalent set of field equations, namely the Klein–Gordon equation

$$D^2\kappa + \frac{1}{\rho}D\rho D\kappa - \frac{1}{\kappa}D\kappa^2 + \frac{\alpha^2}{4f}\left(\kappa^3 D\psi^2 - \frac{1}{\kappa}D\chi^2\right) = 0,$$

the Maxwell equations

$$D^2\psi + \left(\frac{D\rho}{\rho} + 2\frac{D\kappa}{\kappa} - \frac{Df}{f}\right)D\psi - \frac{1}{\kappa^2 f}(D\varepsilon - \psi D\chi)D\chi = 0, \quad (5)$$

$$D^2\chi + \left(\frac{D\rho}{\rho} - 2\frac{D\kappa}{\kappa} - \frac{Df}{f}\right)D\chi + \frac{\kappa^2}{f}(D\varepsilon - \psi D\chi)D\psi = 0, \quad (6)$$

and the Einstein equations

$$D^2 f + \frac{1}{f}[(D\varepsilon - \psi D\chi)^2 - Df^2] + \frac{D\rho}{\rho}Df - \frac{1}{2\kappa^2}(\kappa^4 D\psi^2 + D\chi^2) = 0, \quad (7)$$

$$D^2\varepsilon - D\psi D\chi - \psi D^2\chi + (D\varepsilon - \psi D\chi)\left(\frac{D\rho}{\rho} - 2\frac{Df}{f}\right) = 0. \quad (8)$$

Eq. (8) can be cast into first order differential equations defining

$$A_1 = \frac{1}{2f}[f_{,z} - i(\varepsilon_{,z} - \psi\chi_{,z})], \qquad D_1 = \frac{1}{\alpha^2}(\ln\kappa)_{,z},$$

$$B_1 = \frac{1}{2f}[f_{,z} + i(\varepsilon_{,z} - \psi\,\chi_{,z})], \qquad C_1 = (\ln\kappa)_{,z}\,,$$
$$E_1 = -\frac{1}{2}f^{-1/2}\left[\kappa\,\psi_{,z} - i\frac{\chi_{,z}}{\kappa}\right],$$
$$F_1 = \frac{1}{2}f^{-1/2}\left[\kappa\,\psi_{,z} + \frac{i\chi_{,z}}{\kappa}\right], \tag{9}$$

$(z = \rho + i\,\zeta)$ and A_2, B_2... etc. with $\bar{z}$ in place of z, cf. [4] and [5]. In terms of the potentials (9), the field equations (2) now read

$$
\begin{aligned}
A_{1,\bar{z}} &= A_1A_2 - A_1B_2 - \tfrac{1}{2}C_2A_1 - \tfrac{1}{2}C_1A_2 - E_1F_2\,,\\
A_{2,z} &= A_1A_2 - A_2B_1 - \tfrac{1}{2}C_2A_1 - \tfrac{1}{2}C_1A_2 - E_2F_1\,,\\
B_{1,\bar{z}} &= B_1B_2 - A_2B_1 - \tfrac{1}{2}C_2B_1 - \tfrac{1}{2}C_1B_2 - E_2F_1\,,\\
B_{2,z} &= B_1B_2 - A_1B_2 - \tfrac{1}{2}C_2B_1 - \tfrac{1}{2}C_1B_2 - E_1F_2\,,\\
E_{1,\bar{z}} &= A_1E_2 + \tfrac{1}{2}A_2E_1 - \tfrac{1}{2}B_2E_1 - \tfrac{1}{2}C_1E_2 - \tfrac{1}{2}C_2E_1 + \alpha^2D_1F_2\,,\\
E_{2,z} &= A_2E_1 + \tfrac{1}{2}A_1E_2 - \tfrac{1}{2}B_1E_2 - \tfrac{1}{2}C_1E_2 - \tfrac{1}{2}C_2E_1 + \alpha^2D_2F_1\,,\\
F_{1,\bar{z}} &= B_1F_2 + \tfrac{1}{2}B_2F_1 - \tfrac{1}{2}A_2F_1 - \tfrac{1}{2}C_1F_2 - \tfrac{1}{2}C_2F_1 + \alpha^2D_1E_2\,,\\
F_{2,z} &= B_2F_1 + \tfrac{1}{2}B_1F_2 - \tfrac{1}{2}A_1F_2 - \tfrac{1}{2}C_1F_2 - \tfrac{1}{2}C_2F_1 + \alpha^2D_2E_1\,,\\
D_{1,\bar{z}} &= -(E_1E_2 + F_1F_2) - \tfrac{1}{2}C_1D_2 - \tfrac{1}{2}C_2D_1\,,\\
D_{2,z} &= -(E_1E_2 + F_1F_2) - \tfrac{1}{2}C_1D_2 - \tfrac{1}{2}C_2D_1\,,
\end{aligned} \tag{10}
$$

which transforms system (8) into 10 non-linear first order partial differential equations in place of five of second order. There exist a Lax pair representation of (10) only for $\alpha = 0$ (see Ref. 6) and $\sqrt{3}$ (see Ref. 4). If we want to extract information from eqs. (10) for α arbitrary, we must find another method for solving this system.

3. HARMONIC MAPS ANSATZ

In this work we apply an alternative method for finding exact solutions of the field equations (10), which we call the harmonic maps ansatz [7,8,5]. We briefly explain it. Let λ^i, $i = 1, ..., p$ be harmonic maps

$$(\rho\lambda^i_{,z})_{,\bar{z}} + (\rho\lambda^i_{,\bar{z}})_{,z} + 2\rho\Gamma^i_{jk}\lambda^j_{,z}\lambda^k_{,\bar{z}} = 0,$$

where Γ^i_{jk} are the Christoffel symbols of a Riemannian space V_p. Suppose a Riemannian space V_2, $ds^2 = d\lambda\, d\tau/(1 - l\lambda\tau)$, $l = 0, 1$, with

$$A_1 = a_1(\lambda)\lambda_{,z} + a_2(\tau)\tau_{,z}\,, \qquad B_1 = b_1(\lambda)\lambda_{,z} + b_2(\tau)\tau_{,z}\,,$$
$$E_1 = e_1(\lambda)\lambda_{,z} + e_2(\tau)\tau_{,z}\,, \qquad F_1 = f_1(\lambda)\lambda_{,z} + f_2(\tau)\tau_{,z}\,,$$
$$D_1 = d_1(\lambda)\lambda_{,z} + d_2(\tau)\tau_{,z}\,,$$

and A_2, B_2... etc. with $\bar{z}$ in place of z. If we substitute this ansatz into (10), we obtain

$$
\begin{aligned}
a_{1;\lambda} &= a_1^2 - a_1 b_1 - e_1 f_1 , \\
b_{1;\lambda} &= b_1^2 - a_1 b_1 - e_1 f_1 , \\
e_{1;\lambda} &= \frac{3}{2} a_1 e_1 - \frac{1}{2} b_1 e_1 + d_1 f_1 , \\
f_{1;\lambda} &= \frac{3}{2} b_1 f_1 - \frac{1}{2} a_1 f_1 + d_1 e_1 , \\
d_{1;\lambda} &= -\frac{\alpha^2}{2} (e_1^2 + f_1^2),
\end{aligned} \tag{11}
$$

where $a_{;\lambda} = a_{,\lambda} - 4 l a \tau / (1 - l \lambda \tau)$ and a similar system for a_2, b_2, etc. with $\bar{z}$ in place of z and τ in place of λ. For $l = 0$ we have five non-linear first order ordinary differential equations in place of 5 coupled of second order. This equations are of course much more easier to solve because they are uncoupled.

4. EXACT SOLUTIONS

We present here a set of static exact solutions of (11) with $l = 0$. For this case the line element reads

$$
ds^2 = f \, dt^2 - f^{-1} [e^{2k} (d\rho^2 + d\zeta^2) + \rho^2 d\varphi^2],
$$

where f and k are functions of z and $\bar{z}$ only.

We write the line element in terms of the harmonic maps λ and τ, that means, in terms of two functions fulfilling the harmonic map equation

$$
\lambda_{,\rho\rho} + \frac{1}{\rho} \lambda_{,\rho} + \lambda_{,\zeta\zeta} = 0, \qquad \tau_{,\rho\rho} + \frac{1}{\rho} \tau_{,\rho} + \tau_{,\zeta\zeta} = 0 .
$$

In what follows we give some magnetostatic (electrostatic) solutions. In any case they can be transformed into electrostatic (magnetostatic) ones because the field equations are invariant under the transformations

$$
\Phi \rightarrow -\Phi, \qquad F_{\mu\nu} \rightarrow F^*_{\mu\nu} = \tfrac{1}{2} e^{-2\alpha\Phi} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma},
$$

where $\varepsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita pseudotensor. In [3] we have given some static exact solutions of (11), and here we complete the scheme. The class of solutions we study here is

$$f = \frac{e^{\lambda}}{g^{\gamma}}, \qquad \kappa^2 = \frac{e^{-\lambda + (p+q)\tau}}{g^{\beta}},$$

$$\psi = 0, \qquad \chi = \frac{l_1 \tau + l_2}{g}, \qquad g = l_3 \tau + l_4, \tag{12}$$

where l_1, l_2..., p, q, and κ_0, satisfy the relationship $4l_1^2 - \kappa_0^2(1 + \alpha^2)(l_1 l_4 - l_2 l_3)^2 = 0$. We substitute these solutions into (2), and the function k can be obtained from the differential equation

$$k_{,z} = \frac{\rho}{2}\left[(\lambda_{,z})^2 + \frac{1}{\alpha^2}((\lambda_{,z} - (p+q)\tau_{,z})^2 - 2pq\beta(\tau_{,z})^2)\right]. \tag{13}$$

In what follows we classify the solutions in subcases. Let us separate the function k as

$$k = k_g + k_e + k_s, \tag{14}$$

where we called

$$k_{g,z} = \frac{\rho}{2}(\lambda_{,z})^2, \qquad k_{s,z} = \frac{\rho}{2\alpha^2}(\lambda_{,z} - (p+q)\tau_{,z})^2,$$

$$k_{e,z} = \frac{\rho}{\alpha^2} pq\beta(\tau_{,z})^2. \tag{15}$$

It is important to note that the magnetostatic potential χ is completely determined by the harmonic function τ. This means that the magnetostatic (electrostatic) potential is determined by τ only, so we can obtain solutions with arbitrary electromagnetic fields. The most important well-known solutions can be derived from this method. Some examples are given in [3] and [10,11].

We are interested in models for a star. Stars have to be very similar to the Schwarzschild solution. In order to obtain Schwarzschild-like solutions, (stars-like fields), we take $\lambda = \ln(1 - 2m/r)$. The metric then reads

$$ds^2 = e^{2(k_s + k_e)} g^{\gamma} \frac{dr^2}{1 - 2m/r} + g^{\gamma} r^2 (e^{2(k_s + k_e)} d\theta^2 + sin^2\theta d\varphi^2) - \frac{(1 - 2m/r)}{g^{\gamma}} dt^2. \tag{16}$$

The differential equation for k_s and k_e can be integrated separately. Doing so and using the separation of the function k, we can carry out the following classification of the solutions:

(i) First class.
$\tau = 0$, which implies $g = 1$, $\chi = \psi = 0$. The metric is

$$ds^2 = e^{2k_s}\frac{dr^2}{1 - 2m/r} + r^2(e^{2k_s}d\theta^2 + sin^2\theta d\varphi^2) - (1 - 2m/r)dt^2. \quad (17)$$

(ii) Second class.
$\tau \neq 0$, but $g = 1$, $\psi = 0$. Then we have

$$ds^2 = e^{2(k_s + k_e)}\frac{dr^2}{(1 - 2m/r)} + r^2(e^{2(k_s + k_e)}d\theta^2 + sin^2\theta d\varphi^2) - (1 - 2m/r)dt^2. \quad (18)$$

For $k_s + k_e = 0$, then $\tau = \lambda$, $(p + q - 1)^2 - 2pq\beta = 0$ and we recover the Schwarzschild line element.

(iii) Third class.
This important case with $\tau = \lambda$ and $(p + q - 1)^2 - 2pq\beta = 0$, is that of spherical symmetry. In this case we recover as particular solution the (GHS) black hole [13], the metric is

$$ds^2 = g^\gamma\frac{dr^2}{1 - 2m/r} + g^\gamma r^2(d\theta^2 + sin^2\theta d\varphi^2) - \frac{(1 - 2m/r)}{g^\gamma}dt^2. \quad (19)$$

(iv) Fourth class
$\tau \neq 0$, but $k_s = 0$, $p + q = 1$ and $\tau = \lambda$. The metric is

$$ds^2 = e^{2k_e}g^\gamma\frac{dr^2}{1 - 2m/r} + g^\gamma r^2(e^{2k_e}d\theta^2 + \sin^2\theta d\varphi^2) - \frac{(1 - 2m/r)}{g^\gamma}dt^2. \quad (20)$$

All these solutions have the form of the Schwarzschild one, and one expects to recover the same physical behavior for it; that means, we expect that all these solutions could be models for the exterior field of a star or of a compact object.

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REFERENCES

1. Buchdahl, H. A. (1959). *Phys. Rev.* **115**, 1325.
2. Neugebauer, G. (1969). *Habilitationsschrift* (Universität Jena Press, Jena).
3. Matos, T., Nuñez, D., and Quevedo, H. (1995). *Phys. Rev. D***51**, R310.
4. Matos, T. (1986). *Astron. Nachr.* **307**, 317.
5. Matos, T. (1994). *J. Math. Phys.* **35**, 1302.
6. Kramer, D., Stephani, H., MacCallum, M. A. H., and Herlt, E. (1980). *Exact Solutions of Einstein's Field Equations* (VEB Deutscher Verlag der Wissenschaften, Berlin).
7. Matos, T., and Becerril, R. (1992). *Rev. Mex. Fis.* **38**, 69.
8. Matos, T. (1995). *Math. Notes* **58**, 1178.
9. Matos, T., and Rios, M. (1999). In preparation
10. Matos, T., and Macias, A. (1994). *Mod. Phys. Lett.* **A9**, 3707.
11. Macias, A., and Matos, T. (1994). *Class. Quantum Grav.* **13**, 345.
12. Matos, T. (1994). *Phys. Rev. D***48**, 4296.
13. Gibbons, G., and Maeda, K. (1988). *Nucl. Phys. B***298**, 741; Garfinkle, D., Horowitz, G. T., and Strominger, A. (1991). *Phys. Rev. D***43**, 3140.