Differential and Integral Equations Volume 12, Number 1, January 1999, pp. 23–40

LARGE TIME BEHAVIOR OF SOLUTIONS OF HIGHER ORDER NONLINEAR DISPERSIVE EQUATIONS OF KDV TYPE WITH WEAK NONLINEARITY

Nakao Hayashi
Department of Applied Mathematics, Science University of Tokyo
1-3, Kagurazaka, Shinjuku-ku, Tokyo 162, Japan

Tonatiuh Matos and Pavel I. Naumkin
Instituto de Física y Matemáticas, Universidad Michoacana
AP 2-82, CP 58040, Morelia, Michoacana, Mexico

(Submitted by: Sergiu Klainerman)

Abstract. We study the asymptotic behavior for large time of solutions to the Cauchy problem for the higher-order dispersive equations of Korteweg-de Vries type with weak nonlinearity (wKdV):

$$u_t + \partial_x f(u) + \mathcal{K}u = 0,$$

where $x, t \in \mathbf{R}$, $f(u) = |u|^{\rho-1}u$ if $\rho > \nu$ or $f(u) = u^\rho$ if $\rho > \nu$ is integer, the operator $\mathcal{K}$ is a pseudodifferential operator with a homogeneous and conservative symbol $K(p)$ of order $\nu > 3$, namely, $\mathcal{K}u = \mathcal{F}^{-1}K(p)\hat{u}(p), K(p) = -\frac{i}{\nu}|p|^{\nu-1}p$, $\mathcal{F}\phi$ or $\hat{\phi}$ is the Fourier transformation of ϕ and $\mathcal{F}^{-1}\phi$ is the inverse Fourier transformation of ϕ. If the power ρ of the nonlinearity is greater than ν, then the solution of the Cauchy problem has a quasilinear asymptotic behavior for large time. More precisely, we show that the solution $u(t)$ satisfies the decay estimate

$$\|u(t)\|_{L^\beta} \le C(1+t)^{-\frac{1}{\nu}(1-\frac{1}{\beta})} \quad \text{for} \quad \beta \in (\frac{2\nu-2}{\nu-2}, \infty],$$

$$\|uu_x(t)\|_{L^\infty} \le Ct^{-2/\nu}(1+t)^{-1/\nu}$$

and using these estimates we prove the existence of the scattering state $u_+ \in L^2$ such that

$$\|u(t) - U(t)u_+\|_{L^2} \le Ct^{-\frac{\rho-\nu}{\nu}} \quad \text{and} \quad \|u(t) - U(t)u_+\|_{L^\infty} \le Ct^{-\frac{1+\rho-\nu}{\nu}}$$

Received for publication December 1997.
AMS Subject Classifications: 35Q55.

for any small initial data belonging to the weighted Sobolev space $H^{1,1} = \{\phi \in L^2; \|(1+|x|^2)^{1/2}(1-\partial_x^2)^{1/2}\phi\|_{L^2} < \infty\}$, where $U(t)$ is the free evolution group, associated with corresponding linear equation.

1. Introduction. In this paper we study the asymptotic behavior in time of solutions of the Cauchy problem for the following nonlinear dispersive equation of the Korteweg-de Vries type with weak nonlinearity (wKdV)

$$\begin{cases} u_t + \partial_x f(u) + \mathcal{K}u = 0, & t, x \in \mathbf{R}, \\ u(0,x) = u_0(x), & x \in \mathbf{R}, \end{cases} \tag{1.1}$$

where u_0 is a real valued function and the nonlinear term is $f(u) = |u|^{\rho-1}u$ if $\rho > \nu$ or we can consider $f(u) = u^\rho$ if $\rho > \nu$ is integer. The operator $\mathcal{K}$ is a pseudodifferential operator with a homogeneous and conservative symbol $K(p)$ of order $\nu > 3$, namely $\mathcal{K}u = \mathcal{F}^{-1}K(p)\hat{u}(p), K(p) = -\frac{i}{\nu}|p|^{\nu-1}p$. Here and below $\mathcal{F}\phi$ or $\hat{\phi}$ is the Fourier transformation of ϕ defined by $\mathcal{F}\phi(\xi) = \frac{1}{\sqrt{2\pi}}\int e^{-ix\xi}\phi(x)dx$. The inverse Fourier transformation $\mathcal{F}^{-1}$ is given by $\mathcal{F}^{-1}\phi(x) = \frac{1}{\sqrt{2\pi}}\int e^{ix\xi}\phi(\xi)d\xi$. The nonlinearity in equation (1.1) is weak from the point of view of large time behavior of solutions since the nonlinear term decays faster than the linear summands in the equation and therefore does not disturb essentially the character of the large time behavior of the solution. Note that if $\nu = 3$, then we get from (1.1) the well studied generalized Korteweg - de Vries (gKdV) equation, if $\nu = 2$, then (1.1) represents the generalized Benjamin - Ono equation, and if $\nu = 5$ we get the Kawahara equation (some other interesting examples can be found in the book [21]). The Cauchy problem (1.1) was intensively studied by many authors. The existence and uniqueness of solutions to (1.1) in different Sobolev spaces were proved in [1, 9, 10, 13 - 17, 19, 20, 24, 30]. The smoothing properties of solutions were studied in [5, 6, 7] and the blow-up effect for the slowly decaying solutions of the Cauchy problem (1.1) was found in [4]. Our purpose in the present paper is to study the large time asymptotic behavior of solutions to the Cauchy problem (1.1) in the super critical case $\rho > \nu$ when the initial data u_0 are small enough in a certain weighted Sobolev space. For the special cases of the KdV equation itself and the modified KdV equation ($\rho = 3$, $\nu = 3$ in (1.1)) the Cauchy problem was solved by the Inverse Scattering Transform (IST) method and the large time asymptotic behavior of solutions was found (see [2]). Here we consider the case $\nu > 3$ since the case $\nu = 3$, which corresponds to the generalized Korteweg- deVries equation was considered in papers [12, 16, 23, 25, 26, 28, 29]. In the present paper we develop the method of the papers [11, 12] to the case of equation (1.1) with higher - order derivatives. Our method here is also close to that of the paper [12] to the extent that we use the estimates of the following operator

$$I\phi(t,x) = x\phi - \nu t \int_{-\infty}^{x} \partial_t \phi(t,y)\, dy$$

which almost commutes with the linear part of equation (1.1) $L = \partial_t + \mathcal{K}$. The operator I is related to the operator

$$J\phi(t,x) = U(-t)xU(t)\phi(t,x) = (x + \nu t \int_{-\infty}^{x} \mathcal{K})\phi(t,x)$$

previously applied for studying the smoothing properties of the solutions [5]. Also our considerations in the present paper are essentially based on the sharp decay estimates of the solutions (see Lemma 2.2) obtained due to the known precise asymptotics of the free evolution group defined by

$$\begin{aligned} U(t)\phi &= \mathcal{F}^{-1} e^{-tK(\xi)} \hat{\phi}(\xi) \\ &= \frac{1}{2\pi} \int dy \phi(y) \int d\xi e^{i\xi(x-y) - tK(\xi)} = \frac{1}{\sqrt[\nu]{t}} \mathrm{Re} \int \mathrm{A}(\frac{x-y}{\sqrt[\nu]{t}}) \phi(y) dy, \end{aligned}$$

where $\mathrm{A}(x) = \frac{1}{\pi} \int_0^\infty e^{ixz - K(z)} dz$.

Before stating our results we give the following:

Notation and function spaces. We introduce some function spaces. $L^p = \{\phi \in \mathcal{S}'; \|\phi\|_p < \infty\}$, where $\|\phi\|_p = (\int |\phi(x)|^p dx)^{1/p}$ if $1 \le p < \infty$ and $\|\phi\|_\infty$ =ess.sup$_{x\in\mathbf{R}} |\phi(x)|$ if $p = \infty$. For simplicity, we let $\|\phi\| = \|\phi\|_2$. Weighted Sobolev space $H^{m,s}$ is defined by $H^{m,s} = \{\phi \in \mathcal{S}'; \|\phi\|_{m,s} = \|(1+|x|^2)^{s/2}(1-\partial_x^2)^{m/2}\phi\| < \infty\}$, $m, s \in \mathbf{R}$. We denote by $\dot{B}^s_{p,q}$ the homogeneous Besov space with the semi-norm

$$\|\psi\|_{\dot{B}^s_{p,q}} = \left(\int_0^\infty y^{-1-\zeta q} \sup_{|z| \le y} \sum_{k \le [s]} \|\partial^k (\psi_{(z)} - \psi)\|_p^q dy \right)^{1/q},$$

where $s = [s] + \zeta$, $0 < \zeta < 1$, $\psi_{(z)}(x) = \psi(x+z)$ and $[s]$ is the largest integer less than s, $\partial = \partial_x = \frac{\partial}{\partial x}$ is the partial differentiation with respect to the space variable x. We note that the norm of the homogeneous Sobolev space $\dot{H}^{\gamma,0}$ is equivalent to that of $\dot{B}^{\gamma}_{2,2}$ (see [3]). We introduce the operator

$$D^{\alpha}\phi = \mathcal{F}^{-1}\xi^{\alpha}e^{-\frac{i\pi}{2}(1+\alpha)}\mathcal{F}\phi = \frac{2\pi}{\Gamma(1-\alpha)}\int_0^{\infty}(\phi(x+y)-\phi(x))\frac{dy}{y^{\alpha+1}},$$

where $\alpha \in (0,1)$ and ξ^{α} is the main value of the complex function, i.e., $\xi^{\alpha} = |\xi|^{\alpha}\exp(i\alpha \arg\xi)$. Therefore, we can write $\|\phi\|_{\dot{H}^{\alpha,0}} = \|D^{\alpha}\phi\|$. We let $(\psi,\varphi) = \int \psi\cdot\overline{\varphi}dx$ and let $C(I;H)$ be the space of continuous functions from an interval I to a Banach space H. Different positive constants might be denoted by the same letter C.

Since equation (1.1) is invariant with respect to the change of variables $x \to -x'$, and $t \to -t'$, it is sufficient to consider the positive time t only.

We now state our results in this paper.

Theorem 1.1. *We assume that the initial data u_0 are real, $u_0 \in H^{1,1}$ and $\|u_0\|_{1,1} = \epsilon$, where ϵ is sufficiently small. Then there exists a unique global solution $u \in C(\mathbf{R};H^{1,1})$, of the Cauchy problem (1.1) with $\rho > \nu > 3$ such that*

$$\|u(t)\|_{\beta} \le \frac{C\epsilon}{(1+t)^{\frac{1}{\nu}-\frac{1}{\nu\beta}}}, \quad \|uu_x(t)\|_{\infty} \le \frac{C\epsilon^2}{t^{\frac{2}{\nu}}(1+t)^{\frac{1}{\nu}}},$$

for all $t > 0$ and for every $\beta \in (\frac{2\nu-2}{\nu-2},\infty]$.

Theorem 1.2. *Let u be the solution of (1.1) with $\rho > \nu > 3$ obtained in Theorem 1.1. Then for any $u_0 \in H^{1,1}$ there exists a unique function $V \in L^{\infty}\cap L^2$ such that $V(0) = \hat{u}_0(0)$ and*

$$\begin{cases} \|\mathcal{F}U(-t)u(t) - V\| \le C\epsilon t^{-\frac{\rho-\nu}{\nu}} \\ \|\mathcal{F}U(-t)u(t) - V\|_{\infty} \le C\epsilon t^{-\frac{2\gamma(\rho-\nu)}{\nu(1+3\gamma)}} \end{cases} \quad \text{for} \quad t \ge 1, \tag{1.2}$$

where $\gamma \in (0,\min(\frac{1}{2},\frac{\rho-\nu}{\nu}))$. Furthermore we have the asymptotic formula for large time t uniformly with respect to the space variable x

$$u(t,x) = \frac{1}{\sqrt[\nu]{t}}ReA(\frac{x}{\sqrt[\nu]{t}})V(\chi) + O(\epsilon t^{-\frac{1+\gamma}{\nu}}(1+|x|/\sqrt[\nu]{t})^{-\frac{\nu-2}{2\nu-2}}), \tag{1.3}$$

where $\chi = \sqrt[\nu-1]{-x/t}$ for $x \le 0$ and $\chi = 0$ for $x \ge 0$.

Remark 1.1. The inequality (1.2) shows that the function V can be calculated approximately via the initial data u_0 and (1.2) means the existence of the scattering states in the noncritical case $\rho > \nu$. Since $V(0) = \hat{u}_0(0)$ the asymptotics of solutions has a linear character in the domain $|x| << t$.

Remark 1.2. Since we consider small amplitude solutions which decay in time, therefore we can also apply our method to the equation (1.1) with the nonlinearity $f(u) = O(|u|^{\rho-1}u)$ for small u, where $\rho > \nu$ provided that some regularity conditions on f are fulfilled.

We organize our paper as follows. In Section 2 we give some preliminary estimates. The Sobolev inequality is stated in Lemma 2.1. Lemma 2.2 states that the time decay of the function $u(t,x)$ can be represented by the free evolution group $U(t)$ and is estimated by the value $\|u(t)\|_{1,0} + \|\partial Ju(t)\| + \|D^\alpha Ju(t)\|$, where $\alpha = 1/2 - \gamma, \gamma \in (0, \min(\frac{1}{2}, \frac{\rho-\nu}{\nu}))$. Lemma 2.3 is necessary for estimates of nonlinearity in the fractional order Sobolev space $\dot{H}^{\alpha,1}$. In Section 3 we prove Theorems 1.1-1.2. In Lemma 3.1 we prove a priori estimates of local solutions to (1.1) in a functional space X_T, which yields Theorem 1.1. Theorem 1.2 comes from Theorem 1.1, Lemma 2.1 and the integral equation associated with (1.1). The function space X_T is the following:

$$X_T = \{\phi \in C([0,T]; \mathcal{S}'); ||\phi||_{X_T} = \sup_{t\in[0,T]} \mathcal{M}\phi(t) < \infty\},$$

where $\mathcal{M}\phi(t) = \|\phi(t)\|_{1,0} + \|D^\alpha J\phi(t)\| + \|\partial J\phi(t)\|$.

2. Preliminaries. We prepare three lemmas which are necessary to obtain our results.

Lemma 2.1. *Let q, r be any numbers satisfying $1 \le q, r \le \infty$, and let j, m be any real numbers satisfying $0 \le j < m$. If $u \in H_r^{m,0} \cap L^q$, then the following inequality is valid*

$$\|(-\partial_x^2)^{j/2} u\|_p \le C \|(-\partial_x^2)^{m/2} u\|_r^a \|u\|_q^{1-a},$$

where C is a constant depending only on m, j, q, r, a, here $\frac{1}{p} = j + a(\frac{1}{r} - m) + (1-a)\frac{1}{q}$ and a is any real number from the interval $\frac{j}{m} \le a \le 1$, with the following exception: if $m - j - \frac{1}{r}$ is nonnegative and an integer, then the value a must be from the interval $\frac{j}{m} \le a < 1$.

For Lemma 2.1 see, e.g., [8, 27].

Lemma 2.2. *Let $u(t,x)$ be a smooth real valued function such that the value $\mathcal{M}u(t) = \|u(t)\|_{1,0} + \|\partial Ju(t)\| + \|D^\alpha Ju(t)\|$ is bounded for all $t > 0$, where $\alpha = 1/2 - \gamma, \gamma \in (0, 1/2)$. Then we have the following estimates for all $t > 0$*

$$\|u(t)\|_\beta \le C(1+t)^{-1/\nu+1/(\nu\beta)}\mathcal{M}u(t), \tag{2.1}$$

and

$$\|uu_x\|_\infty \le Ct^{-2/\nu}(1+t)^{-1/\nu}(\mathcal{M}u(t))^2, \tag{2.2}$$

where $\beta \in (\frac{2\nu-2}{\nu-2}, \infty]$, $\nu > 3$. Moreover, we have the following asymptotics for large time $t \ge 1$ uniformly with respect to $x \in \mathbf{R}$

$$u(t,x) = \frac{1}{\sqrt[\nu]{t}} ReA(\frac{x}{\sqrt[\nu]{t}})\hat{v}(t,\chi) + O(t^{-\frac{1+\gamma}{\nu}}(1+|x|/\sqrt[\nu]{t})^{-\frac{\nu-2}{2\nu-2}}\|D^\alpha Ju(t)\|), \tag{2.3}$$

where $\chi = \sqrt[\sigma]{-x/t}$, $\sigma = \nu - 1 > 2$ for $x \le 0$ and $\chi = 0$ for $x \ge 0$; the function $v = U(-t)u(t)$, and $U(t)$ is the free evolution group associated with the linearized equation (1.1).

Proof. Note that by Lemma 2.1 we have the estimate $\|u\|_\beta \le C\|u\|_{1,0}$ therefore inequality (2.1) is valid for $0 \le t \le 1$. Now let us prove the estimate (2.1) and asymptotics (2.3) for $t \ge 1$. We have the identity with $v(t) = U(-t)u(t)$

$$\begin{aligned} u(t,x) = U(t)v(t) &= \frac{1}{\pi}\mathrm{Re}\int_0^\infty e^{ipx-K(p)t}\hat{v}(t,p)dp \\ &= \frac{1}{\pi\sqrt[\nu]{t}}\mathrm{Re}\int_0^\infty e^{iq\eta+iq^\nu/\nu}(\hat{v}(t,\chi) + (\hat{v}(t,\frac{q}{\sqrt[\nu]{t}}) - \hat{v}(t,\chi)))dq \\ &= \frac{1}{\sqrt[\nu]{t}}\mathrm{ReA}(\frac{x}{\sqrt[\nu]{t}})\hat{v}(t,\chi) + \mathcal{R}(t,x), \end{aligned} \tag{2.4}$$

where

$$\mathcal{R}(t,x) = \frac{1}{\pi\sqrt[\nu]{t}}\mathrm{Re}\int_0^\infty e^{iq\eta+iq^\nu/\nu}(\hat{v}(t,\frac{q}{\sqrt[\nu]{t}}) - \hat{v}(t,\chi))\,dq,$$

and we made a change of variables $q = p\sqrt[\nu]{t}, \eta = x/\sqrt[\nu]{t}$. Consider the case $x \ge 0$, i.e. $\eta \ge 0$ and $\chi = 0$. Using the identity with $\mu = \sqrt[\sigma]{|\eta|}$, $\sigma = \nu - 1 > 2$

$$e^{iq\eta+iq^\nu/\nu} = \frac{1}{1+iq(q^\sigma+\mu^\sigma)}\frac{\partial}{\partial q}(qe^{iq\eta+iq^\nu/\nu}) \tag{2.5}$$

we integrate by parts with respect to q in the remainder term $\mathcal{R}(t,x)$

$$\mathcal{R}(t,x) = \frac{1}{\pi\sqrt[\nu]{t}}\mathrm{Re}\int_0^\infty \Big(\frac{iq(\nu q^\sigma + \mu^\sigma)(\hat{v}(t,\frac{q}{\sqrt[\nu]{t}}) - \hat{v}(t,0))}{1+iq(q^\sigma+\mu^\sigma)} - \frac{q}{\sqrt[\nu]{t}}\hat{v}_p(t,\frac{q}{\sqrt[\nu]{t}})\Big)\frac{e^{iq\eta+iq^\nu/\nu}dq}{1+iq(q^\sigma+\mu^\sigma)}. \tag{2.6}$$

By the Cauchy inequality we have

$$\begin{aligned}|\hat{v}(t,p) - \hat{v}(t,0)| &\le \int_0^p |\hat{v}_p(t,p)|dp \le \Big(\int_0^p |p|^{-2\alpha}dp\int_0^p |\hat{v}_p(t,p)|^2|p|^{2\alpha}dp\Big)^{\frac12}\\ &\le C|p|^\gamma \||p|^\alpha \hat{v}_p(t,p)\| \le C|p|^\gamma\|D^\alpha Ju(t)\|.\end{aligned} \tag{2.7}$$

Making a change of variable $z = q(1+\mu^\sigma)$, we obtain

$$\begin{aligned}\int_0^\infty \frac{(1+q)^{1+3\gamma}dq}{(1+q(q^\sigma+\mu^\sigma))^2} &\le C\int_0^\infty \frac{(1+q)^{1+3\gamma}dq}{(1+q(1+\mu^\sigma))^{\frac53-\gamma}(1+q^\nu)^{\frac13+\gamma}}\\ &\le C(1+\mu^\sigma)^{-1}\int_0^\infty \frac{dz}{(1+z)^{\frac53-\gamma}} \le C(1+\mu)^{-\sigma},\end{aligned} \tag{2.8}$$

where we have used the inequality $1+q(q^\sigma+\mu^\sigma) > \frac12(1+q(1+\mu^\sigma))$. By the Cauchy inequality we get from (2.8)

$$\begin{aligned}\int_0^\infty \frac{q^\gamma dq}{1+q(q^\sigma+\mu^\sigma)} &\le \Big(\int_0^\infty \frac{(1+q)^{1+3\gamma}dq}{(1+q(q^\sigma+\mu^\sigma))^2}\int_0^\infty \frac{dq}{(1+q)^{1+\gamma}}\Big)^{\frac12}\\ &\le \frac{C}{(1+\mu)^{\sigma/2}}.\end{aligned} \tag{2.9}$$

Via (2.7), (2.8) and (2.9) we get from (2.6)

$$\begin{aligned}|\mathcal{R}(t,x)| &\le \frac{C}{\sqrt[\nu]{t}}\int_0^\infty \frac{(|\hat{v}(t,\frac{q}{\sqrt[\nu]{t}}) - \hat{v}(t,0)| + \frac{q}{\sqrt[\nu]{t}}|\hat{v}_p(t,\frac{q}{\sqrt[\nu]{t}})|)dq}{|1+iq(q^\sigma+\mu^\sigma)|}\\ &\le \frac{C}{t^{\frac{\gamma+1}{\nu}}}\|D^\alpha Ju(t)\|\int_0^\infty \frac{q^\gamma dq}{1+q(q^\sigma+\mu^\sigma)}\\ &+ \frac{C}{\sqrt[\nu]{t^2}}\Big(\int_0^\infty q^{2\alpha}|\hat{v}_p(t,\frac{q}{\sqrt[\nu]{t}})|^2dq\int_0^\infty \frac{q^{1+2\gamma}dq}{(1+q(q^\sigma+\mu^\sigma))^2}\Big)^{\frac12}\\ &\le \frac{C\|D^\alpha Ju(t)\|}{(1+\mu)^{\sigma/2}t^{\frac{\gamma+1}{\nu}}}.\end{aligned} \tag{2.10}$$

Now let us consider the second case $x \le 0$, that is $\eta = -\mu^\sigma \le 0$ (we remind that $\mu = \frac{\sqrt[\sigma]{|x|}}{\sigma \sqrt[\nu]{t}}$, $\chi = \frac{\mu}{\sqrt[\nu]{t}}$). Using the identity

$$e^{iq\eta + iq^\nu/\nu} = \frac{1}{1 + i(q-\mu)(q^\sigma - \mu^\sigma)} \frac{\partial}{\partial q}((q-\mu)e^{iq\eta + iq^\nu/\nu}) \tag{2.11}$$

we integrate by parts with respect to q in the remainder term $\mathcal{R}(t,x)$

$$\begin{aligned}\mathcal{R}(t,x) = \frac{1}{\pi \sqrt[\nu]{t}} \mathrm{Re} \int_0^\infty \Big(&\frac{i(q-\mu)(\nu(q^\sigma - \mu^\sigma) - \sigma\mu(q^{\sigma-1} - \mu^{\sigma-1}))}{1 + i(q-\mu)(q^\sigma - \mu^\sigma)} \\ &\times (\hat{v}(t, \frac{q}{\sqrt[\nu]{t}}) - \hat{v}(t,\chi)) - \frac{q-\mu}{\sqrt[\nu]{t}} \hat{v}_p(t, \frac{q}{\sqrt[\nu]{t}}) \Big) \frac{e^{iq\eta + iq^\nu/\nu}}{1 + i(q-\mu)(q^\sigma - \mu^\sigma)} dq.\end{aligned} \tag{2.12}$$

Making a change of the variable of integration $z = (q-\mu)\mu^{\frac{\nu}{2}-1}$, $y = q - \mu$ we get

$$\begin{aligned}\int_0^\infty \frac{|q-\mu|^\gamma dq}{1 + (q-\mu)(q^\sigma - \mu^\sigma)} &\le C \int_0^{2\mu} \frac{|q-\mu|^\gamma dq}{1 + (q-\mu)^2 \mu^{\sigma-1}} + C \int_{2\mu}^\infty \frac{(q-\mu)^\gamma dq}{1 + (q-\mu)^\nu} \\ &\le \frac{C}{\mu^{(1+\gamma)(\frac{\nu}{2}-1)}} \int_{-\mu^{\nu/2}}^{\mu^{\nu/2}} \frac{|z|^\gamma dz}{1+z^2} + C \int_\mu^\infty \frac{y^\gamma dy}{1+y^\nu} \le \frac{C}{(1+\mu)^{\nu/2-1}},\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}&\int_0^\infty \frac{(q-\mu)^2 dq}{(1 + (q-\mu)(q^\sigma - \mu^\sigma))^2 q^{2\alpha}} \\ &\le C\Big[\frac{\mu^2}{(1+\mu^\nu)^2} \int_0^{\mu/2} \frac{dq}{q^{2\alpha}} + \int_{\mu/2}^{2\mu} \frac{|q-\mu|^{2-2\alpha} dq}{(1 + (q-\mu)^2 \mu^{\sigma-1})^2} + \int_{2\mu}^\infty \frac{(q-\mu)^{2-2\alpha} dq}{(1 + (q-\mu)^\nu)^2}\Big] \\ &\le C(1+\mu)^{2+2\gamma-2\nu} + \frac{C}{\mu^{(1+\gamma)(\nu-2)}} \int_{-\mu^{\nu/2}}^{\mu^{\nu/2}} \frac{|z|^{2-2\alpha} dz}{(1+z^2)^2} + C \int_\mu^\infty \frac{y^{2-2\alpha} dy}{1 + y^{2\nu}} \\ &\le \frac{C}{(1+\mu)^{\nu-2}}.\end{aligned} \tag{2.14}$$

Since $\sup_{q,\mu \ge 0, q \ne \mu} \frac{\mu(q^{\sigma-1} - \mu^{\sigma-1})}{q^\sigma - \mu^\sigma} < \infty$ in view of (2.13), (2.14) and by the

Cauchy inequality, we obtain from (2.12)

$$|\mathcal{R}(t,x)| \le$$
$$\frac{C}{\sqrt[\nu]{t}}\int_0^\infty \left(|\hat{v}(t,\frac{q}{\sqrt[\nu]{t}}) - \hat{v}(t,\chi)| + \frac{|q-\mu|}{\sqrt[\nu]{t}}|\hat{v}_p(t,\frac{q}{\sqrt[\nu]{t}})|\right)\frac{dq}{|1+i(q-\mu)(q^\sigma-\mu^\sigma)|}$$
$$\le \frac{C}{t^{\frac{\gamma+1}{\nu}}}\|D^\alpha Ju(t)\|\Bigg(\int_0^\infty \frac{|q-\mu|^\gamma dq}{1+(q-\mu)(q^\sigma-\mu^\sigma)}$$
$$+ (\int_0^\infty \frac{(q-\mu)^2 dq}{(1+(q-\mu)(q^\sigma-\mu^\sigma))^2 q^{2\alpha}})^{1/2}\Bigg) \le \frac{C\|D^\alpha Ju(t)\|}{t^{\frac{\gamma+1}{\nu}}(1+\mu)^{\nu/2-1}} \quad (2.15)$$

Thus, by virtue of estimates (2.10) and (2.15) we obtain the asymptotic formula (2.3). Using the estimate $|\mathrm{A}(\eta)| \le C(1+|\eta|)^{-\frac{\nu-2}{2\nu-2}}$ we get from (2.4), (2.7), (2.10) and (2.15)

$$|u(t,x)| \le C(1+t)^{-1/\nu}(1+|\eta|)^{-\frac{\nu-2}{2\nu-2}}\mathcal{M}u(t), \quad (2.16)$$

since, as in (2.7) we get

$$\|\hat{v}\|_\infty \le C\||p|^\alpha \hat{v}_p(t,\cdot)\| + C\|p\hat{v}_p(t,\cdot)\| \le C\mathcal{M}u.$$

The first estimate (2.1) with $\beta = \infty$ follows from (2.16) immediately. Making a change of variable $\eta = \frac{x}{\sqrt[\nu]{t}}$ and using inequality (2.16) we get estimate (2.1) for all $\beta \in (\frac{2\nu-2}{\nu-2}, \infty)$ in the following manner

$$\|u(t,x)\|_{L^\beta_x} \le C(1+t)^{-1/\nu}\mathcal{M}u(t)\|(1+\frac{|x|}{\sqrt[\nu]{t}})^{-\frac{\beta(\nu-2)}{2\nu-2}}\|^{1/\beta}_{L^1_x}$$
$$\le C(1+t)^{-1/\nu+1/(\nu\beta)}\mathcal{M}u(t)\|(1+|\eta|)^{-\frac{\beta(\nu-2)}{2\nu-2}}\|^{1/\beta}_{L^1_\eta}$$
$$\le C(1+t)^{-1/\nu+1/(\nu\beta)}\mathcal{M}u(t).$$

As in (2.4) we have for the derivative u_x

$$u_x(t,x) = \frac{1}{\pi\sqrt[\nu]{t^2}}\mathrm{Re}\int_0^\infty e^{iq\eta+iq^\nu/\nu}\hat{v}(t,\frac{q}{\sqrt[\nu]{t}})iqdq.$$

In the domain $x \ge 0$ using the identity (2.5) we obtain analogously to (2.10)

$$|u_x(t,x)| \le Ct^{-2/\nu}\|\hat{v}\|_\infty\int_0^\infty \frac{qdq}{|1+iq(q^\sigma+\mu^\sigma)|} + Ct^{-3/\nu}\int_0^\infty \frac{q^2|\hat{v}_p(t,\frac{q}{\sqrt[\nu]{t}})|dq}{|1+iq(q^\sigma+\mu^\sigma)|}$$
$$\le \frac{C}{\sqrt[\nu]{t^2}}(\|\hat{v}\|_\infty + \||p|^\alpha\hat{v}_p(t,p)\| + \|p\hat{v}_p(t,p)\|)$$
$$\le \frac{C}{\sqrt[\nu]{t^2}}(\|\partial Ju(t)\| + \|D^\alpha Ju(t)\|) \quad (2.17)$$

for all $t > 0$. And in the domain $x \leq 0$ using the identity (2.11) we get analogously to inequality (2.15), for $t \geq 1$

$$\begin{aligned}
|u_x(t,x)| &\leq \frac{C}{\sqrt[\nu]{t^2}} \int_0^\infty (\|\hat{v}\|_\infty + \frac{|q-\mu|}{\sqrt[\nu]{t}} |\hat{v}_p(t, \frac{q}{\sqrt[\nu]{t}})| \frac{qdq}{|1 + i(q-\mu)(q^\sigma - \mu^\sigma)|} \\
&\leq \frac{C\|\hat{v}\|_\infty}{\sqrt[\nu]{t^2}} \int_0^\infty \frac{qdq}{1 + (q-\mu)(q^\sigma - \mu^\sigma)} \\
&+ \frac{C}{\sqrt[\nu]{t^2}} \||p|^\alpha \hat{v}_p(t,p)\| (\int_0^\infty \frac{q^{2-2\alpha}(q-\mu)^2 dq}{(1 + (q-\mu)(q^\sigma - \mu^\sigma))^2})^{1/2} \\
&\leq \frac{C}{\sqrt[\nu]{t^2}} (\|\partial Ju(t)\| + \|D^\alpha Ju(t)\|)(1 + |\eta|)^{\frac{4-\nu}{2\nu-2}}, \qquad (2.18)
\end{aligned}$$

since making a change of variables $z = (q-\mu)\mu^{\frac{\nu}{2}-1}, y = q - \mu$ we have analogously to (2.13), (2.14)

$$\begin{aligned}
&\int_0^\infty \frac{qdq}{1 + (q-\mu)(q^\sigma - \mu^\sigma)} \leq C\mu \int_0^{2\mu} \frac{dq}{1 + (q-\mu)^2 \mu^{\sigma-1}} + C \int_{2\mu}^\infty \frac{(q-\mu)dq}{1 + (q-\mu)^\nu} \\
&\quad \leq C\mu^{2-\frac{\nu}{2}} \int_{-\mu^{\nu/2}}^{\mu^{\nu/2}} \frac{dz}{1+z^2} + C \int_0^\infty \frac{ydy}{1+y^\nu} \leq C(1+\mu)^{2-\frac{\nu}{2}},
\end{aligned}$$

and since $q^\sigma - \mu^\sigma \geq C(q-\mu)q^{\sigma-1}$

$$\begin{aligned}
\int_0^\infty & \frac{q^{2-2\alpha}(q-\mu)^2 dq}{(1 + (q-\mu)(q^\sigma - \mu^\sigma))^2} \leq C \int_0^\infty \frac{q^{1-2\alpha} q(q-\mu)^2 dq}{(1 + (q-\mu)(q^\sigma - \mu^\sigma))^2} \\
&\leq C \int_0^\infty \frac{q^{2\gamma} dq}{1 + (q-\mu)(q^\sigma - \mu^\sigma)} \\
&\leq \int_0^{2\mu} \frac{C\mu^{2\gamma} dq}{1 + (q-\mu)^2 \mu^{\sigma-1}} + C \int_{2\mu}^\infty \frac{(q-\mu)^{2\gamma} dq}{1 + (q-\mu)^\nu} \\
&\leq \frac{C}{\mu^{\frac{\nu}{2}-1-2\gamma}} \int_{-\mu^{\nu/2}}^{\mu^{\nu/2}} \frac{dz}{1+z^2} + C \int_0^\infty \frac{ydy}{1+y^\nu} \leq C.
\end{aligned}$$

Now estimate (2.2) with $t \geq 1$ follows from (2.16), (2.17) and (2.18). And for the case $0 < t < 1$, $x \leq 0$ we get

$$
\begin{aligned}
|u_x(t,x)| &\leq \frac{C}{\sqrt[\nu]{t}} \int_0^\infty (\hat{v}(t, \frac{q}{\sqrt[\nu]{t}}) + \frac{|q-\mu|}{\sqrt[\nu]{t}} |\hat{v}_p(t, \frac{q}{\sqrt[\nu]{t}})| \frac{\frac{q}{\sqrt[\nu]{t}} dq}{|1 + i(q-\mu)(q^\sigma - \mu^\sigma)|} \\
&\leq \frac{C}{\sqrt[2\nu]{t}} (\int_0^\infty |\hat{v}(t,p)|^2 p^2 dp)^{1/2} (\int_0^\infty \frac{dq}{(1 + (q-\mu)(q^\sigma - \mu^\sigma))^2})^{1/2} \\
&+ \frac{C}{\sqrt[\nu]{t^2}} \| \sqrt{|p|} \hat{v}_p(t,p) \| (\int_0^\infty \frac{q(q-\mu)^2 dq}{(1 + (q-\mu)(q^\sigma - \mu^\sigma))^2})^{1/2} \\
&\leq \frac{C}{\sqrt[\nu]{t^2}} (\| \partial u(t) \| + \| D^\alpha J u(t) \|),
\end{aligned}
$$

whence and from (2.17) we obtain (2.2) for $t \leq 1$. Lemma 2.2 is proved.

Lemma 2.3. *The following estimates*

$$
\| D^\alpha |u|^{\rho-1} u \|^2 \leq C \| |u|^{\rho-1} \|^2 (\| u u_x \|_\infty + \| u \|_\infty^{6\gamma} \| u u_x \|_\infty^{1-3\gamma}), \tag{2.19}
$$

$$
\begin{aligned}
&|(D^\alpha h, D^\alpha |u|^{\rho-1} h_x)| \leq C \| D^\alpha h \| (\| D^\alpha h \| + \| \partial h \|)(\| u \|_\infty^{\rho-3} \| u u_x \|_\infty \\
&+ \| u \|_\infty^{\rho-3-2\gamma} \| u \|^{2\gamma} \| u u_x \|_\infty + \| u \|_\infty^{\rho-3+2\gamma} \| u u_x \|_\infty^{1-\gamma})
\end{aligned} \tag{2.20}
$$

are valid if the right-hand sides of the inequalities (2.19) *and* (2.20) *are bounded. Here* $\alpha = 1/2 - \gamma$, $\gamma \in (0, \min(\frac{1}{2}, \frac{\rho-\nu}{\nu}))$, $\rho > \nu > 3$.

Remark 2.1. Lemma 2.3 implies that we need higher order α of the derivative (closer to $\frac{1}{2}$) if the power ρ of the nonlinearity is near the critical value ν.

Proof. It is easy to get the following estimates

$$
|f - f_{(z)}| \leq \rho |z| \| u u_x \|_\infty (|u|^{\rho-2} + |u_{(z)}|^{\rho-2})
$$

and

$$
|f - f_{(z)}| \leq (|u|^\rho + |u_{(z)}|^\rho)
$$

with $f(u) = |u|^{\rho-1} u$, or $f(u) = u^\rho$ if $\rho > \nu > 3$ is an integer (below we consider only the first case, the second is the same), whence we get for any $\delta \in [0, 2]$

$$
|f(u) - f(u_{(z)})|^2 \leq C |z|^\delta \| u u_x \|_\infty^\delta (|u|^{2(\rho-\delta)} + |u_{(z)}|^{2(\rho-\delta)}), \tag{2.21}
$$

where $u_{(z)}(x) = u(x+z)$, $z \geq 0$. Integrating with respect to x, we get

$$\|f(u) - f(u_{(z)})\|^2 \leq Cz^{\delta}\||u|^{\rho-\delta}\|^2\|uu_x\|_\infty^{\delta} \tag{2.22}$$

for all $z \geq 0$. Applying (2.22) with $\delta = 1$ and $\delta = 1 - 3\gamma$, we obtain

$$\begin{aligned}
\|D^\alpha f(u)\|^2 &= C\|\int_0^\infty (f(x+z) - f(x))\frac{dz}{z^{\alpha+1}}\|_{L_x^2}^2 \\
&\leq C(\int_0^\infty \|(f_{(z)} - f)\|\frac{dz}{z^{\alpha+1}})^2 \leq C\||u|^{\rho-1}\|^2\|uu_x\|_\infty(\int_0^1 \frac{dz}{z^{1-\gamma}})^2 \\
&+ C\||u|^{\rho-1+3\gamma}\|^2\|uu_x\|_\infty^{1-3\gamma}(\int_1^\infty \frac{dz}{z^{1+\gamma/2}})^2) \\
&\leq C\||u|^{\rho-1}\|^2(\|uu_x\|_\infty + \|u\|^{6\gamma}\|uu_x\|_\infty^{1-3\gamma}).
\end{aligned}$$

Thus, the first estimate (2.19) is proved.

We have the identity

$$\begin{aligned}
&(D^\alpha h, D^\alpha |u|^{\rho-1}h_x) \\
&= (D^\alpha h, |u|^{\rho-1}D^\alpha h_x) + (D^\alpha h,(D^\alpha |u|^{\rho-1}h_x - |u|^{\rho-1}D^\alpha h_x)) = Q_1 + Q_2.
\end{aligned} \tag{2.23}$$

Integration by parts yields

$$|Q_1| = | - \tfrac{1}{2}(D^\alpha h, (|u|^{\rho-1})_x D^\alpha h)| \leq C\|D^\alpha h\|^2\|u\|_\infty^{\rho-3}\|uu_x\|_\infty. \tag{2.24}$$

Again, integrating by parts, we obtain the identity

$$\begin{aligned}
&D^\alpha |u|^{\rho-1}h_x - |u|^{\rho-1}D^\alpha h_x \\
&\qquad = C\int_0^\infty (|u(x+z)|^{\rho-1} - |u(x)|^{\rho-1})h_z(x+z)\frac{dz}{z^{1+\alpha}} \\
&\qquad = C\int_0^\infty (|u(x+z)|^{\rho-1} - |u(x)|^{\rho-1})(h(x+z) - h(x))_z\frac{dz}{z^{1+\alpha}} \\
&\qquad = C\int_0^\infty (|u(x+z)|^{\rho-1} - |u(x)|^{\rho-1})(h(x+z) - h(x))\frac{dz}{z^{2+\alpha}} \\
&\qquad + C\int_0^\infty (|u|^{\rho-3}uu_x)(x+z)(h(x+z) - h(x))\frac{dz}{z^{1+\alpha}},
\end{aligned}$$

whence using the Cauchy inequality we get

$$
\begin{aligned}
|Q_2| &\le C\|D^\alpha h\|(\|\int_0^\infty (|u|^{\rho-3}uu_x)_{(z)}(h_{(z)}-h)\frac{dz}{z^{1+\alpha}}\| \\
&\quad + \|\int_0^\infty (|u_{(z)}|^{\rho-1}-|u|^{\rho-1})(h_{(z)}-h)\frac{dz}{z^{2+\alpha}}\|) \\
&\le C\|D^\alpha h\|\|\int_0^\infty \frac{(h_{(z)}-h)^2dz}{z^{1+\gamma+2\alpha}}\|_{L^1}^{\frac{1}{2}}(\|\int_0^\infty (|u|^{2\rho-6}(uu_x)^2)_{(z)}\frac{dz}{z^{1-\gamma}}\|_\infty^{\frac{1}{2}} \\
&\quad + \|\int_0^\infty ((|u_{(z)}|^{\rho-1})-|u|^{\rho-1})^2\frac{dz}{z^{3-\gamma}}\|_\infty^{\frac{1}{2}}).
\end{aligned} \tag{2.25}
$$

By the Hölder inequality we obtain

$$
\begin{aligned}
&\|\int_0^\infty (|u|^{2\rho-6}(uu_x)^2)_{(z)}\frac{dz}{z^{1-\gamma}}\|_\infty = \|\int_0^1 + \int_1^\infty\|_\infty \\
&\le \|u\|_\infty^{2\rho-6}\|uu_x\|_\infty^2\int_0^1\frac{dz}{z^{1-\gamma}} + \||u|^{\frac{2\rho-6}{4\gamma}}\|^{4\gamma}\|uu_x\|_\infty^2(\int_1^\infty \frac{dz}{z^{\frac{1-\gamma}{1-2\gamma}}})^{1-2\gamma} \\
&\le C\|u\|_\infty^{2\rho-6-4\gamma}\|uu_x\|_\infty^2(\|u\|_\infty^{4\gamma}+\|u\|^{4\gamma}).
\end{aligned} \tag{2.26}
$$

Analogously to (2.21) and (2.22) we get the following inequalities

$$
\|(|u_{(z)}|^{\rho-1})-|u|^{\rho-1}\|_\infty \le C|z|\|u\|_\infty^{\rho-3}\|uu_x\|_\infty
$$

if $|z|\le 1$ and

$$
\|(|u_{(z)}|^{\rho-1})-|u|^{\rho-1}\|_\infty \le C|z|^{1-\gamma}\|u\|_\infty^{\rho-3+2\gamma}\|uu_x\|_\infty^{1-\gamma}
$$

if $|z|>1$. Then we obtain

$$
\begin{aligned}
&\|\int_0^\infty ((|u_{(z)}|^{\rho-1})-|u|^{\rho-1})^2\frac{dz}{z^{3-\gamma}}\|_\infty = \|\int_0^1+\int_1^\infty\|_\infty \\
&\qquad \le C\|u\|_\infty^{2\rho-6}\|uu_x\|_\infty^{2-2\gamma}(\|uu_x\|_\infty^{2\gamma}+\|u\|_\infty^{4\gamma}).
\end{aligned} \tag{2.27}
$$

Substitution of (2.26) and (2.27) to (2.25) yields

$$
\begin{aligned}
|Q_2| &\le C\|D^\alpha h\|(\|D^\alpha h\|+\|\partial h\|)(\|u\|_\infty^{\rho-3}\|uu_x\|_\infty \\
&\quad + \|u\|_\infty^{\rho-3-2\gamma}\|u\|^{2\gamma}\|uu_x\|_\infty + \|u\|_\infty^{\rho-3+2\gamma}\|uu_x\|_\infty^{1-\gamma})
\end{aligned} \tag{2.28}
$$

since by interpolation we have

$$\|\int_0^\infty \frac{(h_{(z)}-h)^2 dz}{z^{1+\gamma+2\alpha}}\|_{L^1}^{\frac{1}{2}} \le (\int_0^\infty \frac{\|h_{(z)}-h\|^2 dz}{z^{1+\gamma+2\alpha}})^{1/2} \le \|h\|_{\dot{B}_{2,2}^{\alpha+\gamma/2}}$$
$$\le C\|h\|_{\dot{H}^{\alpha+\gamma/2}} = C\|D^{\alpha+\gamma/2}h\| \le C(\|D^\alpha h\| + \|\partial h\|).$$

Estimates (2.24) and (2.28) with the identity (2.23) gives us the second estimate (2.20) of the lemma. Lemma 2.3 is proved.

3. Proofs of Theorems 1.1–1.2. To clarify the idea of the proof of the Theorems we only show a priori estimates of local solutions to equation (1.1). For that purpose we assume that the following local existence theorem holds.

Theorem 3.1. *We assume that $\|u_0\|_{1,1} = \epsilon \le \epsilon'$ and ϵ' is sufficiently small. Then there exists a finite time interval $[0,T]$ with $T > 1$ and a unique solution u of (1.1) with $\rho > \nu$ such that $||u||_{X_T} \le C\epsilon'$.*

For the proof of Theorem 3.1, see, e.g., [1, 7, 9, 10, 13 - 18, 30]. Now we prove the following lemma.

Lemma 3.1. *Let u be the local solutions of the Cauchy problem (1.1) with $\rho > \nu$ stated in Theorem 3.1. Then we have for any $t \in [0,T]$ $\mathcal{M}u(t) \le C\epsilon$, where $\mathcal{M}u(t) = \|u(t)\|_{1,0} + \|D^\alpha Ju(t)\| + \|\partial Ju(t)\|$ and the constant C does not depend on the time T of existence of solutions.*

Proof. Denote $Lu = \partial_t u + \mathcal{K}u$, then we can write down equation (1.1) as $Lu = -\partial_x f(u)$. First of all, we note that the two conservation laws $\|u\| = \|u_0\|$ and $|\hat{u}(t,0)| = |\hat{u}_0(0)|$ take place. Then we differentiate the equation (1.1) with respect to x to get $Lu_x = -\partial_x^2 f(u)$. Multiplying both sides of this equation by u_x and integrating by parts we obtain

$$\frac{d}{dt}\|u_x\|^2 \le C\|u\|_\infty^{\rho-3}\|uu_x\|_\infty\|u_x\|^2. \tag{3.1}$$

Using estimates (2.1) - (2.3) of Lemma 2.2 and Theorem 3.1 we get

$$\|u(t)\|_\beta \le \frac{C\epsilon'}{(1+t)^{\frac{1}{\nu}-\frac{1}{\nu\beta}}}, \quad \|uu_x(t)\|_\infty \le \frac{C\epsilon'}{t^{\frac{2}{\nu}}(1+t)^{\frac{1}{\nu}}}, \tag{3.2}$$

where $\beta \in (\frac{2\nu-2}{\nu-2}, \infty]$. Therefore, from (3.1), (3.2) and the Gronwall inequality, we obtain $\|u_x\|^2 \le C\|u_{0x}\|^2 \le C\epsilon^2$. Thus, we get

$$\|u\|_{1,0} \le C\epsilon. \tag{3.3}$$

Let us denote the dilation operator $I\phi = x\phi - \nu t \int_{-\infty}^{x} \partial_t \phi dx'$ and the operator $J\phi = x\phi + \nu t \int_{-\infty}^{x} \mathcal{K}\phi dx'$. Their difference is equal to

$$I\phi - J\phi = -\nu t \int_{-\infty}^{x} L\phi dx' \tag{3.4}$$

Also we note that the following commutator representations

$$[L, J]\phi = 0, \quad [L, I]\phi = -\nu \int_{-\infty}^{x} L\phi dx', \quad [J, \partial_x]\phi = [I, \partial_x]\phi = -\phi \tag{3.5}$$

are valid. Using (3.5) we get

$$\begin{aligned} I(|u|^{\rho-1}u)_x &= x\partial_x(|u|^{\rho-1}u) - \nu t \partial_t(|u|^{\rho-1}u) \\ &= \rho|u|^{\rho-1}(x\partial_x u - \nu t \partial_t u) = \rho|u|^{\rho-1} I u_x. \end{aligned} \tag{3.6}$$

Therefore, applying the operator I to the equation (1.1) we find

$$\begin{aligned} LIu = ILu - \nu \int_{-\infty}^{x} Lu dx' = -I(|u|^{\rho-1}u)_x + \nu|u|^{\rho-1}u \\ = -\rho|u|^{\rho-1} I u_x + \nu|u|^{\rho-1}u = -\rho|u|^{\rho-1}(Iu)_x + (\nu+\rho)|u|^{\rho-1}u. \end{aligned} \tag{3.7}$$

Applying the operator D^α to (3.7), multiplying the result by $D^\alpha Iu$ and using Lemma 2.3 with $h = Iu$ we get

$$\begin{aligned} \frac{d}{dt}\|D^\alpha Iu\|^2 &= -2(D^\alpha Iu, \rho D^\alpha |u|^{\rho-1}(Iu)_x - (3+\rho)D^\alpha(|u|^{\rho-1}u)) \\ &\le C\|D^\alpha Iu\|\{(\|D^\alpha Iu\| + \|\partial Iu\|)(\|u\|_\infty^{\rho-3}\|uu_x\|_\infty \\ &+ \|u\|_\infty^{\rho-3-2\gamma}\|u\|^{2\gamma}\|uu_x\|_\infty + \|u\|_\infty^{\rho-3+2\gamma}\|uu_x\|_\infty^{1-\gamma}) \\ &+ \||u|^{\rho-1}\|(\|uu_x\|_\infty^{1/2} + \|u\|_\infty^{3\gamma}\|uu_x\|_\infty^{\frac{1-3\gamma}{2}})\}. \end{aligned} \tag{3.8}$$

Analogously to (3.7), using (3.5) and (3.6) we have

$$\begin{aligned} LIu_x = ILu_x - \nu Lu = -I(|u|^{\rho-1}u)_{xx} + \nu(|u|^{\rho-1}u)_x \\ = -(I(|u|^{\rho-1}u)_x)_x + (\nu+1)(|u|^{\rho-1}u)_x \\ = -\rho(|u|^{\rho-1}(Iu_x)_x + (\rho-1)|u|^{\rho-3}uu_x Iu_x - (\nu+1)|u|^{\rho-1}u_x). \end{aligned} \tag{3.9}$$

Multiplying (3.9) by Iu_x and integrating by parts, we obtain

$$\frac{d}{dt}\|Iu_x\|^2 \leq C\|u\|_\infty^{\rho-3}\|uu_x\|_\infty(\|Iu_x\|+\|u\|)\|Iu_x\|. \tag{3.10}$$

Applying (3.2), (3.3) and the Gronwall inequality to (3.8) and (3.10) we obtain the estimate $\|D^\alpha Iu\| + \|Iu_x\| \leq C\|u_0\|_{1,1} \leq C\epsilon$, whence by (3.2), (3.3), the identity (3.4) and Lemma 2.3 we get

$$\begin{aligned}\|D^\alpha Ju\| + \|\partial Ju\| &\leq \|D^\alpha Iu\| + \|Iu_x\| + \|u\| \\ &\quad + \nu t(\|D^\alpha|u|^{\rho-1}u\| + \rho\||u|^{\rho-1}u_x\|) \leq C\epsilon.\end{aligned} \tag{3.11}$$

The lemma follows from (3.3) and (3.11). □

We are now in a position to prove Theorems 1.1-1.2.

Proof of Theorem 1.1. We have by Lemma 3.1 $||u||_{X_T} \leq C\epsilon$ for $t \in [0,T]$. We take ϵ satisfying $C\epsilon \leq \epsilon'$. Then a standard continuation argument yields the result because our constant C does not depend on the time T. □

Proof of Theorem 1.2. We have by (1.1) $(U(-t)u)_t = -U(-t)\partial_x f(u)$, whence via (3.2), (3.3) we get

$$\begin{aligned}\|U(-t)u(t) - U(-s)u(s)\| &\leq C\int_s^t \|u(\tau)\|_\infty^{\rho-3}\|uu_x(\tau)\|_\infty\|u(\tau)\|d\tau \\ &\leq C\epsilon\int_s^t \tau^{-\rho/\nu}d\tau \leq C\epsilon s^{-(\rho-\nu)/\nu}\end{aligned} \tag{3.12}$$

for $1 < s < t$. And by virtue of Lemma 2.1 with $p = \infty, j = 0, m = 1, r = 1+\gamma, q = 2, a = \frac{1+\gamma}{1+3\gamma}$ and by the Hölder inequality, we get

$$\begin{aligned}&\|\mathcal{F}U(-t)u(t) - \mathcal{F}U(-s)u(s)\|_\infty \leq C(\|\partial\mathcal{F}U(-t)u(t)\|_r^a \\ &+ \|\partial\mathcal{F}U(-s)u(s)\|_r^a)\|\mathcal{F}U(-t)u(t) - \mathcal{F}U(-s)u(s)\|^{1-a} \\ &\leq C(\|\mathcal{F}U(-t)Ju(t)\|_r^a + \|\mathcal{F}U(-s)Ju(s))\|_r^a)s^{-\frac{2\gamma(\rho-\nu)}{\nu(1+3\gamma)}}\epsilon^{1-a} \\ &\leq Cs^{-\frac{2\gamma(\rho-\nu)}{\nu(1+3\gamma)}}((\mathcal{M}u(t))^a + (\mathcal{M}u(s))^a)(\int_0^\infty (p^\alpha+p)^{-\frac{2+2\gamma}{1-\gamma}}dp)^{\frac{1-\gamma}{2+2\gamma}}\epsilon^{1-a} \\ &\leq C\epsilon s^{-\frac{2\gamma(\rho-\nu)}{\nu(1+3\gamma)}}.\end{aligned} \tag{3.13}$$

By (3.12), (3.13) we find that there exists a unique function $V \in L^\infty \cap L^2$ such that $\lim_{t\to\infty}(\|\mathcal{F}U(-t)u(t) - V\|_\infty + \|\mathcal{F}U(-t)u(t) - V\|) = 0$. And

(3.12), (3.13) imply (1.2). The asymptotic formula (1.3) follows from the identity (2.4), (1.2) and the estimate (3.11). Using the conservation law $\hat{u}(t,0) = \hat{u}_0(0)$ we easily get the equality $V(0) = \hat{u}_0(0)$. This completes the proof of Theorem 1.2. $\square$

Acknowledgments. One of the authors (P.I.N.) wishes to express his deep gratitude to Consejo Nacional de Ciencia y Tecnologia de Mexico (Conacyt) for the support. He is also grateful to Instituto de Física y Matemáticas de Universidad Michoacana for the kind hospitality.

REFERENCES

[1] L. Abdelouhab, J. L. Bona, M. Felland and J. - C. Saut, *Nonlocal models for nonlinear, dispersive waves*, Physica D **40** (1989), 360-392.
[2] M.J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, SIAM, Philadelphia, 1981.
[3] J.Bergh and J. Löfström, *Interpolation spaces*, New York, Springer, 1976.
[4] J.L. Bona and J.-C. Saut, *Dispersive blow-up of solutions of generalized Korteweg-de Vries equation*, J. Diff. Eq. **103** (1993), 3-57.
[5] A. de Bouard, N. Hayashi and K. Kato, *Gevrey regularizing effect for the (generalized) Korteweg - de Vries equation and nonlinear Schrödinger equations*, Ann. Inst. Henri Poincaré, Analyse non linéaire **12** (1995), 673-725.
[6] P. Constantin and J.-C. Saut, *Local smoothing properties of dispersive equations*, J. Amer. Math. Soc. **1** (1988), 413-446.
[7] W. Craig, K.Kapeller and W.A. Strauss, *Gain of regularity for solutions of KdV type*, Ann. Inst. Henri Poincaré, Analyse non linéaire **9** (1992), 147-186.
[8] A.Friedman, *Partial Differential Equations*, Holt-Rinehart and Winston, 1969.
[9] J. Ginibre and G. Velo, *Smoothing properties and retarded estimates for some dispersive evolution equations*, Comm. Math. Phys. **144** (1992), 163-188.
[10] N. Hayashi, *Analyticity of solutions of the Korteweg-de Vries equation*, SIAM J. Math. Anal. **22** (1991), 1738-1745.
[11] N.Hayashi and P.I.Naumkin, *Large time asymptotics of solutions to the generalized Benjamin-Ono equation*, Trans. Amer. Math. Soc., to appear.
[12] N.Hayashi and P.I.Naumkin, *Large time asymptotics of solutions to the generalized Korteweg-de Vries equation*, J. Funct. Anal., to appear.
[13] T. Kato, *On the Cauchy problem for the (generalized) Korteweg-de Vries equation*, Studies in Applied Mathematics (V.Guillemin, eds.), Advances in Mathematics Supplementary Studies, vol. 8, Berlin, 1983, pp. 93-128.
[14] C.E.Kenig, G.Ponce and L.Vega, *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. Math. J. **40** (1991), 33-69.
[15] C.E.Kenig, G.Ponce and L.Vega, *Higher - order nonlinear dispersive equations*, Proc. AMS **122** (1994), 157 - 166.
[16] S. Klainerman, *Long time behavior of solutions to nonlinear evolution equations*, Arch. Rat. Mech. Anal. **78** (1982), 73-89.
[17] S. Klainerman and G. Ponce, *Global small amplitude solutions to nonlinear evolution equations*, Comm. Pure Appl. Math. **36** (1983), 133-141.

[18] S.N. Kruzhkov, A.V. Faminskii, *Generalized solutions of the Cauchy problem for the Korteweg - de Vries equation*, Math. USSR, Sbornik **48** (1984), 391-421.

[19] J.L. Lions, *Quelques méthodes de résolution des problémas aux limites non linéaires*, Paris: Dunod and Gauthier Villars, 1969.

[20] T. Matos, *Exact solutions of G-invariant chiral equations*, Math. Notes **58** (1995), 710.

[21] P.I. Naumkin and I.A. Shishmarev, *Nonlinear Nonlocal equations in the Theory of Waves*, Translations of Monographs, vol. 133, A.M.S., Providence, R.I., 1994.

[22] G. Ponce, *Regularity of solutions to nonlinear dispersive equations*, J. Diff. Eq. **78** (1989), 122-135.

[23] M.A. Rammaha, *On the asymptotic behavior of solutions of generalized Korteweg - de Vries equation*, J. Math. Anal. Appl. **140** (1989), 228-240.

[24] J.-C. Saut, *Sur quelque généralisations de l'equation de Korteweg-de Vries*, J.Math. Pure Appl. **58** (1979), 21-61.

[25] A. Sidi, C. Sulem and P.L. Sulem, *On the long time behavior of a generalized KdV equation*, Acta Applicandae Math. **7** (1986), 35-47.

[26] J. Shatah, *Global existence of small solutions to nonlinear evolution equations*, J. Diff. Eq. **46** (1982), 409-425.

[27] E.M.Stein, *Singular Integral and Differentiability Properties of Functions*, vol. 30, Princeton Univ. Press, Princeton Math. Series, 1970.

[28] W.A. Strauss, *Dispersion of low-energy waves for two conservative equations*, Arch. Rat. Mech. Anal. **55** (1974), 86-92.

[29] W.A. Strauss, *Nonlinear scattering theory at low energy*, J. Funct. Anal. **41** (1981), 110-133.

[30] M. Tsutsumi, *On global solutions of the generalized Korteweg - de Vries equation*, Publ. Res. Inst. Math. Sci. **7** (1972), 329-344.