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Hydrodynamics of galactic dark matter

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Abstract
We consider simple hydrodynamical models of galactic dark matter in which the galactic halo is a self-gravitating and self-interacting gas that dominates the dynamics of the galaxy. Modelling this halo as a spherically symmetric and static perfect fluid satisfying the field equations of general relativity, visible baryonic matter can be treated as 'test particles' in the geometry of this field. We show that the assumption of an empirical 'universal rotation curve' that fits a wide variety of galaxies is compatible, under suitable approximations, with state variables characteristic of a non-relativistic Maxwell–Boltzmann gas that becomes an isothermal sphere in the Newtonian limit. Consistency criteria lead to a minimal bound for particle masses in the range 30 eV $< m <$ 60 eV and to a constraint between the central temperature and the particle mass. The allowed mass range includes popular supersymmetric particle candidates, such as the neutralino, axino and gravitino, as well as lighter particles ($m \approx$ keV) proposed by numerical n-body simulations associated with self-interactive CDM and WDM structure formation theories.

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1. Introduction

The presence of large amounts of dark matter at the galactic lengthscale is already an established fact. Assuming that this dark matter is a gas (or gas mixture) of various particle species, the established classification criteria label possible dark matter forms as 'cold' or 'hot', depending on the relativistic or non-relativistic nature of the particles' energetic spectrum at

Table 1. Particle candidates for a MB dark matter gas.

SCDM/WDM light candidates	Mass (keV)	References
Light gravitino	$\sim$0.5	[35]
	$\sim$0.75–1.5	[36, 37]
	$\sim$2.6–5	[38]
Sterile neutrino	$<$40	[39]
	1–100	[40]
Standard neutrinos	$\sim$1	[41, 42]
Dilaton	$\sim$0.5	[43]
Light axino	$\sim$100	[44]
Majoron	$\sim$1	[45–47]
Mirror neutrinos	$\sim$1	[48, 49]
CDM heavy candidates	**Mass (GeV)**	**References**
Neutralino	$>$32.3	[50]
	$>$46	[51]
Axino	$\sim$10	[52, 53]
Gravitino	$\lesssim$100	[54]

their decoupling from the cosmic mixture [1–3]. Hot dark matter (HDM) scenarios seem to be incompatible with current theories of structure formation and thus are not favoured dark matter candidates [2–4]. Cold dark matter (CDM), usually examined within a Newtonian framework, can be considered as non-interactive (a self-gravitating gas of collisionless particles) or self-interactive [5]. CDM models are often developed in terms of n-body numerical simulations [6–9]. Non-interactive CDM models present the following discrepancies with observations at the galactic scale [10, 11]: (a) the 'substructure problem' related to excess clustering on sub-galactic scales, (b) the 'cusp problem' characterized by a monotonic increase of density towards the centre of halos, leading to excessively concentrated cores. These problems appear in the more recent numerical simulations (see [7–9]). In order to deal with these problems, the possibility of self-interacting dark matter has been considered, so that nonzero pressure or thermal effects can emerge, thus leading to self-interactive models of CDM (i.e., SCDM) [12–16] and 'warm' dark matter (WDM) models [17–22] that challenge the duality CDM versus HDM. Other proposed dark matter sources consist of replacing the gas of particles approach by scalar fields [23, 24] and even more 'exotic' sources [25].

Whether based on SCDM or WDM, current theories of structure formation point towards dark matter characterized by particles having a mass of the order of at least keVs (see [12–22]), thus suggesting that massive but light particles, such as electron neutrinos and axions (see table 1), should be eliminated as primary dark matter candidates (though there is no reason to assume that these particles would be absent in galactic halos). Of all possible particle candidates (denoted as WIMPs (weakly interactive massive particles)) complying with the required mass value of relic gases, only the massive neutrinos (the muon or tau neutrinos), have been detected, whereas other WIMPs (gravitino, sterile neutrino, axino, etc) are speculative. See [26–29] and table 1 for a list of candidate particles and appropriate references.

Even if the dynamics of visible matter in galaxies can be described successfully with Newtonian gravity, we believe that general relativity is an appropriate framework for understanding basic features of galactic dark matter, a gravitational field source whose precise physical nature still remains an open question. If the results obtained with GR coincide with Newtonian results, then there is no harm done from a pragmatic calculations-oriented point of view. However, from a formal-theoretical approach, we believe it is beneficial to broaden the scope of the study of galactic dynamics by incorporating it into a more general gravitational theory. In particular, in this paper we aim at testing the compatibility between observed galactic rotation curves and simple thermodynamical assumptions under the framework of GR.

Since the dark matter halo probably constitutes an overwhelming majority (90%) of the galactic mass, an alternative approach to numerical simulations and Newtonian hydrodynamics follows by a general relativistic model describing the gravitational field of the galaxy as a spacetime geometry generated by the dark matter halo (as a self-gravitating gas), hence visible matter becomes test particles that evolve along stable geodesics of this spacetime. There is strong empiric evidence that the radial profile of rotational velocities ('rotation curves') in most galaxies roughly fits a 'universal rotation curve' (URC) [30, 31]. This URC is characterized by a 'flattening' effect whereby rotation velocities tend to a constant 'terminal' velocity whose value depends on the type of galaxy (between 125 km s^{-1} and 250 km s^{-1}). The profile of rotation curves identifies two main contributions of galactic matter: visible matter (the disc), showing a Keplerian decay, and dark matter (the halo), explaining the flattening effect. This kinematic evidence might allow us to determine (at least partially) the geometry of the spacetime associated with a self-gravitating galaxy. In other words, our approach somehow inverts the standard initial value procedure in general relativistic hydrodynamics: instead of prescribing initial data based on physically motivated sources and then finding the geometry of spacetime and the trajectories of test particles after solving Einstein's equations, we provide first constraints on the geometry of spacetime (from symmetry criteria and empirical kinematic data) and then find, with the help of the field equations, the corresponding momentum–energy tensor of the sources. This approach to galactic dark matter has been used in connection with scalar fields [23].

Bearing in mind that the dark matter halo overwhelmingly dominates the galactic matter content (at least in the halo region), we shall assume that the galactic halo (as a self-gravitating gas) is the unique matter source of the galactic spacetime. Visible matter then becomes test observers that follow stable circular geodesic orbits (the galactic rotation curves) of this spacetime. Following the 'inverse' approach described above, we propose to use the empiric law governing the form of the URC for the galactic halo (see [30, 31]) in order to make specific assertions on the nature of the sources of the galactic spacetime. Considering the self-gravitating galactic halo gas to be self-interactive (instead of collisionless matter or a scalar field), we aim at verifying if the assumption of the URC profile for the rotation velocity of geodesic observers (rotation curves) is compatible with the assumption that the galactic halo gas is a simple self-gravitating and self-interactive gas in thermodynamical equilibrium. For this purpose, we consider the galactic halo to be a spacetime characterized as: (a) spherically symmetric, (b) its energy–momentum tensor is that of a perfect fluid satisfying the equation of state of an equilibrium Maxwell–Boltzmann gas in its non-relativistic limit [32, 33]. Assumption (a) is supported by observations in the halo of galaxies, while (b) is the central hypothesis in the present work. Assumption (b) requires the spacetime to be stationary (static if rotation vanishes) and leads to a law relating temperature gradients with the 4-acceleration (Tolman law). Then, since we are assuming the validity of the empiric URC for the galactic halo, we need to cast the field equations and the conditions imposed by the thermodynamics in terms of this rotation velocity (i.e., the velocity of test particles in stable geodesic orbits), considered now as a dynamical variable. Using the URC empiric law as an ansatz for this velocity immediately leads to expressions for the state variables that are (under suitable approximations) consistent with the thermodynamics of the non-relativistic

Maxwell–Boltzmann ideal gas. From these expressions and bearing in mind numerical estimates of the empirical parameters appearing in the URC ('terminal' rotation velocities and the 'core radius'), we obtain: (1) a constraint on the ratio of the particle mass to temperature for this gas, (2) the criterion of applicability of the Maxwell–Boltzmann distribution (i.e., the non-degeneracy criterion) [34], leading to a minimal bound of about 30–60 eV for the mass of the gas particles. Therefore, the assumption of a Maxwell–Boltzmann gas (SCDM or WDM) model for the galactic halo leads to an acceptable value for the particle's mass lying in the range $m > 0.5\,\text{keV}$. We provide in table 1 a list of particle candidates that could be accommodated according to the criteria (1) and (2) above, namely: neutralino, photino, light gravitino, sterile neutrino, dilaton, axino, majoron, mirror neutrino and possibly standard massive neutrinos. As mentioned previously, this mass range is compatible with predictions of current work based on SCDM and WDM structure formation models. We find it interesting to remark that baryons and electrons comply with criterion (2) above, but criterion (1) would imply gas temperatures of the order of 10^3–10^6 K. A gas of baryons or electrons at such temperatures would certainly not be 'dark'. HDM or WDM models based on less massive particles, such as the electron neutrino, remain outside the scope of the present work, since these particles might require assuming a fully relativistic Maxwell–Boltzmann gas or a degenerate gas (possibly relativistic) complying with Fermi–Dirac or Bose–Einstein statistics. The axion, as well as other non-thermal relic sources, are also beyond the scope of this paper and their study requires a different approach.

The paper is organized as follows. In the following section, we present the field equations for a static spherically symmetric spacetime with a perfect fluid source. In section 3, we provide a review of the thermodynamics of an equilibrium Maxwell–Boltzmann gas. Then, in section 4, we rewrite the field equations in terms of the orbital velocity of stable circular geodesics and then assume for this velocity the empirical ansatz given by the URC. This leads to forms of the state variables that will be compatible, under suitable series approximations, with the thermodynamics of the Maxwell–Boltzmann gas. Putting all these results together, we discuss in the last section the possible ranges for the mass of the particles of the dark halo gas and suggest future lines of research.

2. Field equations

As mentioned above, we consider the halo to be spherically symmetric and the determining component of the stress–energy tensor determining the geometry. Thus, we consider the line element of a spherically symmetric spacetime:

$$\mathrm{d}s^2 = -B^2(r)c^2\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-2M(r)/r} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2). \tag{2.1}$$

Assuming as the source of such a line element a static perfect fluid momentum–energy tensor $T^{ab} = (\rho + p)u^a u^b + pg^{ab}$, with $u^a = B^{-1}\delta^a_{ct}$, we obtain the following field equations:

$$-G^t{}_t = \kappa\rho = \frac{2M'}{r^2}, \tag{2.2}$$

$$G^r{}_r = \kappa p = \frac{2}{r}\left(1 - \frac{2M}{r}\right)\frac{B'}{B} - \frac{2M}{r^3}, \tag{2.3}$$

$$G^r{}_r - G^\theta{}_\theta = 0 \quad\Rightarrow\quad \left(1-\frac{2M}{r}\right)\frac{B''}{B} - \frac{1}{r}\left(1 + M' - \frac{3M}{r}\right)\frac{B'}{B} + \frac{3M}{r^3} - \frac{M'}{r^2} = 0, \tag{2.4}$$

where $\kappa = 8\pi G/c^4$ and the prime denotes a derivative with respect to r. As mentioned in the introduction (see also [23, 24]), it is possible (by working with Einstein's equations backwards) to impose constraints on the geometry of the spacetime that yield valuable information regarding the type of matter sources curving such spacetime. We apply this reasoning to the observed velocity profile of stars orbiting around a galaxy, considered as test particles moving in stable circular geodesics. Knowing the specific form of the velocity profiles around these geodesics should allow us to infer at least the basic features of the nature of the sources producing the galactic field. For a spherically symmetric spacetime (2.1), the energy, E, and the angular momentum, L, are conserved quantities for any particle moving in a geodesic, hence the geodesic equation for the radial motion has the following form:

$$\dot{r}^2 - \frac{1}{B^2}\left(\frac{E^2}{B^2} - \frac{L^2}{r^2} - 1\right) = 0, \tag{2.5}$$

where a dot denotes a derivative with respect to the affine parameter of the geodesic. For a test particle to be in stable circular geodesic motion in any static spherically symmetric spacetime, its energy and angular momentum must satisfy

$$E^2 = \frac{B^3}{B - rB'}, \tag{2.6}$$

$$L^2 = \frac{r^3 B'}{B - rB'}. \tag{2.7}$$

The tangential velocity of these test particles, $V(r)$, can also be expressed in terms of the metric coefficients as

$$\frac{V^2}{c^2} \equiv v^2(r) = \frac{rB'}{B}. \tag{2.8}$$

In previous works (see [23–25]), this tangential velocity was assumed to be constant along the full domain of the solution. In the present paper, we consider $v = v(r)$ and eliminate the metric coefficient, $B(r)$, and its derivatives in (2.2)–(2.4) in terms of the tangential velocity, $v^2(r)$, as given by equation (2.8). This leads to a form for the field equations in which $v(r)$ becomes a dynamical variable replacing $B(r)$.

After some algebraic manipulation, equation (2.4) becomes the following constraint relating the metric function M and the tangential velocity:

$$M' + \frac{(-3 - 5v^2 + 4vv'r + 2v^4)M}{r(1+v^2)} - \frac{v(-2v + 2v'r + v^3)}{1+v^2} = 0. \tag{2.9}$$

Substitution of this last equation into equations (2.2) and (2.3) provides the following expressions for the density and pressure of the fluid in terms of M, v and v':

$$\kappa p = 2\frac{-M - 2Mv^2 + v^2 r}{r^3}, \tag{2.10}$$

$$\kappa\rho = \frac{[-8vv'r - 2(2v^2+1)(v^2-3)]M + 4r^2vv' + 2v^2(-2+v^2)r}{r^3(1+v^2)}. \tag{2.11}$$

It is remarkable to see how the replacement of B by v considerably simplifies the field equations for a general static and spherically symmetric field with a perfect fluid source. Writing the field equations in terms of the orbital velocity, v, provides an useful insight into how an (in principle) observable quantity relates to spacetime curvature and to physical quantities (state variables) which characterize the source of spacetime. Thus, given an empirical functional form for $v(r)$ (a rotation 'profile' for test particles), we can obtain $M(r)$ by integrating the constraint (2.9) and thus arrive at the fully determined forms of ρ and p in (2.11).

3. Thermodynamics

We aim at verifying if the empirical laws associated with galactic rotation curves (hence, associated with $v(r)$) can be compatible with matter sources that satisfy basic physical considerations and principles. Since galactic dark matter is, most probably, non-relativistic and the assumption of a perfect fluid source for the static metric (2.1) points to an equilibrium configuration, it is tempting to verify if ρ and p associated with a given empirical form for $v(r)$ correspond to state variables characteristic of simple, non-relativistic systems in thermodynamic equilibrium, such as a suitable ideal gas in its non-relativistic limit [32, 33].

If we assume that the self-gravitating ideal 'dark' gas exists in physical conditions far from those in which the quantum properties of the gas particles are relevant, we would be demanding that these particles comply with Maxwell–Boltzmann (MB) statistics. Following [34], the 'non-degeneracy' condition that justifies an MB distribution is given by

$$\frac{n\hbar^3}{(mk_{\mathrm{B}}T)^{3/2}} \ll 1, \tag{3.1}$$

where $n, T, \hbar$ and k_{B} are, respectively, the particle number density, absolute temperature, Planck's and Boltzmann's constants. If constraint (3.1) holds and we further assume thermodynamical equilibrium and non-relativistic conditions, the ideal dark gas must satisfy the equation of state of a non-relativistic monatomic ideal gas

$$\rho = mc^2 n + \tfrac{3}{2} n k_{\mathrm{B}} T, \qquad p = n k_{\mathrm{B}} T, \tag{3.2}$$

whose macroscopic state variables can be obtained from a MB distribution function under an equilibrium kinetic theory approach (the non-relativistic and non-degenerate limit of the Jüttner distribution) [32]. An equilibrium MB distribution restricts the geometry of spacetime [33], resulting in the existence of a timelike Killing vector field $\beta^a = \beta u^a$, where $\beta \equiv mc^2 \big/ k_{\mathrm{B}} T$, as well as the following relation (Tolman's law) between the 4-acceleration and the temperature gradient

$$\dot{u}_a + h_a^b (\ln T)_{,b} = 0, \qquad h_a^b = u_a u^b + \delta_a^b, \tag{3.3}$$

leading to

$$\frac{B'}{B} + \frac{T'}{T} = 0 \quad \Rightarrow \quad T \propto B^{-1}. \tag{3.4}$$

The particle number density n trivially satisfies the conservation law $J^a{}_{;a} = 0$, where $J^a = n u^a$; thus, the number of dark particles is conserved. Note that given (2.11), the equation of state (3.2) and the temperature from the Tolman law (3.4), we have two different expressions for n

$$n = \frac{p}{k_{\mathrm{B}} T} \propto pB, \tag{3.5}$$

$$n = \frac{1}{mc^2}\left[\rho - \frac{3}{2} p\right]. \tag{3.6}$$

The quantity $mc^2 n$ in (3.6) follows directly from equations (2.11)

$$\kappa mc^2 n = \frac{[-8vv'r + (v^2+9)(2v^2+1)]M + 4vv'r^2 - v^2(7+v^2)r}{r^3(1+v^2)}, \tag{3.7}$$

while n in (3.5) also follows from p in (2.11) with $B \propto \exp[\int (v^2/r)\,\mathrm{d}r]$. Consistency requires that (3.5) and (3.6) yield the same expression for n.

4. Dark fluid hydrodynamics

So far we have expressed the field equations and the thermodynamics of the fluid source in terms of the tangential velocities $v(r)$ and the effective mass–energy $M(r)$. If the dark matter component dominates the dynamics of the fluid, we can ignore the contribution from visible matter (baryons) and assume that the matter source is made exclusively of this dark matter component. An useful strategy to follow is then to prescribe, as a functional form for $v(r)$, the empirical ansatz of the radial profile of tangential velocities of the dark matter halo obtained from the 'universal rotation curve' (URC) that roughly fits observed galactic rotation curves [30, 31]. This empirical form can be used in equations (2.9) and (2.11). The function $M(r)$ follows by solving the constraint (2.9), subject to the appropriate boundary conditions. The obtained M together with the prescribed v yield fully determined forms for ρ, p and the metric coefficient $B(r)$. Next, we can verify the compatibility of n obtained from (3.5) and (3.6). Finally, we should be able to estimate the ratio $\beta = mc^2/k_B T$ which in turn, from estimations of masses from particle physics, leads to an estimation of the temperature of the dark gas in terms of the particles' mass.

We shall assume for v^2 the empiric dark halo rotation velocity law given by Persic and Salucci [30, 31]

$$v^2 = \frac{v_0^2 x^2}{a^2 + x^2}, \qquad x \equiv \frac{r}{r_{\rm opt}}, \tag{4.1}$$

where $r_{\rm opt}$ is the 'optical radius' containing 83% of the galactic luminosity, whereas the empirical parameters a and v_0, respectively, the ratio of 'halo core radius' to $r_{\rm opt}$ and the 'terminal' rotation velocity, depend on the galactic luminosity. For spiral galaxies we have: $v_0^2 = v_{\rm opt}^2(1-\beta_*)(1+a^2)$, where $v_{\rm opt} = v(r_{\rm opt})$, and the best fit to rotation curves is obtained for: $a = 1.5(L/L_*)^{1/5}$ and $\beta_* = 0.72 + 0.44\,\log_{10}(L/L_*)$, where $L_* = 10^{10.4}L_\odot$. The range of these parameters for spiral galaxies is $125\ \mathrm{km\ s^{-1}} < v_0 < 250\ \mathrm{km\ s^{-1}}$ and $0.6 < a < 2.3$.[4] The field equation (2.9) becomes the following linear first-order ODE:

$$\frac{\mathrm{d}M}{\mathrm{d}x} + \frac{\left(1+2v_0{}^2\right)\left(v_0{}^2-3\right)x^4 - a^2\left(v_0{}^2+6\right)x^2 - 3a^4}{x(b^2+x^2)\left(a^2+x^2+v_0{}^2x^2\right)}M + \frac{v_0{}^2x^4\left(2-v_0{}^2\right)r_{\rm opt}}{(a^2+x^2)\left(a^2+x^2+v_0{}^2x^2\right)} = 0, \tag{4.2}$$

the state variables (2.11) become

$$\frac{1}{2}\kappa\rho = \frac{\left[\left(1+2v_0{}^2\right)\left(3-v_0{}^2\right)x^4 + a^2\left(v_0{}^2+6\right)x^2 + 3a^4\right]M - v_0{}^2\left(2-v_0{}^2\right)x^5 r_{\rm opt}}{(a^2+x^2)\left[a^2+\left(1+v_0{}^2\right)x^2\right]x^3 r_{\rm opt}^3}, \tag{4.3}$$

$$\frac{1}{2}\kappa p = \frac{-\left[a^2+\left(1+2v_0{}^2\right)x^2\right]M + v_0{}^2x^3 r_{\rm opt}}{(a^2+x^2)x^3 r_{\rm opt}^3}, \tag{4.4}$$

while the metric coefficient B takes the form

$$B = [1+x^2]^{v_0^2/2} \quad\Rightarrow\quad T = T_c[1+x^2]^{-v_0^2/2}, \tag{4.5}$$

so that T is the temperature complying with the Tolman law and $T_c = T(0)$. Given a solution $M = M(x)$ of (4.2), all state variables get determined as functions of x and v_0. The solution of (4.2) is the following quadrature:

$$M = \frac{\left(v_0^2-2\right)(a^2+x^2)^{2-v_0^2}v_0^2x^3}{\left[a^2+\left(1+v_0^2\right)x^2\right]^{2/(1+v_0^2)}} r_{\rm opt}\int \frac{\left[a^2+\left(1+v_0^2\right)x^2\right]^{(1-v_0^2)/(1+v_0^2)}x\,\mathrm{d}x}{(a^2+x^2)^{3-v_0^2}}, \tag{4.6}$$

[4] The URC given by (4.1) fits also elliptic and irregular galaxies.

where we have set an integration constant to zero in order to comply with the consistency requirement that $v_0 = 0$ implies flat spacetime ($B = 1, M = 0$). Since the velocities of rotation curves are Newtonian, $v_0 \ll c$ (typical values are $v_0/c \approx 0.5 \times 10^{-3}$), instead of evaluating (4.6) we will expand this quadrature around v_0/c (in order to keep the notation simple, we write v_0 instead of v_0/c). This yields

$$M = \frac{x^3 r_{\rm opt}}{a^2+x^2} v_0^2 \left[1 - \frac{5x^2+2a^2}{2(a^2+x^2)} v_0^2 + \frac{12x^4+11a^2x^2+3a^4}{2(a^2+x^2)^2} v_0^4 + \mathcal{O}\big(v_0^6\big)\right], \tag{4.7}$$

we obtain the expanded forms of ρ and p by inserting (4.6) into (4.3) and (4.4) and then expanding around v_0, leading to

$$\kappa\rho r_{\rm opt}^2 = \frac{2(3a^2+x^2)}{(a^2+x^2)^2} v_0^2 - \frac{5x^4+23a^2x^2+6a^4}{(a^2+x^2)^3} v_0^4 + \mathcal{O}\left(v_0^6\right), \tag{4.8}$$

$$\kappa p r_{\rm opt}^2 = \frac{2a^2+x^2}{(a^2+x^2)^2} v_0^4 - \frac{2x^4+7a^2x^2+3a^4}{(a^2+x^2)^3} v_0^6 + \mathcal{O}\left(v_0^8\right), \tag{4.9}$$

while the expanded form for T follows from (4.5)

$$T = T_c\left[1 - \tfrac{1}{2}\ln(1+x^2)v_0^2 + \tfrac{1}{8}\ln^2(1+x^2)v_0^4 - \mathcal{O}\left(v_0^6\right)\right]. \tag{4.10}$$

In order to compare n obtained from (3.5) and (3.6), we substitute (4.1) and (4.6) into (3.7) and expand around v_0, leading to

$$n r_{\rm opt}^2 = \frac{1}{\kappa m c^2}\left[\frac{2(3a^2+x^2)}{(a^2+x^2)^2} v_0^2 - \frac{18x^4+55a^2x^2+13a^4}{2(a^2+x^2)^3} v_0^4 + \mathcal{O}\left(v_0^6\right)\right], \tag{4.11}$$

while n in (3.5) follows by substituting (4.6) into (4.9), using T from (4.5) and then expanding around v_0. This yields

$$n r_{\rm opt}^2 = \frac{1}{\kappa k_{\rm B} T_c}\left[\frac{2a^2+x^2}{(a^2+x^2)^2} v_0^4 \right. \\ \left. + \frac{(2a^2+x^2)(a^2+x^2)\ln(1+x^2) - 2(a^2+2x^2)(3a^2+x^2)}{2(a^2+x^2)^3} v_0^6 + \mathcal{O}\left(v_0^8\right)\right]. \tag{4.12}$$

Since $v_0/c \ll 1$, a reasonable approximation is obtained if the leading terms of n from (4.11) and (4.12) coincide. By looking at these equations, it is evident that this consistency requirement implies

$$\tfrac{1}{2} m v_0^2 = \tfrac{3}{2} k_{\rm B} T_c, \tag{4.13}$$

where v_0 denotes a velocity (km s^{-1}) and not the adimensional ratio v_0/c. Equation (4.13) is analogous to the condition that defines the so-called 'virialized temperature' in the context of cooling of a baryon gas, though in the present approach such a temperature corresponds to the dark matter gas (see section 17.3 of [3]). Since higher-order terms in v_0/c have a minor contribution, the two forms of n are approximately equal. This is shown in figure 1 displaying the adimensional quantity $\kappa m c^2 n r_{\rm opt}^2$ from (4.11) and (4.12) as functions of x for typical values $v_0 = 200$ km s^{-1}, $a = 1$ and eliminating T_c with (4.13). Equation (4.6) shows how 'flattened' rotation curves, as obtained from the empirical form (4.1), lead to $M \propto r^3$ for $r \approx 0$ and $M \propto r$ for large r. Equations (4.3)–(4.13) represent a relativistic generalization of the 'isothermal sphere' that follows as the Newtonian limit of an ideal Maxwell–Boltzmann characterized by $\rho \approx mc^2 n$, $p \ll \rho$ and $T \approx T_c$. In fact, using Newtonian hydrodynamics, we would have obtained only the leading terms of equations (4.3)–(4.13). It is still interesting to find out that the isothermal sphere can be obtained from general relativity in the limit $v_0/c \ll 1$

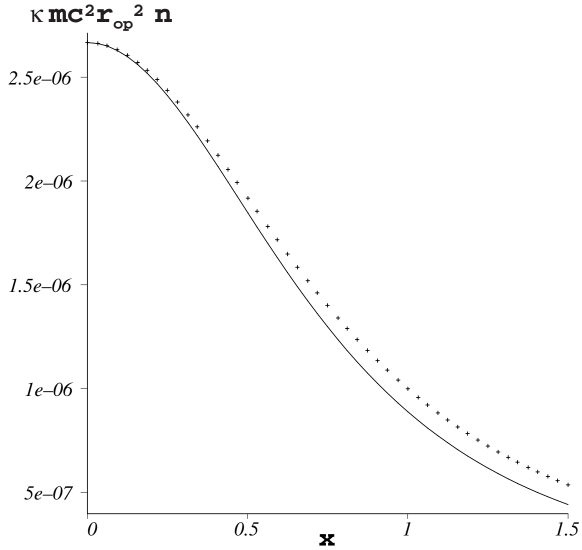

Figure 1. Comparison of n obtained from (4.11) and (4.12). This plot displays the adimensional quantity $\kappa mc^2 r_{\rm op}^2 n$, as a function of x, obtained by truncating the right-hand sides of (4.11) (solid curve) and (4.12) (dotted curve with crosses) up to fourth order in v_0/c and assuming the consistency condition (4.13). Since we are using an empiric law for observed galactic rotation curves (the URC given by (4.1)), the fact that these two expressions for n are so close to each other provides an empirical justification for the compatibility between the MB distribution and these rotation curves.

by demanding that rotation curves have a form like (4.1). The total mass of the galactic halo is usually given as M evaluated at the radius $r = r_{200}$ (the radius at which ρ is 200 times the mean cosmic density). Assuming this density to be $\approx 10^{-29}$ gm cm^{-3} together with typical values $v_0 = 200$ km s^{-1} and $a = 1$ yields $r_{200} \approx 150$ kpc. Evaluating M at this values yields about $10^{17} M_\odot$, while M evaluated at a typical $r_{\rm opt} = 15$ kpc leads to about $10^{12} M_\odot$, an order of magnitude larger than the galactic mass due to visible matter. These values are consistent with [30, 31].

5. Discussion

So far we have found a reasonable approximation for galactic dark matter to be described by a self-gravitating Maxwell–Boltzmann gas, under the assumption of the empirical rotation velocity law (4.1). The following consistency relations emerge from equations (4.11), (4.12) and (4.13):

$$n_c \approx \frac{3 v_0^2}{4\pi G m a^2 r_{\rm opt}^2}, \qquad T_c \approx \frac{m v_0^2}{3 k_{\rm B}} \tag{5.1}$$

hence, bearing in mind that $n \leqslant n_c$ and $T \approx T_c$, the condition (3.1) for the validity of the MB distribution can be written as

$$\frac{n\hbar^3}{(m k_{\rm B} T)^{3/2}} \leqslant \frac{n_c \hbar^3}{(m k_{\rm B} T_c)^{3/2}} \ll 1. \tag{5.2}$$

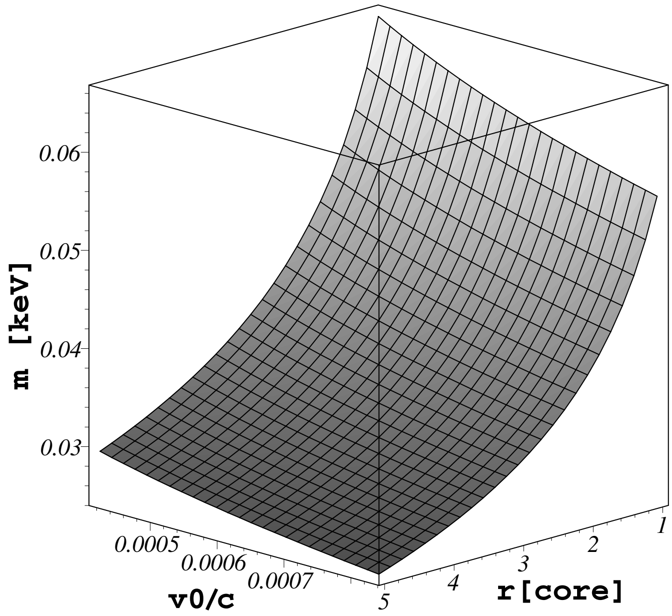

Figure 2. Minimal mass for which the Maxwell–Boltzmann distribution is applicable. This graph displays m (in keV) as a function of v_0/c and $b = ax$, respectively, the terminal velocity and 'halo core radius' associated with the URC given in (4.1). Assuming typical ranges for spiral galaxies: $125\ \mathrm{km\ s^{-1}} < v_0 < 250\ \mathrm{km\ s^{-1}}$ and $1\ \mathrm{kpc} < b < 5\ \mathrm{kpc}$, we obtain masses in the range of $30\ \mathrm{eV} < m < 60\ \mathrm{eV}$ that follow from the right-hand side of the relation (5.3), providing the criterion for applicability of the Maxwell–Bolzmann distribution. Dark matter particle candidates complying with an MB distribution must have much larger mass than the plotted values $30\ \mathrm{eV} < m < 60\ \mathrm{eV}$.

Inserting (5.1) into (5.2) yields the condition

$$m \gg \left[\frac{3^{5/2}\hbar^3}{4\pi G a^2 r_{\mathrm{opt}}^2 v_0}\right]^{1/4}, \tag{5.3}$$

a criterion of applicability of the MB distribution (non-degeneracy) that is entirely given in terms of m, the fundamental constants $G, \hbar$ and the empirical parameters v_0 and $b \equiv a r_{\mathrm{opt}}$ (the 'terminal' rotation velocity and the 'core radius'). For dark-matter-dominated galaxies (spiral and low surface brightness (LSB)) [30] these parameters have a small variation range: $r_{\mathrm{opt}} \approx 15$ kpc, $1\ \mathrm{kpc} < b < 5\ \mathrm{kpc}$ and $125\ \mathrm{km\ s^{-1}} < v_0 < 250\ \mathrm{km\ s^{-1}}$, the constraint (5.3) does provide a tight estimate of the minimal value for the mass of the particles under the assumption that these particles form a self-gravitating ideal dark gas complying with MB statistics. As shown in figure 2, this minimal value lies between 30 and 60 eV, thus implying that appropriate particle candidates must have a much larger mass than this range of values. This minimal bound excludes, for instance, light mass particles such as the electron neutrino ($m_{\nu_e} < 2.2$ eV) or the axion ($m_A \approx 10^{-5}$ eV). The currently accepted estimations of cosmological bounds on the sum of masses for the three active neutrino species is about 24 eV [28], a value that would apparently rule out all neutrino flavours. However, recent estimations of these cosmological bounds have raised this sum to about 1 keV [42], hence more massive neutrinos could also be accommodated as dark matter particle candidates. Estimates of masses of various particle candidates are displayed in table 1.

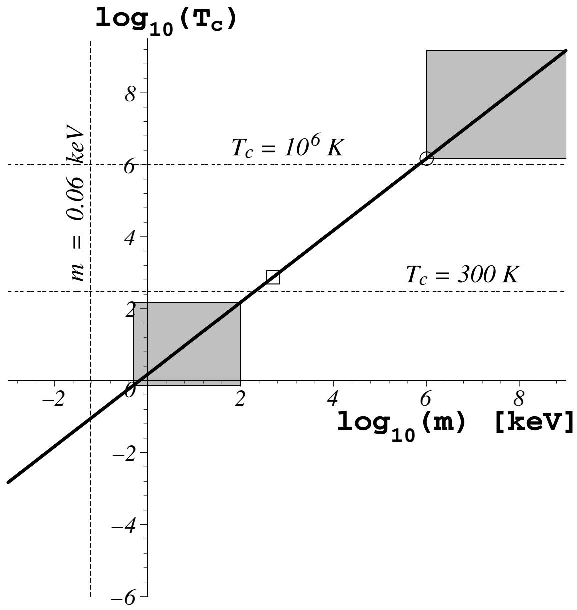

Figure 3. Relation between particle mass and central temperature. This graph displays the relation between $\log_{10}(T_c)$ (in K) and $\log_{10}(m)$ (in keV) that follows from equation (5.4) for a terminal velocity $v_0 = 200$ km s^{-1}. Almost identical plots are obtained for other velocities in the observed range 125 km s^{-1} $< v_0 <$ 250 km s^{-1}. The circle and box symbols respectively denote the proton and electron mass yielding central temperatures of the order $T_c \approx 10^6, 10^3$ K. The central temperature for light particles in the range 0.5 keV $< m <$ 100 keV is less than 300 K (rectangle on the left), while for massive sypersymmetric particles in the range 1 GeV $< m <$ 100 GeV, we have T_c as large as 10^9 K (rectangle on the right). However, such high temperatures cannot rule out these weakly interactive particles as components of the dark matter MB gas.

Since $T \approx T_c$, the consistency condition (4.13) provides the following constraint on the temperature and particle mass of the dark gas:

$$\frac{m}{T_c} = \frac{3k_{\mathrm{B}}}{v_0^2} \approx 0.4 \times 10^3 \frac{\mathrm{eV}}{\mathrm{K}}, \tag{5.4}$$

where we have taken $v_0 = 200\,\mathrm{km\,s}^{-1}$.[5] Considering in (5.4) the minimal mass range that follows from (5.3), we would obtain gas temperatures consistent with the assumed typical temperatures of relic gases: $T_c \approx 2$–4 K. However, as long as we do not have more information on the interaction and physical properties of various particle candidates, we cannot rule out a given large mass value on the grounds that the corresponding gas temperature could be too high. However, if we assume that the ideal dark gas is made up of electrons or baryons, so that $m = m_p$ or $m = m_e$, then condition (5.3) for applicability of the MB distribution is certainly satisfied, but (5.4) implies a temperature of the order of $T_c \approx 10^3$ K for electrons and $T_c \approx 10^6$ K for baryons! Obviously, baryons or electrons at such high temperatures would certainly not remain unobservably 'dark'. We can rule them out, but we cannot rule out more massive particles (in the range of 1–100 GeV) characterized by weak interaction even if the gas temperature is in the range of 10^8–10^9 K. Figure 3 illustrates, for various particle candidates, the relation between T_c and m contained in (5.4). The main novelty of the present paper

[5] The variation of v_0 in the observed ranges for spiral galaxies does not alter significantly the numerical value in the rhs of equation (5.4).

is the fact that it is based on a general relativistic hydrodynamics, as opposed to numerical simulations [7–9], Newtonian or kinetic theory perturbative approaches (see [12–22]).

Finally, the fact that under the assumption of MB distribution, we have obtained a minimal mass in the range $30\ \text{eV} < m < 60\ \text{eV}$ that seems to discriminate against thermal relic gases composed of lighter particles (electron neutrino, etc) coincides with the fact that these particle candidates tend to be ruled out because of their inability to produce sufficient matter clustering [2, 3]. In spite of these arguments, if either of these particles constitute self-gravitating gases accounting for a galactic halo it would be inconsistent to model such a gas as SCDM in the context of a classical ideal gas complying with MB statistics. It would be necessary to examine these cases as either HDM or WDM, by using either a relativistic MB distribution (very light particles can be relativistic even at low temperatures) and/or a distribution that takes into account (depending on the particle) Fermi–Dirac or Bose–Einstein statistics. Non-thermal axions are very light particles ($m \sim 10^{-5}$ eV); however, this type of relic source cannot be treated as a Maxwell–Boltzmann gas. Thus the lower mass limit that we have obtained does not apply. This and other non-thermal sources [23, 24] require a wholly different approach. These studies will be undertaken in future papers.

Acknowledgments

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References

[1] Trimble V 1987 *Ann. Rev. Astron.* **25** 425
Trimble V 1988 *Contemp. Phys.* **29** 373
Dolgov A D 1999 Invited talk at *Gamow Memorial International Conference* (*Preprint* hep-ph/9910532)
[2] Padmanabhan T 1995 *Structure Formation in the Universe* (Cambridge: Cambridge University Press)
[3] Peacock C 1999 *Cosmological Physics* (Cambridge: Cambridge University Press)
[4] Boehm C, Fayet P and Schaeffer R 2000 *Preprint* astro-ph/0012404
[5] Peebles P J E 1984 *Astrophys. J.* **277** 470
[6] Frenk C S *et al* 1988 *Astrophys. J.* **327** 507
[7] Navarro J F, Frenk C S and White S D M 1996 *Astrophys. J.* **462** 563
Navarro J F, Frenk C S and White S D M 1997 *Astrophys. J.* **490** 493
[8] Moore B *et al* 1999 *Mon. Not. R. Astron. Soc.* **310** 1147
[9] Ghigna S *et al* 1999 *Preprint* astro-ph/9910166
[10] Moore B 1994 *Nature* **370** 629
[11] Flores R and Primack J P 1994 *Astrophys. J.* **427** L1
[12] Spergel D N and Steinhardt P J 2000 *Phys. Rev. Lett.* **84** 3760
[13] Burkert A 2000 *Astrophys. J. Lett.* **534** 143
[14] Firmani C *et al* 2000 *Mon. Not. R. Astron. Soc.* **315** 29
[15] Mohapatra R N, Nussinov S and Teplitz V L 2001 *Preprint* hep-ph/0111381
[16] Mohapatra R N and Teplitz V L 2000 *Phys. Rev.* D **62** 063506
[17] Colombi S, Dodelson S and Widrow L 1996 *Astrophys. J.* **458** 1
[18] Schaeffer R and Silk J 1998 *Astrophys. J.* **332** 1
[19] Hogan C J 1999 *Preprint* astro-ph/9912549
[20] Hannestad S and Scherrer R 2000 *Phys. Rev.* D **62** 043522
[21] Dalcanton J J and Hogan C J 2000 *Preprint* astro-ph/0004381
[22] Hogan C J and Dalcanton J J 2000 *Preprint* astro-ph/0002330

[23] Matos T and Guzmán F S 2000 *Class. Quantum Grav.* **17** L9–L16 (*Preprint* gr-qc/9810028)
Matos T and Ureña L A 2001 *Phys. Rev.* D **63** 063506 (*Preprint* astro-ph/0006024)
[24] Matos T, Guzmán F S and Núñez D 2000 *Phys. Rev.* D **62** 061301(R) (*Preprint* astro-ph/0003398)
Matos T, Núñez D, Guzmán F S and Ramirez E 2002 *Gen. Rel. Grav.* **34** 283–305 (*Preprint* astro-ph/0005528)
Matos T and Guzmán F S 2001 *Class. Quantum Grav.* **18** 5055–64 (*Preprint* gr-qc/0108027)
[25] Nucamendi U, Salgado M and Sudarsky D 2000 *Phys. Rev. Lett.* **84** 3037
[26] Pal P B 1999 Particle dark matter: an overview *Invited talk at Workshop on Cosmology: Observations Confront Theories (West Bengal, India, 11–17 Jan. 1999)*
Pal P B 1999 *Pramana* **53** 1053–59 (*Preprint* astro-ph/9906261)
[27] Ahmad Q R *et al* (SNO Collaboration) 2001 *Phys. Rev. Lett.* **87** 071301
[28] Groom D E *et al* (Particle Data Group) 2000 The 2000 review of particle physics *Eur. Phys. J.* C **15** 1 and 2001 partial update for edition 2002 (webpage http://pdg.lbl.gov)
[29] Ellis John 2002 Summary of DARK 2002 *4th Int. Heidelberg Conf. on Dark Matter in Astro and Particle Physics (Cape Town, South Africa, 4–9 Feb. 2002)* (*Preprint* astro-ph/0204059)
[30] Salucci P and Persic M 1997 Dark and visible matter in galaxies *Proc. Sesto D M 1996 Conf. (Astron. Soc. Pac. Conf. Ser.* **117**) (*Preprint* astro-ph/9703027)
[31] Battaner E and Florido E 2000 The rotation curve of spiral galaxies and its cosmological implications *Fund. Cosmic Phys.* **21** 1–154 (*Preprint* astro-ph/0010475)
[32] de Groot S R, van Leeuwen W A and van Weert Ch G 1980 *Relativistic Kinetic Theory. Principles and Applications* (Amsterdam: North-Holland) pp 46–55
[33] Maartens R 1996 *Causal Thermodynamics in Relativity* Lectures given at the *Hanno Rund Workshop on Relativity and Thermodynamics (Natal University, Durban, South Africa)* (*Preprint* astro-ph/9609110)
[34] Landau L D and Lifshitz E M 1986 *Física Estadística* Parte II Vol 9 d *Curso de Física Teórica,* Editorial Reverté, SA (see equation (4.65))
[35] Kawasaki M, Sugiyama N and Yanagida T 1997 *Mod. Phys. Lett.* A **12** 1275
[36] Baltz E A and Murayama H 2001 *Preprint* astro-ph/0108172
[37] Burns S D 1997 *Preprint* astro-ph/9711304
[38] Hansen S H, Lesgourgues J, Pastor S and Silk J 2001 *Preprint* astro-ph/0106108
[39] Dolgov A D and Hansen S H 2000 *Preprint* hep-ph/0009083
[40] Abazajian K, Fuller G M and Patel M 2001 *Phys. Rev.* D **64** 023501
[41] Liu C and Song J 2001 *Phys. Lett.* B **512** 247
[42] Giudice G F, Kolb E W, Riotto A, Semikoz D V and Tkachev I I 2001 *Phys. Rev.* D **64** 043512
[43] Cho Y M and Keum Y Y 1998 *Mod. Phys. Lett.* A **13** 109
[44] Bonometto S A, Gabbiani F and Masiero A 1994 *Phys. Rev.* D **49** 3918
[45] Berezinsky V and Valle J W F 1993 *Phys. Lett.* B **318** 360
[46] Babu K S, Rothstein I Z and Seckel D 1993 *Nucl. Phys.* B **403** 725
[47] Dolgov A, Pastor S and Valle J W F 1995 *Preprint* astro-ph/9506011
[48] Berezhiani Z G and Mohapatra R N 1995 *Phys. Rev.* D **52** 6607
[49] Berezhiani Z G 1996 *Acta Phys. Pol.* B **27** 1503
[50] Abreu P *et al* 2000 *Phys. Lett.* B **489** 38–5
[51] Ellis J, Falk T, Ganis G and Olive K A 2000 *Phys. Rev.* D **62** 075010
[52] Covi L, Kim H B, Kim J E and Roszkowski L 2001 *J. High Energy Phys.* JHEP05(2001)033
[53] Roszkowski L 2000 Invited plenary talk at the *4th Int. Workshop on Particle Physics and the Early Universe* (COSMO-2000) *(Cheju, Cheju Island, Korea, 4–8 Sept. 2000)*
Roszkowski L 2000 *3rd Int. Workshop on the Identification of Dark Matter* (IDM-2000) *(York, England, 18–22 Sept. 2000)* (*Preprint* hep-ph/0102325)
[54] Kawasaki M and Moroi T 1995 *Prog. Theor. Phys.* **93** 879–900