Hydrodynamical description of Galactic Dark Matter*

Luis G. Cabral-Rosetti,† Darío Núñez,‡ and Roberto A. Sussman
Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México
Apartado postal 70-543, 94510 México, D.F., Mexico
†-‡-† e-mail: †luis@nuclecu.unam.mx; ‡nunez@nuclecu.unam.mx, nunez@gravity.phys.psu.edu; ‡sussman@nuclecu.unam.mx

Tonatiuh Matos
Departamento de Física, Centro de Investigación y Escuelas Avanzadas del Instituto Politécnico Nacional
Apartado postal 14-740, México D.F., Mexico
e-mail: tmatos@fis.cinvestav.mx

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We consider simple hydrodynamical models of galactic dark matter in which the galactic halo is a self-gravitating and self-interacting gas that dominates the dynamics of the galaxy. Modeling this halo as a spherically symmetric and static perfect fluid satisfying the field equations of General Relativity, visible baryonic matter can be treated as “test particles” in the geometry of this field. We show that the assumption of an empirical “universal rotation curve” that fits a wide variety of galaxies is compatible, under suitable approximations, with state variables characteristic of a non-relativistic Maxwell-Boltzmann gas that becomes an isothermal sphere in the Newtonian limit. Consistency criteria lead to a minimal bound for particle masses in the range 30 eV \( \leq m \leq 60 \) eV and to a constraint between the central temperature and the particles mass. The allowed mass range includes popular supersymmetric particle candidates, such as the neutralino, axino and gravitino, as well as lighter particles \((m \sim \text{keV})\) proposed by numerical \(N\)-body simulations associated with self-interactive “cold” and “warm” dark matter structure formation theories.

Keywords: Hydrodynamics; dark matter; thermodynamic properties; models beyond the standard model

Consideramos modelos hidrodinámicos simples de materia oscura galáctica en los cuales el halo galáctico es un gas autogravitante y autointeractivo que domina la dinámica de la galaxia. Modelando este halo como un fluido perfecto estático y esféricamente simétrico que satisface las ecuaciones de relatividad general, la materia visible bariónica puede tratarse como partículas de prueba en la geometría de este campo. Mostramos que la suposición de una curva de rotación universal que ajusta una amplia variedad de galaxias es compatible, bajo aproximaciones apropiadas, con variables de estado características de un gas no relativista de Maxwell-Boltzmann que en el límite Newtoniano se convierte en una esfera isotérmica. Criterios de consistencia dan una cota mínima para las masas de las partículas de 30 eV \( \leq m \leq 60 \) eV y una contricción entre la temperatura central y la temperatura de las partículas. El rango de masas permitido incluye candidatos supersimétricos populares como el neutralino, axino y gravitino, así como partículas más ligeras \((m \sim \text{keV})\) propuestas por simulaciones numéricas asociadas con teorías de formación de estructura de materia oscura fría y tibia.

Descriptores: Hidrodinámica; materia oscura; propiedades termodinámicas; modelos alternativos del modelo standard

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1. Introduction

The presence of large amounts of dark matter at the galactic lengthscale is already an established fact. It is currently thought that this dark matter is made of relic self-gravitating gases which are labeled as “cold” (CDM) or “hot” (HDM), depending on the relativistic or non-relativistic nature of the particles energetic spectrum at their decoupling from the cosmic mixture [1–3]. HDM scenarios are not favoured, as they seem to be incompatible with current theories of structure formation [2–4]. CDM, usually examined within a newtonian framework, can be considered as non-interactive (collisionless particles) or self-interactive [5]. \(N\)-body numerical simmulations are often used for modeling CDM gases [6–9]. However, in recent numerical simulations (see Refs. 7–9) non-interactive CDM models present the following discrepancies with observations at the galactic scal-
eliminated as primary dark matter candidates (though there is no reason to assume that these particles would be absent in galactic halos). Of all possible weakly interactive massive particles (WIMPS), complying with the required mass value of relic gases, only the massive Neutrinos (the muon or tau neutrinos), have been detected, whereas other WIMPS (neutralino, gravitino, photino, sterile neutrino, axino, etc.) are speculative. See Ref. 26–28 and Table I for a list of candidate particles and appropriate references.

In this paper we develop an alternative description of galactic DM. Since the dark matter halo constitutes about 90% of the galactic mass, we consider the galactic gravitational field as a spacetime whose sole, self-gravitating, source is the halo, described as a perfect fluid. Assuming this galactic field as a spacetime whose sole, self-gravitating, source is the source of spacetime. Since the dark matter halo constitutes about 90% of the galactic mass, we consider the galactic gravitational field as a spacetime whose sole, self-gravitating, source is the halo, described as a perfect fluid. Assuming this galactic field as a spacetime whose sole, self-gravitating, source is the source of spacetime.

### 2. Field equations

Considering the line element of a static spherically symmetric space time

\[
ds^2 = -A^2(r) \, c^2 \, dt^2 + \frac{dr^2}{1 - 2 \, M(r)/r} + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right),
\]

the tangential velocity of test particles along stable circular geodesic orbits can be expressed in terms of the metric coefficients as

\[
\frac{V^2}{c^2} \equiv v^2(r) = \frac{r \, A'}{A}.
\]

Becomes a dynamical variable replacing \( A(r) \). Assuming as source of (1) a perfect fluid momentum energy tensor: \( T^{ab} = (\rho + p) \, u^a \, u^b + p \, g^{ab} \), with \( u^a = A^{-1} \delta^a_{\text{ct}} \), the following field equations in terms of (2) become

\[
M' + \frac{(-3 - 5 \, v^2 + 4 \, v \, v' \, r + 2 \, v^4) \, M}{r \,(1 + v^2)} - \frac{v \,(2 \, v + 2 \, v' \, r + v^2)}{1 + v^2} = 0.
\]

\[
\kappa \, p = 2 \, \frac{M - 2 \, M \, v^2 + v^2 \, r}{r^3},
\]

\[
\kappa \, p = \frac{[ -8 \, v \, v' \, r - 2 \,(2 \, v^2 + 1) \,(v^2 - 3)] \, M}{r^3 \,(1 + v^2)} + \frac{4 \, r^2 \, v \, v' + 2 \, v^2 \,( -2 + v^2) \, r}{r^3 \,(1 + v^2)},
\]

where \( \kappa = 8 \pi G / c^4 \) and a prime denotes derivative with respect to \( r \). Writing the field equations in terms of the orbital velocity, \( v \), provides a useful insight into how an (in principle) observable quantity relates to spacetime curvature and with physical quantities (state variables) which characterize the source of spacetime.

### 3. Thermodynamics

If we assume that the self gravitating ideal “dark” gas exists in physical conditions far from those in which the quantum properties of the gas particles are relevant, we would be demanding that these particles comply with Maxwell-Boltzmann (MB) statistics. Following Ref. 34, the condition that justifies an MB distribution is given by

\[
\frac{n \, h^3}{(m \, k_B \, T)^{3/2}} \ll 1,
\]

where \( n \), \( T \), \( h \) and \( k_B \) are, respectively, the particle number density, absolute temperature, Planck’s and Boltzmann’s constants. If the constraint (6) holds and we further assume thermodynamical equilibrium and non-relativistic conditions, the ideal dark gas must satisfy the equation of state
of a non-relativistic monatomic ideal gas

$$\rho = m c^2 n + \frac{3}{2} n k_B T, \quad p = n k_B T,$$

(7)

whose macroscopic state variables can be obtained from a MB distribution function under an equilibrium Kinetic theory approach (the non-relativistic and non-degenerate limit of the Jüttner distribution) [32]. An equilibrium MB distribution restricts the geometry of spacetime [33], resulting in the existence of a timelike Killing vector field $\beta^a = \beta u^a$, where $\beta \equiv m c^2/k_B T$, as well as the following relation (Tolman’s law) between the 4-acceleration and the temperature gradient

$$u_a + h_a^b (\ln T)_b = 0, \quad h_a^b = u_a u^b + \delta_a^b,$$

(8)

leading to

$$\frac{A'}{A} + \frac{T'}{T} = 0 \quad \Rightarrow \quad T \propto A^{-1}.$$

(9)

The particle number density $n$ trivially satisfies the conservation law $J^a = 0$ where $J^a = n u^a$, thus the number of dark particles is conserved. Notice that given (4) and (5), the equation of state (7) and the temperature from the Tolman law (9), we have two different expressions for $n$

$$n = \frac{p}{k_B T} \propto p A,$$

$$n = \frac{1}{m c^2} \left[ \frac{\rho - 3}{2} \right] \propto \frac{\rho}{A} \left[ 1 - \frac{3}{2} \frac{p}{\rho} \right].$$

(10)

The quantity $mc^2 n$ in (11) follows directly from Eqs. (4) and (5), while in (10) also follows from $p$ in (4) with $A \propto \exp[\int (v^2/r)dr]$. Consistency requires that (10) and (11) yield the same expression for $n$.

### 4. Dark fluid hydrodynamics

We shall assume for $v^2$ the empiric dark halo rotation velocity law given by Persic and Salucci [30, 31]

$$v^2 = \frac{v_0^2}{a^2 + x^2}, \quad x \equiv \frac{r}{r_{\text{opt}}}$$

(12)

where $r_{\text{opt}}$ is the “optical radius” containing 83% of the galactic luminosity, whereas the empiric parameters $a$ and $v_0$, respectively, the ratio of “halo core radius” to $r_{\text{opt}}$ and the “terminal” rotation velocity, depend on the galactic luminosity. For spiral galaxies we have: $v_0^2 = v_{\text{opt}}^2 (1 - \beta) (1 + a^2)$, where $v_{\text{opt}} = v(r_{\text{opt}})$ and the best fit to rotation curves is obtained for: $a = 1.5 (L/L_\star)^{1/5}$ and $\beta = 0.72 + 0.44 \log v_0 (L/L_\star)$, where $L_\star = 10^{10.4}\ L_\odot$. The range of these parameters for spiral galaxies is 125 km/sec ≤ $v_0$ ≤ 250 km/sec and 0.6 ≤ $a$ ≤ 2.3.

Inserting (12) into (2) and (3) we obtain

$$A = \left[ 1 + x^2 \right] \frac{v_0}{a^2 + x^2} \Rightarrow T = T_e \left[ 1 + x^2 \right] \frac{v_0}{a^2 + x^2},$$

(13)

$$M = \frac{\left( v_0^2 - 2 \right) (a^2 + x^2)^{2 - v_0^2} v_0^2}{\left[ a^2 + (1 + v_0^2) x^2 \right]^{2/(1 + v_0^2)}}$$

$$\times r_{\text{opt}} \int \frac{\left( a^2 + (1 + v_0^2) x^2 \right)^{(1 - v_0^2)/(1 + v_0^2)}}{\left( a^2 + x^2 \right)^{3 - v_0^2}} xdx,$$

(14)

where $T_e = T(0)$ and we have set an integration constant to zero in order to comply with the consistency requirement that $v_0 = 0$ implies flat spacetime ($A = 1, M = 0$). Since the velocities of rotation curves are newtonian, $v_0 \ll c$ (typical values are $v_0/c \approx 0.5 \times 10^{-3}$), instead of evaluating (14) we will expand this quadrature around $v_0/c$ (in order to keep the notation simple, we write $v_0$ instead of $v_0/c$). This yields

$$M = \frac{x^3 r_{\text{opt}} v_0^2}{a^2 + x^2} \left[ 1 - \frac{5x^2 + 2a^2}{2(a^2 + x^2)} v_0^2 \right. + \left. \frac{12x^4 + 11a^2x^2 + 3a^4}{2(a^2 + x^2)^2} v_0^4 + \mathcal{O}(v_0^6) \right],$$

(15)

we obtain the expanded forms of $\rho$ and $p$ by inserting (12) and (14) into (4) and (5) and then expanding around $v_0$, leading to

$$\kappa \rho v_{\text{opt}}^2 = \frac{2(3a^2 + x^2)}{(a^2 + x^2)^2} v_0^2$$

$$- \frac{5x^4 + 23a^2x^2 + 6a^4}{(a^2 + x^2)^3} v_0^4 + \mathcal{O}(v_0^6),$$

(16)

$$\kappa p v_{\text{opt}}^2 = \frac{2a^2 + x^2}{(a^2 + x^2)^2} v_0^4$$

$$- \frac{2x^4 + 7a^2x^2 + 3a^4}{(a^2 + x^2)^3} v_0^6 + \mathcal{O}(v_0^8),$$

(17)

while the expanded form for $T$ follows from (13)

$$T = T_e \left[ 1 - \frac{1}{2} \ln (1 + x^2) \right] v_0^2$$

$$+ \frac{1}{8} \ln^2 (1 + x^2) v_0^4 + \mathcal{O}(v_0^6).$$

(18)

In order to compare $n$ obtained from (10) and (11), we substitute (12) and (14) into (11) and expand around $v_0$, leading to

$$n v_{\text{opt}}^2 = \frac{1}{\kappa m c^2} \left[ \frac{2(3a^2 + x^2)}{(a^2 + x^2)^2} v_0^2 \right.$$

$$\left. - \frac{18x^4 + 55a^2x^2 + 13a^4}{2(a^2 + x^2)^3} v_0^4 + \mathcal{O}(v_0^6) \right],$$

(19)

while $n$ in (10) follows by substituting (14) into (17), using $T$ from (13) and then expanding around $v_0$. This yields

$$m_{\text{opt}}^2 = \frac{1}{\kappa k_B T_e} \left[ \frac{2a^2 + x^2}{(a^2 + x^2)^2} v_0^4 ight. \
+ \left. \frac{2(a^2 + 2x^2)(3a^2 + x^2)}{2(a^2 + x^2)^3} v_0^6 + O(v_0^8) \right]. \tag{20}$$

Since $v_0/c \ll 1$, a reasonable approximation is obtained if the leading terms of $n$ from (19) and (20) coincide. By looking at these equations, it is evident that this consistency requirement implies

$$\frac{1}{2} m_{\text{opt}}^2 = \frac{3}{2} k_B T_e, \tag{21}$$

where $v_0$ denotes a velocity (cm/sec) and not the adimensional ratio $n_0/v_0$. Since higher order terms in $v_0/c$ have a minor contribution, the two forms of $n$ are approximately equal. This is shown in Fig. 1 displaying the adimensional quantity $\kappa m \bar{c}^2 n r_{\text{opt}}^2$ from (19) and (20) as functions of $x$ for typical values $v_0/c = 0.0006$, $a = 1$ and eliminating $T_e$ with (21). Equation (14) shows how “flattened” rotation curves, as obtained from the empiric form (12), lead to $M \propto r^3$ for $r \approx 0$ and $M \propto r$ for large $r$. Equations (15) to (21) represent a relativistic generalization of the “isothermal sphere” that follows as the newtonian limit of an ideal Maxwell-Boltzmann gas characterized by $\rho \approx m \bar{c}^2 n, p \ll \rho$ and $T \approx T_e$. In fact, using newtonian hydrodynamics we would have obtained only the leading terms of Eqs. (15) to (21). It is still interesting to find out that the isothermal sphere can be obtained from General Relativity in the limit $v_0/c \ll 1$ by demanding that rotation curves have a form like (12). The total mass of the galactic halo, usually given as $M$ evaluated at the radius $r = r_{200}$ (the radius at which $\rho$ is 200 times the mean cosmic density). Assuming this density to be $\rho \approx 10^{-29}$ gm/cm$^3$ together with typical values $v_0 = 200$ km/sec and $\alpha = 1$ yields $r_{200} \approx 150$ kpc. Evaluating $M$ at this values yields about $10^{12} M_\odot$, while $M$ evaluated at a typical “optical radius” $r = 15$ kpc leads to about $10^{10} M_\odot$, an order of magnitude larger than the galactic mass due to visible matter.

5. Discussion

So far we have found a reasonable approximation for galactic dark matter to be described by a self gravitating Maxwell-Boltzmann gas, under the assumption of the empiric rotation velocity law (12). The following consistency relations emerge from Eqs. (19), (20) and (21)

$$n_c \approx \frac{3 v_0^2}{4 \pi G m a^2 r_{\text{opt}}^2}, \quad T_c \approx \frac{m v_0^2}{3 k_B} \tag{22}$$

hence, bearing in mind that $n \leq n_c$ and $T \approx T_c$, the condition (6) for the validity of the MB distribution together with (22) yields the condition

$$m \gg \left[ \frac{3^{5/2} \hbar^3}{4 \pi G m a^2 r_{\text{opt}}^2 v_0} \right]^{1/4}, \tag{23}$$

a criteria of applicability of the MB distribution that is entirely given in terms of $m$, the fundamental constants $G$, $\hbar$ and the empiric parameters $v_0$ and $a r_{\text{opt}}$ (the “terminal” rotation velocity and the “core radius”) [29]. For dark matter dominated galaxies (spiral and low surface brightness (LSB)) [30] these parameters have a small variation range: $r_{\text{opt}} \approx 15$ kpc, $0.6 \leq \alpha \leq 2.3$ and $125$ km/sec $\leq v_0 \leq 300$ km/sec, the constraint (23) does provide a tight estimate of the minimal value for the mass of the particles under the assumption that these particles form a self gravitating ideal dark gas complying with MB statistics. As shown in Fig. 2, this minimal value lies between 30 and 60 eV, thus implying that appropriate particle candidates must have a much larger mass than this figure [29]. This minimal bound excludes, for instance, light mass thermal particles such as the electron neutrino ($m_{\nu_e} < 2.2$ eV). The axion is also very light ($m_\chi \approx 10^{-5}$ eV) but it is not a thermal relic and so we cannot study it under the present framework. The currently accepted estimations of cosmological bounds on the sum of

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However, recent estimations of these cosmological bounds have raised this sum to about 1 keV [42], hence more massive neutrinos could also be accomodated as dark matter particle candidates. Estimates of masses of various particle candidates complying with an MB distribution must have much larger mass than the plotted values 30 eV \( \leq m \leq 60 \) eV.

masses for the three active neutrino species is about 24 eV, a value that would apparently rule out all neutrino flavours. However, recent estimations of these cosmological bounds have raised this sum to about 1 keV [42], hence more massive neutrinos could also be accomodated as dark matter particle candidates. Estimates of masses of various particle candidates are displayed in Table I.

Since \( T \approx T_c \), the consistency condition (21) provides the following constraint on the temperature and particles mass of the dark gas

\[
\frac{m}{T_c} = \frac{3 k_B}{v_0^2} \approx 0.4 \times 10^3 \frac{\text{eV}}{\text{K}},
\]

where we have taken \( v_0 = 300 \) km/sec. Considering in (24) the minimal mass range that follows from (23), we would obtain gas temperatures consistent with the assumed typical temperatures of relic gases: \( T_c \approx 2 \) to 4 K. However, since we have no way of inferring a value for the temperature of the ideal dark gas, we have no clear cut criterion for the estimate of a maximal bound for this mass. If we assume that the ideal dark gas is made of electrons or barions, so that \( m = m_e \) or \( m = m_p \), then condition (23) for applicability of the MB distribution is certainly satisfied and (24) implies a temperature of the order of \( T_c \approx 10^3 \) K for electrons and \( T_c \approx 10^6 \) K for barions. Obviously, barions or electrons at such a high temperatures would radiate and certainly not remain unobservably “dark”. However, as long as the interaction is weak and the particles are not charged, we cannot rule out any other particle candidate only on the basis of the gas temperature, even if this temperature is very high (see Fig. 3) as in the case of massive supersymmetric particles. As shown in Table I, a wide range of weakly interactive particles can be considered as possible main components of a MB dark gas, including popular supersymmetric particles (the neutralino), as well as hypothetical light particles predicted by current literature based on WDM models of structure formation [2, 3]. The main novelty of the present paper is the fact that it is based on a general relativistic hydrodynamics, as opposed to numerical simulations [7–9], Newtonian or Kinetic Theory perturbative approaches (see Refs. 12–22).

Finally, the fact that we have obtained a minimal mass on the range 30–60 eV, that seems to discriminate against very
light thermal particles like the electron neutrino, coincides with the fact that these HDM particle candidates tend to be ruled out because of their inability to produce sufficient matter clustering \[1, 3, 29\]. In spite of these arguments, a self gravitating gas of this type of particles accounting for a galactic halo, would have to be modeled, either as a relativistic MB gas (very light particles can be relativistic even at low temperatures) and/or in terms of a distribution that takes into account Fermi-Dirac or Bose-Einstein statistics. These studies will be undertaken in future papers \[29\].

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