

Unified models of inflation and quintessenceAndro Gonzalez,^{1,3} Tonatiuh Matos,² and Israel Quiros^{2,3}¹*Universidad de Ciego de Avila, Ciego de Avila, Cuba*²*Departamento de Física, Centro de Investigación y de Estudios Avanzados del IPN, AP 14-740, 07000 México DF, México*³*Universidad Central de Las Villas, Santa Clara CP 54830, Cuba*

(Received 10 October 2004; published 28 April 2005)

We apply an extended version of the method developed by Chimento and Jakubi [Int. J. Mod. Phys. D **5**, 71 (1996)], to derive exact cosmological (flat) Friedmann-Robertson-Walker solutions in RS2 brane models with a perfect fluid of ordinary matter plus a scalar field fluid trapped on the brane. We found new exact solutions, which could serve to unify inflation and quintessence in a common theoretical framework.

DOI: 10.1103/PhysRevD.71.084029

PACS numbers: 04.20.Jb, 04.50.+h, 11.25.Wx, 98.80.-k

I. INTRODUCTION

Recent observations from the Wilkinson Microwave Anisotropy Probe (WMAP) [1] offer strong supporting evidence in favor of the “Inflationary Paradigm.”¹ In the most simple models in this kind, the energy density of the universe is dominated by the potential energy of a single (inflaton) scalar field that slowly rolls down its self-interaction potential [2]. Restrictions imposed upon the kinds of potentials that can lead to realistic inflationary scenarios are dictated by the slow-roll approximation, and result in that only sufficiently flat potentials can drive inflation. In order for the potential to be sufficiently flat, these conventional inflationary models should be fine-tuned. This simple picture of the early-time cosmic evolution can be drastically changed if one considers models of inflation inspired in “Unified Theories,” like the Super String or M-theory. One of the most appealing models of this kind is the Randall-Sundrum braneworld model of type 2 (RS2) [3]. In this model a single codimension 1 brane with positive tension is embedded in a five-dimensional anti-de Sitter (AdS) (bulk) space-time, which is infinite in the direction perpendicular to the brane. In general, the standard model (SM) matter degrees of freedom are confined to the brane; meanwhile, gravitation can propagate in the bulk. However, in the low-energy limit, due to the curvature of the bulk, the graviton is confined to the brane, and standard (four-dimensional) general relativity (GR) laws are recovered.

RS2 braneworld models have an appreciable impact on early universe cosmology, in particular, for the inflationary paradigm. In effect, a distinctive feature of cosmology with a scalar field confined to a RS2 brane is that the expansion rate of the universe differs at high energy from that predicted by standard GR. Actually, due to a term that is quadratic in the energy density, the friction acting on the

scalar field is enhanced. This means that, in RS2 brane-world cosmology, inflation is possible for a wider region of parameter space than in standard cosmology [4].² Even potentials that are not sufficiently flat from the point of view of the conventional inflationary paradigm can produce successful inflation. At sufficiently low energies (much less than the brane tension), the standard cosmic behavior is recovered prior to primordial nucleosynthesis scale ($T \sim 1$ MeV) and a natural exit from inflation ensues as the field accelerates down its potential [5]. In this scenario, reheating arises naturally even for potentials without a global minimum and radiation is created through gravitational particle production [6] and/or through curvaton reheating [7]. This last ingredient improves the brane “steep” inflationary picture [8]. Other mechanisms such as preheating, for instance, have also been explored [9].

Another interesting feature of this scenario is that the inflaton does not necessarily need to decay and it may survive through the present epoch in the cosmic evolution. Therefore, it may also play the role of the quintessence field, which is a necessary ingredient to explain the current acceleration in the expansion of the universe. Such a unified theoretical framework for the description of both inflaton and quintessence with the help of just one single scalar field has been the target of some works (see for instance Refs. [5,10–12]).³ However, in general, it is very difficult to solve the system of Einstein’s differential equations that model a RS2 brane with the Friedmann-Robertson-Walker (FRW) metric on it, even if only a scalar field matter degree is confined to the brane and the slow-roll assumptions are invoked [4]. In consequence, it is of interest to develop generating techniques for deriving exact solutions to Einstein’s field equations on the brane for the aforementioned problem [4], or to use well-known techniques that have been used to generate solutions in standard GR, but not in a brane context [14,15]. In Ref. [4], for instance, the authors develop algorithms for generating a

¹As predicted theoretically within the framework of the conventional inflationary model of the early universe, these observations signal at a Gaussian, adiabatic, and nearly scale-invariant form of the cosmic microwave background (CMB) power spectrum and at zero spatial curvature (spatially flat universe).

²This is true for braneworld cosmology in general.

³These include models of quintessential inflation without branes as in [13].

class of exact braneworld cosmologies, where a self-interacting scalar field ϕ is confined to a RS2 brane.⁴ They found a number of exact solutions, and in the perfect fluid model the resulting potential is identified. It behaves like $V(\phi) \sim \sinh^{-2}[\lambda\phi]$. The algorithms developed are valid only during inflation, where the field is monotonically rolling down its potential. In Ref. [16], the authors study a RS2 braneworld cosmology for a universe filled with a perfect fluid of ordinary matter and a scalar field with a power-law potential $V(\phi) \sim \phi^\alpha$. The index α can be either positive or negative. The authors describe scaling solutions. Radiation dominated and scalar field dominated solutions to this model were studied in [17]. The author studied the region in the parameter space where a realistic quintessence model is possible.

The aim of the present paper is to generalize the method for generating exact FRW cosmologies developed in Ref. [14], with the purpose to derive exact solutions in a RS2 braneworld with a two-component perfect fluid, consisting of an ordinary fluid plus a self-interacting scalar field. This method has been used to generate FRW solutions in standard (four-dimensional) GR [14,15], and here we generalize it for the brane contexts. We are able to obtain new exact solutions that generalize other existing braneworld FRW solutions. A salient feature of these solutions is that they can accommodate unified models of both early-time inflation and late-time accelerated expansion. Since the solutions are quite general, we are led to conjecture that such a unified pattern of inflation is generic of RS2 cosmological models.

We want to note that assuming low-energy (4D) brane cosmology in a cosmological setup with a scalar field with a fairly simple potential and a matter background is no more than an unrealistic simplification of the physics. However, sometimes this kind of phenomenological approach is very useful. Perhaps the “right way” could be studying these extra dimensional scenarios from a more fundamental point of view, i.e., starting off from a fundamental theory and working out the low-energy physics and studying its cosmology.

The paper has been organized in the following way. In Sec. II, we expose the details of the braneworld model we are about to investigate and we explain the method we use for generating exact solutions. In Sec. III we give the “gallery” of solutions we are able to derive and briefly comment on some of them, making emphasis on their ability to explain, in a unified framework, both early inflation and late-time “quintessential” inflation. Section IV is dedicated to integrate the results in the previous section within a unified picture of inflation. In the final section we summarize the main results we obtained.

We are interested in the region in the parameter space where the classical solution is still valid, but where qua-

dratic (brane) corrections could become significant. This means that the brane tension could be much less than the energy density of the matter degrees of freedom which, in turn, is much less than $M_{(5)}^4$, where $M_{(5)}$ represents the five-dimensional Planck scale.

II. THE SETUP

We study a brane cosmology setup based on the RS2 braneworld model [3], in which the brane is filled both with a scalar field fluid and a perfect fluid of “ordinary” matter, i.e., the total energy density, coming from the matter degrees of freedom that are trapped on the brane, is given by $\rho_T = \rho_\phi + \rho_M$, where $\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$ is the scalar field energy density and V is the self-interaction potential, and $\rho_M = Ma^{-3\gamma}$, where γ is the barotropic index of the ordinary matter perfect fluid. The Einstein field equations (EFE) for a FRW brane universe are then [18]: the Friedmann equation,

$$H^2 = \frac{k_0}{3}\rho_T(1 + \sigma_b\rho_T) + \frac{\epsilon}{a^4} - \frac{k}{a^2} + \frac{\Lambda}{3}; \quad (1)$$

the Raychaudhuri equation,

$$2\dot{H} = -k_0(\rho_T + p_T)(1 + 2\sigma_b\rho_T) - \frac{4\epsilon}{a^4} + \frac{2k}{a^2}; \quad (2)$$

and the Klein-Gordon (KG) equation for the scalar field,

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0, \quad (3)$$

where $k_0 = 8\pi/m_{pl}^2$ (m_{pl} is the effective 4D Planck mass), $k = 0, \pm 1$ is the spatial curvature, ϵ is a constant parameter related with black hole mass in the bulk [the corresponding term in (1) is known as dark radiation]⁵, and $\sigma_b = 1/2\lambda_b$ (λ_b is the brane tension). The parameter σ_b is related to k_0 through the relation $k_0\sigma_b = (16\pi^2)/(3M_{(5)}^6)$ [19]. The pressure of the scalar field fluid is $p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$. Note that, if we drop the dark radiation term, when $\rho_T \ll 1/\sigma_b$ the standard GR Friedmann equation is recovered.

We will look for flat ($k = 0$) FRW solutions to the brane EFE’s (1)–(3) generalizing the method developed by Chimento and Jakubi in Ref. [14].

A. Method for generating exact solutions

The idea behind the method in Ref. [14] is the following. If one rewrites the self-interaction potential as a function of the scale factor a , $V(a) = \frac{F(a)}{a^6}$, where $F(a)$ is an “arbitrary” input function, then it is not difficult to prove that (by working out of the KG equation) the scalar field energy density can be found as a first integral of (2.3):

⁴In [4] the scalar field ϕ is the only matter source.

⁵Dark radiation is a contribution of nonlocal bulk effects onto the brane; it carries scalar modes from bulk gravitons. $\epsilon \neq 0$ for AdS-Schwarzschild bulk, i.e., a AdS bulk with a black hole.

$$\rho_\phi = \frac{1}{a^6} \left\{ 6 \int \frac{da}{a} F(a) + C \right\}, \quad (4)$$

where C is an arbitrary integration constant. Let us introduce the following ‘‘Chimento’s functions’’ (CFs):

$$G(a) \equiv H^2, \quad L(a) \equiv \dot{\phi}^2 = 2 \left\{ \rho_\phi - \frac{F(a)}{a^6} \right\}. \quad (5)$$

Then, using the scale factor as an independent variable, the problem of finding solutions to the EFE can be reduced to quadratures:

$$\Delta t = \int \frac{da}{a} G(a)^{-1/2}, \quad (6)$$

and

$$\Delta \phi = \int \frac{da}{a} \left[\frac{L(a)}{G(a)} \right]^{1/2}, \quad (7)$$

where $\Delta t \equiv t - t_0$, $\Delta \phi \equiv \phi - \phi_0$, t_0 and ϕ_0 being two other arbitrary integration constants. Given an input function $F(a)$, the integration constant C , and the parameters M , γ , k , ϵ , and Λ , one can obtain a solution for the functions $a(t)$ and $\phi(t)$ and identify the potential $V(\phi)$ as well.⁶ As we see the method is fairly general, its only limitation coming from the choice of the input function $F(a)$ [14,15].

III. SOLUTIONS

We will study a RS2 with AdS bulk, i.e., we will not consider here the dark radiation term. During inflation this term rapidly redshifts to zero [5] and, besides, since the decoupling of matter and radiation, this term could also be neglected. At early times during evolution ϵ is no longer a constant [20] and this term does not coincide with that in Eq. (1). We set the arbitrary integration constant in (4) $C = 0$, and the cosmological constant on the brane $\Lambda = 0$. Note that, if quadratic density contribution could be neglected (this seems to be the case since nucleosynthesis epoch), then $\Lambda \neq 0$ could be absorbed into the scalar field self-interaction potential. We consider, also, $k = 0$ to agree with the observational evidence about a spatially flat universe. Following the original paper [14], we set

$$F = Ba^s, \quad (8)$$

where B and s are arbitrary constants. By taking explicitly the integral (4), one finds $\rho_\phi = \rho_0 a^{s-6}$. It is worth noting that this functional dependence of the scale factor leads to the following ‘‘conservation equation’’: $\dot{\rho}_\phi + (6 -$

⁶We want to note here that, unlike in Ref. [14], in the present setup (RS2 with AdS bulk), we were not able to study arbitrary potentials but, more likely, we were able to identify the self-interaction potential after applying the method. So, in a sense, the functional form of the potential is also an outcome of the method.

$s)H\rho_\phi = 0$, so the equation of state $\omega_\phi = \gamma_\phi - 1 = \text{const} \Rightarrow 3\gamma_\phi = 6 - s = \text{const}$. This fact, in turn, leads to the ratio of the scalar field kinetic energy density and the total scalar field energy density being a constant: $\dot{\phi}^2/2\rho_\phi = \gamma_\phi/2 = (6 - s)/6 = \text{const}$.⁷ In consequence,

$$V(a) = \frac{s}{6} \rho_\phi = \frac{2 - \gamma_\phi}{2} \rho_0 a^{-3\gamma_\phi}. \quad (9)$$

Since we do not need to give the functional form of the self-interaction potential (as a function of ϕ) as an input, then, we do not organize the gallery of solutions by grouping under a given kind of potential (as in [14], for instance). Instead we study separately the cases when (i) both ordinary matter and the scalar field are sources of the Einstein equations on the brane, and (ii) only a scalar field fluid is trapped on the brane. The CFs are

$$G(a) = \frac{k_0}{3} \rho_T (1 + \sigma_b \rho_T), \quad L(a) = \gamma_\phi \rho_\phi. \quad (10)$$

In the case when we have ordinary matter and a scalar field fluid trapped on the brane, integrals (6) and (7) with CFs (10) can be explicitly found just for a few particular cases where one fixes a relationship among γ_ϕ and γ . For arbitrary relationships among the barotropic parameters, exact solutions can be found only for the two limiting situations: (a) the ‘‘low-energy’’ limit where $\rho_T \ll 1/\sigma_b$, and (b) the ‘‘high-energy’’ limit, where $\rho_T \gg 1/\sigma_b$. For a universe filled with just a scalar field, we are able to derive a single analytic solution holding ‘‘every time’’ during the cosmic evolution.

A. Universe filled with ordinary matter and a scalar field

1. Low-energy limit ($\rho_T \ll 1/\sigma_b \Rightarrow G(a) \approx \frac{k_0}{3} \rho_T$)

In this case we are neglecting brane effects so we recover a case that has been formerly studied in [14,15]. The integrals (6) and (7) can be explicitly found [14] (recall that $\rho_T = \rho_0 a^{-3\gamma_\phi} + M a^{-3\gamma}$):

$$\begin{aligned} \Delta t = & \sqrt{\frac{4}{3k_0\rho_0}} \frac{a^{3\gamma_\phi/2}}{\gamma_\phi} \\ & \times {}_2F_1 \left(\frac{1}{2}, \frac{\gamma_\phi}{2(\gamma_\phi - \gamma)}, \frac{3\gamma_\phi - 2\gamma}{2(\gamma_\phi - \gamma)}, -\frac{M}{\rho_0} a^{3(\gamma_\phi - \gamma)} \right), \end{aligned} \quad (11)$$

where ${}_2F_1$ is the hypergeometric function, and

$$\Delta \phi = \frac{2\sqrt{(6-s)/k_0}}{3(\gamma - \gamma_\phi)} \sinh^{-1} \left[\sqrt{\rho_0/M} a^{3(\gamma - \gamma_\phi)/2} \right]. \quad (12)$$

⁷In order to study scalar fields with a dynamical equation of state, one should consider the case when the integration constant $C \neq 0$ in Eq. (4).

In consequence, we can find the explicit form of the potential as a function of ϕ :

$$V(\phi) = V_0 \sinh^{2q}[\lambda \Delta \phi], \quad (13)$$

where

$$V_0 \equiv \left(\frac{2 - \gamma_\phi}{2}\right) \left(\frac{M^{\gamma_\phi}}{\rho_0^\gamma}\right)^{1/(\gamma_\phi - \gamma)}, \quad \lambda \equiv \frac{3(\gamma - \gamma_\phi)}{2\sqrt{3}\gamma_\phi/k_0}$$

and⁸

$$q = \gamma_\phi / (\gamma_\phi - \gamma).$$

As explained in [15], this potential is a good quintessential candidate to be the missing energy in the universe. Since it behaves like an inverse power-law potential at early times, then this allows one to avoid the fine-tuning problem. The parameters V_0 , λ , and q can be determined uniquely by the measured values for the equation of state and the amount of vacuum energy to obtain a tracker solution in such a way as to avoid the coincidence problem as well [15]. A good agreement with current observations of SNIa, Angular and Mass power spectrums was reported in Ref. [15].

2. High-energy limit ($\rho_T \gg 1/\sigma_b \Rightarrow G(a) \approx \frac{k_0}{3} \sigma_b \rho_T^2$)

We concentrate in this case since brane effects dominate. Let us introduce a new variable $X \equiv a^{3\gamma}$ and a constant parameter $m = 1 - \frac{\gamma_\phi}{\gamma}$; then the CFs take the following form:

$$\begin{aligned} L(X) &= \gamma_\phi \rho_0 X^{m-1}, \\ G(X) &= (k_0/3) \sigma_b \rho_0^2 X^{-2} (X^m + (M/\rho_0))^2. \end{aligned} \quad (14)$$

In terms of the new variable and constant parameter the integrals (2.6) and (2.7) look like

$$\Delta t = \frac{1}{\gamma \rho_0 \sqrt{3k_0 \sigma_b}} \int \frac{dX}{X^m + (M/\rho_0)}, \quad (15)$$

and

$$\Delta \phi = (1/\gamma) \sqrt{\frac{\gamma_\phi}{3k_0 \sigma_b \rho_0}} \int \frac{dX X^{(m-1)/2}}{X^m + (M/\rho_0)}, \quad (16)$$

respectively. These integrals can now be explicitly taken in the general case to yield

$$\Delta t = \frac{a^{3\gamma}}{\sqrt{3k_0 \sigma_b} \gamma M} {}_2F_1\left(1, \frac{\gamma}{\gamma - \gamma_\phi}, \frac{2\gamma - \gamma_\phi}{\gamma - \gamma_\phi}, -\frac{\rho_0}{M} a^{3(\gamma - \gamma_\phi)}\right), \quad (17)$$

and

⁸When the scalar field dominates the evolution, it is necessary to change $\gamma_\phi \rightarrow \gamma$ in the solution.

$$\begin{aligned} \Delta \phi &= \pm \frac{(2/M)}{2\gamma - \gamma_\phi} \sqrt{\frac{\gamma_\phi \rho_0}{3k_0 \sigma_b}} a^{3\gamma} \\ &\times {}_2F_1\left(1, \frac{2\gamma - \gamma_\phi}{2(\gamma - \gamma_\phi)}, \frac{4\gamma - 3\gamma_\phi}{2(\gamma - \gamma_\phi)}, -\frac{\rho_0}{M} a^{3(\gamma - \gamma_\phi)}\right), \end{aligned} \quad (18)$$

where the “+” sign should be taken if $2\gamma > \gamma_\phi$ and the “−” if the inverse inequality holds. This is a new exact solution to high-energy brane cosmology. Since it is general in the sense that the form of the self-interaction potential is not specified *a priori* and, besides, the method used to derive solutions is general and free of any assumption [except the assumed form of the input function $F(a)$], it generalizes former solutions, for instance, those in [4,5,16]. Besides, in [4,5], since the authors were interested in describing scalar driven inflation, only one (scalar field) fluid was assumed to be confined to the RS2 brane.

Although, in this case, we are not able to identify a self-interaction potential of a simple (analytic) form, we can judge about its asymptotic behavior. In fact, by taking the asymptotic of the hypergeometric function ${}_2F_1(a, b, c, z)$ for small $z \propto a^{3(\gamma - \gamma_\phi)}$ in Eq. (18), in the first approximation,⁹ one obtains that $\Delta \phi \propto a^{3\gamma}$. In consequence, at early times during the evolution, the behavior of the self-interaction potential is like $V(\phi) \propto (\Delta \phi)^{-\gamma_\phi/\gamma}$. This kind of potential has been studied in [5]. It has been established therein that inflation is possible for $\gamma_\phi > 2\gamma$ if

$$\phi < \phi_e \approx \left(\frac{k_0 \sigma_b \gamma^2}{\gamma_\phi^2} V_0\right)^{\gamma/(\gamma_\phi - 2\gamma)},$$

where

$$V_0 \equiv \frac{(\gamma_\phi - 2\gamma)M}{2} \sqrt{\frac{3k_0 \sigma_b}{\gamma_\phi \rho_0}}.$$

It follows then that, if the inverse of the brane tension σ_b ,

$$\sigma_b < \frac{2}{V_0} \left(\frac{\gamma_\phi}{\gamma \sqrt{2k_0}}\right)^{\gamma_\phi/\gamma},$$

the universe inflates for $\phi < \phi_e$.¹⁰ The observational constraints on a model of brane inflation based upon this kind of potential are discussed in detail in Ref. [5]. However, in that reference the authors impose the constraints supposing that this potential can take account of both early inflation and late-time inflation. We should note that, in the case of interest here, this kind of potential emerges just an asymptotic form of a more general potential, and it can take

⁹We are considering that the scale factor a is normalized in such a way that its present value $a_0 = 1$. In consequence, taking small z , i.e., small $a \ll 1$, is consistent with the high-energy limit we are considering in this subsection.

¹⁰Here we are considering the results of [5] but with appropriated identification of the constants with the ones in the present paper.

account only of the early inflation, so some of the constraints imposed in Ref. [5] might not apply in this case; so we get a wider range for the parameters. This point will be discussed in some detail in the next section.

In order to have simpler expressions and to obtain simple explicit functional dependence of V on ϕ , we can explore particular values of m in (14)–(16) (equivalent to taking fixed relationships among the barotropic parameters):

- (i) When $\gamma_\phi = \gamma \Rightarrow m = 0$ in (14) so the integrals (15) and (16) yield

$$\Delta t = \frac{a^{3\gamma}}{\sqrt{3k_0\sigma_b\gamma A}}, \quad (19)$$

and

$$\Delta\phi = \sqrt{\frac{4\rho_0}{3k_0\gamma\sigma_b}} \frac{a^{3\gamma/2}}{A}. \quad (20)$$

The self-interaction potential is a power-law one:

$$V(\phi) = V_0(\Delta\phi)^{-2}, \quad V_0 \equiv \frac{2(2-\gamma)\rho_0^2}{3k_0\sigma_b\gamma A^2}, \quad (21)$$

where, as before, $A = M + \rho_0$.

The case with $m = 1$ ($\gamma_\phi = 0$) is trivial. In this case we have a constant potential $V(\phi) = \rho_0 = \text{const}$. For the time evolution of the scale factor, we have, according to (15),

$$\Delta t = \frac{\ln[a^{3\gamma} + M/\rho_0]}{\sqrt{3k_0\sigma_b\gamma\rho_0}}. \quad (22)$$

For other fixed relationships among the γ s one has

- (ii) $m = -1$ ($\gamma_\phi = 2\gamma$).

In this case

$$\Delta t = \frac{1}{\sqrt{3k_0\sigma_b\gamma M}} \left(M a^{3\gamma} - \frac{\rho_0}{M} \ln[a^{3\gamma} + \rho_0/M] \right), \quad (23)$$

and

$$\Delta\phi = \frac{1}{M} \sqrt{\frac{2\rho_0}{3k_0\gamma\sigma_b}} \ln[a^{3\gamma} + \rho_0/M]. \quad (24)$$

The self-interaction potential is of the following form:

$$V(\phi) = \frac{V_0}{(\exp[\lambda \Delta\phi] - b)^2}, \quad (25)$$

where $V_0 \equiv (1-\gamma)\rho_0$, $\lambda \equiv M\sqrt{\frac{3k_0\sigma_b\gamma}{2\rho_0}}$ and the constant $b \equiv \rho_0/M$. Its asymptotic behavior is like in Sec. III A 2 [see discussion under Eq. (18)] for $\lambda \Delta\phi \ll 1$, $\gamma_\phi = 2\gamma$, and like in (13) for $\lambda \Delta\phi \gg 1$, $q = -1$. Therefore, this potential falls into

the category of ‘‘unified’’ potentials that will be discussed below.

- (iii) $m = -\infty$ ($\gamma = 0$, γ_ϕ arbitrary).

In this particular case the background fluid is a vacuum fluid. In order to avoid using the formal limit $m = -\infty$, it is convenient to change variables $X \rightarrow Y = X^{1-m}$ and $m \rightarrow n = \frac{m}{m-1}$. The integrals (6) and (7) [(15) and (16)] can be cast now, into the following form:

$$\Delta t = \frac{1}{\sqrt{3k_0\sigma_b M \gamma_\phi}} \int \frac{dY}{Y^n + (\rho_0/M)}$$

and

$$\Delta\phi = \frac{1}{M} \sqrt{\frac{\rho_0}{3k_0\sigma_b\gamma_\phi}} \int \frac{dY}{\sqrt{Y(Y^n + (\rho_0/M))}},$$

respectively. After integration for $n = 1$ ($m = -\infty$),

$$\Delta t = \frac{\ln[a^{3\gamma_\phi} + \rho_0/M]}{\sqrt{3k_0\sigma_b\gamma_\phi M}}, \quad (26)$$

$$\Delta\phi = \left(\frac{2}{\sqrt{3k_0\sigma_b\gamma_\phi M}} \right) \arctan \left[\sqrt{\frac{M}{\rho_0}} a^{3\gamma_\phi/2} \right], \quad (27)$$

and the self-interaction potential is of the following form:

$$V(\phi) = V_0 \tan^{-2}[\lambda \Delta\phi], \quad V_0 \equiv \left(\frac{2-\gamma_\phi}{2} \right) M, \\ \lambda \equiv \frac{\sqrt{3k_0\sigma_b\gamma_\phi M}}{2}.$$

This kind of potential has not been formerly studied under the brane perspective, but we leave it for further research.

B. Scalar field dominated solution

Let us now consider the case when the scalar field dominates the matter content. We recall that, since we set to zero the integration constant in Eq. (4), then the equation of state of the scalar field is a constant (meaning constant ratio of kinetic to total energy density of the scalar field). Let us introduce a new variable $Y \equiv a^{3\gamma_\phi}$, then the CFs are

$$G(Y) = A(Y+B)/Y^2, \quad L(Y) = CY^{-1}, \quad (28)$$

where the constants $A \equiv \frac{k_0\rho_0}{3}$, $B \equiv \sigma_b\rho_0$, and $C \equiv \gamma_\phi\rho_0$. Meanwhile, in terms of the new variable Y , the integrals (2.6) and (2.7) can be written as follows:

$$\Delta t = \frac{1}{3\gamma_\phi} \int \frac{dY}{Y\sqrt{G(Y)}}, \quad \Delta\phi = \frac{1}{3\gamma_\phi} \int \frac{dY}{Y} \sqrt{\frac{L(Y)}{G(Y)}}. \quad (29)$$

These integrals can be taken exactly to yield

$$\Delta t = \frac{2}{3\gamma_\phi\sqrt{A}}\sqrt{Y+B}, \quad (30)$$

and

$$\begin{aligned} \Delta\phi &= \frac{2}{3\gamma_\phi}\sqrt{\frac{C}{A}}\ln[\sqrt{Y} + \sqrt{Y+B}] \\ &= \frac{2}{3\gamma_\phi}\sqrt{\frac{C}{A}}\sinh^{-1}\left[\sqrt{\frac{Y}{B}}\right]. \end{aligned} \quad (31)$$

If one realizes that the self-interaction potential can be written as $V(Y) = V_0 Y^{-1}$, where $V_0 \equiv \frac{2-\gamma_\phi}{2\gamma_\phi} C$, then the last equation can be rewritten as

$$\frac{V}{(1 + \sqrt{1 + \frac{2\sigma_b}{2-\gamma_\phi} V})^2} = V_0 e^{-2\lambda\Delta\phi}, \quad (32)$$

where $\lambda \equiv 3\gamma_\phi\sqrt{\frac{A}{C}} = \sqrt{3\gamma_\phi k_0}$. Equation (31) can also be rewritten in a form where the functional form of the self-interaction potential is straightforward:

$$V(\phi) = \bar{V}_0 \sinh[\lambda \Delta\phi]^{-2}, \quad \bar{V}_0 \equiv \frac{2-\gamma_\phi}{2\sigma_b}. \quad (33)$$

This solution has been formerly obtained in [4].¹¹ It is interesting because it reduces to the power-law cosmology driven by an exponential potential at late times (in the low-energy limit); meanwhile, at early times the asymptotic behavior is like $a \sim t^{1/3\gamma_\phi}$, and inflation proceeds for a finite time for $\gamma_\phi < 1/3$ [4]. In this limit (high-energy limit), the self-interaction potential $V \propto \phi^{-2}$. Models with the same asymptotics have been studied in [12], to account for a unified description of early inflation and late-time accelerated expansion.

However, this simple model of the cosmic evolution cannot really account for a unified description of early inflation and quintessential inflation, since a scalar field with a nonevolving equation of state cannot describe correctly a period of cosmic evolution starting in an inflaton dominated regime, followed by a matter dominated regime (including a period of radiation domination) and leading to a cosmological constant dominated era. To have a scalar field with an evolving equation of state, one should consider the integration constant in (4) different from zero or, instead, to study other kinds of input function $F(a)$.

¹¹An appropriated identification of the constants should be done. The constant that appears in [4] $C^2 = 3\gamma_\phi$. Do not confound with the constant C in the present paper.

IV. UNIFIED DESCRIPTION OF INFLATION AND QUINTESSENCE

The main point we want to discuss here is the unified picture that comprises both early inflation and late-time (quintessential) acceleration of the expansion. This idea has been clearly stated in [13] for standard ‘‘nonbrane’’ models and, in brane contexts, for instance, in Refs. [5,11,12]. The fact is that the solutions we found with the help of the general method of Ref. [14] share the same asymptotics of the models studied in [4,5,11,12], which lead to such a unified approach to inflation. The most general solutions we found in the present paper were the limiting low-energy and ‘‘high-energy solutions given in Secs. III A 1 and III A 2, respectively, so we will concentrate our discussion in this case.¹² It is obvious that, to describe the early-time evolution, the high-energy solution is the adequate one. Although a simple behavior of the solution is not at hand, the asymptotic behavior for $a \ll 1$ leads to $V \propto (\Delta\phi)^{-\gamma_\phi/\gamma}$. This is the kind of asymptotics that is able to account for the stage of early inflation by appropriately choosing the free parameters, in order to agree with the observational constraints [4]. We can write this potential as follows:

$$\begin{aligned} V(\phi) &= \frac{M^{4+n}}{\phi^n}, \quad n \equiv \frac{\gamma_\phi}{\gamma}, \\ M^{4+n} &\equiv \left(\frac{2-\gamma_\phi}{2}\right) \left[\frac{(2/M)}{|2/n-1|} \frac{1}{\sqrt{3k_0\sigma_b\gamma_\phi}}\right]^n \rho_0^{1+n/2}. \end{aligned} \quad (34)$$

In a standard inflation chaotic scenario based on GR, inflation proceeds when $\alpha^2 \equiv (1/k_0)(V'/V)^2 \ll 2$ and ends when $\alpha^2 \approx 2$. However, in cosmological FRW brane models of RS2 type, one can introduce an effective slow-roll parameter [5]:

$$\bar{\alpha}^2 \equiv \frac{\alpha^2}{1 + \sigma_b\rho}. \quad (35)$$

Hence, if $\sigma_b\rho \gg 1$, there may be inflation even though $\alpha^2 \gg 2$. The end of inflation is characterized by the condition $\bar{\alpha}^2 \approx 1$, which in the case under study [$\alpha^2 = (n^2/k_0)(1/\phi^2)$] yields that the scalar field value at the end of inflation $\phi_f^{n-2} \approx k_0\sigma_b M^{4+n}/n^2$. The number of e-foldings between a scalar field value ϕ and ϕ_f , $N \approx -k_0\sigma_b \int_\phi^{\phi_f} d\phi V^2/V'$, can be written in the form of a relationship between ϕ and ϕ_f :

$$\left(\frac{\phi}{\phi_f}\right) = \frac{1}{[1 + \frac{(n-2)}{n}N]^{1/(n-2)}}. \quad (36)$$

¹²By ‘‘general’’ we understand here a solution to the field equations with two sources: ordinary fluid and scalar field fluid, which does not depend upon the relationship among the barotropic parameters of the background and scalar field fluids.

An observationally acceptable lower limit of the scalar spectral index ($n_s \approx 1 - [2(2n - 1)/n]/[1 + (\frac{n-2}{n}N)]$) is $n_s \approx 0.9$. This bound on n_s translates into a lower bound on n given a fixed N . Observations constrain the spectral tilt at $N \approx 50$. In this case $n > 7$ [5]. For bigger N this lower limit is smaller. For instance, for $N = 69$, $n > 3$. Another constraint imposed to the model is given by the following condition: $V(\phi) < m_5^4 = \frac{(32\pi^2)}{6\sigma_b k_0}^{2/3}$. Otherwise, the assumption that the inflaton field is confined to the brane may not be valid above the five-dimensional Planck scale [5]. The magnitude of the self-interaction potential $N = 50$ e-foldings before the end of inflation:

$$V_N \approx 8 \left[\frac{(600\pi^2)}{4k_0^4 \sigma_b^2} \frac{n}{2 + (n-2)N} \right]^{1/3} A_{s,\text{COBE}}^{2/3}, \quad (37)$$

where $A_{s,\text{COBE}} = 2 \times 10^{-5}$ is the value of the amplitude of scalar perturbations after constraints due to COBE normalization are considered. Thus, the smallness of the perturbations on COBE scales ensures that the aforementioned condition imposed to the self-interaction potential ($V < m_5^4$) is automatically satisfied and sufficient inflation is possible [5]. Up to this moment, we have discussed about the high-energy limit of the picture, which is suitable to describe early inflation.

On the opposite hand, to describe the late-time stage of the cosmic evolution, the most adequate is the low-energy solution of Sec. III A 1 that, as it has been discussed in [15], is a good quintessential candidate to be the missing energy in the universe. In this case, one chooses the appropriate values for the parameters of the potential in such a way that the solution of III A 1 can only be reached until the matter and the scalar field energies are of the same order, as it happens at present. Thus all parameters can be determined and the solution becomes a tracker one [15]. The behavior of the scalar field for radiation and matter dominated epochs is the same as that found for an inverse power-law potential in [21]. This low-energy model of late-time accelerated inflation was found to be in good agreement with observational evidence for SNIa, Angular and Mass power spectrums [15].

This is the way in which a “unified picture” of cosmic inflation emerges with the help of the general method of generating FRW solutions developed in Ref. [14], when applied to RS2 brane cosmologies (with a AdS bulk geometry). Besides, as in other papers on RS2 cosmological models (for instance [4]), we have not considered the so-called slow-roll approximation, or any other approximation, to describe the early-time inflation epoch.

We want to point out that, in the present case one should be careful about the meaning of “unification” between early and late-time inflation. Actually, a drawback of the present approach is that a consistent solution holding during the whole evolution is not found. Instead we are able to derive exact solutions that hold asymptotically, i.e., during

a given period of the cosmic evolution (either early-time high-energy solution, or late-time low-energy solution). A smooth matching of both pieces of the evolution is lacking. In this sense it is difficult to achieve an actually unified description.

V. CONCLUSIONS

We have used a fairly general method to derive exact cosmological FRW solutions in RS2 brane models with a AdS bulk geometry. The only assumption is given by the choice of the input function $F(a)$. The method was developed in [14], where it was formerly used to generate solutions only in standard GR.

We found new exact solutions that generalize formerly known solutions [4,5,11,12,16] and were able to recover other well-known solutions like in [4,14,15]. The most interesting fact is that the solutions found can accommodate both the inflaton field and quintessence in a common framework (see discussion in the former section). Since, as stated above, the method used to generate cosmological solutions is quite general, it seems that such a description of inflation and quintessence is a generic property of brane inflation cosmological models.

As it was noted in the original paper [14], and has been discussed, for instance in [4], the limitations are clearly imposed by the slow-roll approximation, and most of the solutions we found cannot be obtained by using this approximation. In this sense, the models that emerge, and that are able to describe early-time inflation, are more general than those obtained within the framework of slow-roll inflation.

The method we have used here could be applied to other brane cosmological models and, also, other general input functions [other than $F(a) = Ba^s$ in (8)] could be considered, in order to obtain other solutions (and, correspondingly, other self-interaction potentials) than those obtained in the present paper. Precisely, one of the drawbacks of the present approach is that the most interesting solution, that containing both scalar field and ordinary matter as sources, is not a single solution accounting for the complete cosmic evolution, but consists of two separated pieces: high-energy and low-energy asymptotic solutions, respectively. The matching of both pieces of the cosmic evolution is not clear. In this sense, the study of scalar fields with a dynamical equation of state is of relevance and this is possible only under choices of the input function other than the one taken here, or by relaxing the choice of a zero value for the integration constant in the first integral of the scalar field energy density.

ACKNOWLEDGMENTS

T. M. and I. Q. acknowledge partial financial support by CONACYT México, under Grants No. 32138-E and

No. 42748. T. M. also thanks the support of the Germany-México bilateral project DFG-CONACYT 444 MEX-13/

17/0-1. I. Q. and A. G. acknowledge the MES of Cuba for partial financial support of this research.

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