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ϕ^2 as dark matter

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ABSTRACT

In this paper, we consider ϕ^2 scalar field potential as a candidate to dark matter. If it is an ultralight boson particle, it condensates like a Bose–Einstein system at very early times and forms the basic structure of the Universe. Real scalar fields collapse in equilibrium configurations which oscillate in space–time (oscillatons).The cosmological behaviour of the field equations are solved using the dynamical system formalism. We use the current cosmological parameters as constraints for the present value of the scalar field and reproduce the cosmological predictions of the standard Λ cold dark matter model with this model. Therefore, scalar field dark matter seems to be a good alternative to cold dark matter nature.

Key words: cosmology: theory – dark matter.

1 INTRODUCTION

Scalar fields are one of the most interesting and most mysterious fields in theoretical physics. Fundamental scalar fields are needed in all unification theories; however, there is no experimental evidence of its existence. From the standard model of particles which needs the Higgs boson, to the superstring theory which contains the dilaton, passing through the Kaluza–Klein and the Brans–Dicke theories or through the inflationary model, scalar fields are necessary fields. Doubtless, if they exist, they have some features which make them very special.

The Scalar Field Dark Matter (SFDM) model paradigm has been constructed step by step. One of the first suggestions that a (complex) scalar field could contribute to structure formation of the Universe was given by Press, Ryden & Spergel (1990) and Madsen (1992). Nevertheless, complex scalar fields were used before as matter candidates as boson stars by Ruffini (1969) [for a recent introduction to boson stars, see for example Guzmán (2006)]. One of the first candidates to be scalar field dark matter is the axion, one of the solutions to the strong-charge–parity problem in quantum chromodynamics (see an excellent review in Kolb & Turner 1990). Essentially, the axion is a scalar field with mass restricted by observations to $\sim 10^{-5}$ eV, which has its origin at 10^{-30} s after the big bang, when the energy of the Universe was 10^{12} GeV. This candidate is till now one of the most accepted candidates for the nature of dark matter, if its abundance is about 10^9 particles per cubic centimetre.

The first in suggesting that a dark halo could be a Bose–Einstein condensate (BEC) were Sin (1994) and Ji & Sin (1994), who used the weak field limit to show that a BEC with several nodes can fit the rotation galaxy curves with a very good accuracy. Further investigations in this direction were performed by Lee & Koh (1996), where they incorporated ϕ^4 interactions to the scalar field potential and used the Gross–Pitaevskii equation instead of the Schrödinger one (Lee 2008). Nevertheless, Seidel & Suen (1991, 1994) showed that when the whole BEC is in the ground state, many nodes in the Einstein–Klein–Gordon fields are unstable, since they evolve into the 0-node solution after a while [for a clear explanation to this point see also Guzmán & Ureña-López (2003)]. Thus, the static solutions given by Sin (1994), Ji & Sin (1994) and Lee & Koh (1996) are expected to be unstable.

Later on, Peebles & Vilenkin (1999) proposed that a scalar field driven by inflation can behave as a perfect fluid and can have interesting observational consequences in structure formation. Besides that, they performed a sound waves analysis of this hypothesis giving some qualitative ideas for the evolution of these fields and called it fluid dark matter (Peebles 2000a,b). Independently and in an opposite way, Matos & Guzmán (1999) proposed a scalar field coming from some unification theory can condensate and collapse to form haloes of galaxies. Very early, this scalar field behaves as a perfect fluid, however its ultralight mass means that the bosons condensate at very high temperature and collapse in a very different way to the fluid dark matter of Peebles & Vilenkin (1999). They were able to fit reasonably rotation curves of some galaxies using an exact solution of the Einstein equations with an exponential potential (Matos & Guzmán 1999; Guzmán & Matos 2000; Bernal, Matos & Núñez 2008). The first cosmological study of the SFDM was performed in Matos & Ureña-López (2000a,b) where a cosh scalar field potential was used. The cosmology reproduces all features of the Λ cold dark matter (ΛCDM) model in the linear regime of perturbations.

On the other hand, Lesgourgues, Arbey & Salati (2002) and Arbey, Lesgourgues & Salati (2003) used a complex scalar field with a quartic potential $m^2\phi\phi^\dagger + \lambda(\phi\phi^\dagger)^2$ and solved perturbation

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equations (weak field limit approximation) to fit the rotational curves of dwarf galaxies with a very good accuracy, provided that $m^4/\lambda \sim 50$–75 eV^4.

The importance of scalar fields in the dark sector has been increased; for instance, several authors have investigated the unification of dark matter and dark energy in a single scalar field (Padmanabhan & Choudhury 2002; Arbey, Lesgourgues & Salati 2003; Bertacca, Bartolo & Matarrese 2008). Recently Liddle & Ureña-López (2006) and Liddle, Cédric & Ureña-López (2008) proposed that the landscape of superstring theory can provide the Universe with a $\phi^2 + \Lambda$ scalar field potential. Such a scalar field can inflate the Universe during its early epoch; after that, the scalar field can decay into dark matter. The constant Λ can be interpreted as the cosmological one. This model could explain all unknown components of the Universe in a simple way. Another interesting model in order to explain the scalar fields unification, dark sector and inflation, uses a complex scalar field protected by an internal symmetry (Pérez-Lorenzana, Montesinos & Matos 2008).

In the present work, the main idea is that if scalar fields are fundamental, they live as unified fields in some very early moment at the origin of the Universe. As the Universe expands, the scalar fields cool together with the rest of the particles until they decouple from the rest of the matter. After that, only the expansion of the Universe will keep cooling the scalar fields. If the scalar field fluctuation is under the critical temperature of condensation, the object will collapse as a BEC. After inflation, primordial fluctuations cause the scalar fields to collapse and form haloes of galaxies and galaxy clusters. The cooling of scalar fields continues till the fluctuation separates from the expansion of the Universe.

In this work, we study the most simple model of SFDM, using a ϕ^2 scalar field potential. In Sections 2 and 3 we review the statistical description of a boson gas condensating to form a BEC, focusing on the necessary features for the BEC to form a halo of a galaxy and integrating the Einstein equations with BEC matter. In Section 4, we transform the Einstein field equations in to a dynamical system, then we numerically integrate them and look for the attractor points. We give some conditions on how these field equations can show the right behaviour to reproduce the Universe we observed. Finally, in Section 5, we conclude that this SFDM model could explain the dark matter of the Universe.

2 THE STATISTICS OF A BEC

In this section, we review the condensation of an ideal Bose gas of N particles with mass m contained in a volume V with temperature T and with only a portion ρ_0 of the system in the ground state. In order to see that and to be self-contained, let us start from its grand partition function $\mathcal{Q}$, which is given by

$$\mathcal{Q}(z, V, T) = \prod_{\boldsymbol{p}} \frac{1}{1 - z\mathrm{e}^{-\beta\epsilon_{\boldsymbol{p}}}}, \tag{1}$$

where the fugacity $z \equiv \mathrm{e}^{\beta\mu}$ is defined in terms of the chemical potential μ and $\beta \equiv 1/T$. In this paper, we use the fundamental constants $\hbar = c = k_B = 1$.

Then, the equation of state for an ideal Bose gas is

$$\frac{PV}{T} = \log \mathcal{Q} = -\sum_{\boldsymbol{p}} \log(1 - z\mathrm{e}^{-\beta\epsilon_{\boldsymbol{p}}}). \tag{2}$$

Thus, the grand partition function gives the pressure P directly as a function of z, V and T.

On the other hand, the particle number N and the internal energy U are

$$N = z\frac{\partial}{\partial z} \log \mathcal{Q} = \sum_{\boldsymbol{p}} \frac{z\mathrm{e}^{-\beta\epsilon_{\boldsymbol{p}}}}{1 - z\mathrm{e}^{-\beta\epsilon_{\boldsymbol{p}}}}, \tag{3}$$

$$U = -\frac{\partial}{\partial \beta} \log \mathcal{Q} = \sum_{\boldsymbol{p}} \frac{\epsilon_{\boldsymbol{p}} z\mathrm{e}^{-\beta\epsilon_{\boldsymbol{p}}}}{1 - z\mathrm{e}^{-\beta\epsilon_{\boldsymbol{p}}}}, \tag{4}$$

where $\epsilon_{\boldsymbol{p}}$ is the single-particle energy with momentum $\boldsymbol{p}$ and the average occupation numbers $\langle n_{\boldsymbol{p}}\rangle$ are given by

$$\langle n_{\boldsymbol{p}}\rangle = \frac{z\mathrm{e}^{-\beta\epsilon_{\boldsymbol{p}}}}{1 - z\mathrm{e}^{-\beta\epsilon_{\boldsymbol{p}}}}, \tag{5}$$

which satisfy the conditions

$$N = \sum_{\boldsymbol{p}} \langle n_{\boldsymbol{p}}\rangle, \tag{6}$$

$$U = \sum_{\boldsymbol{p}} \epsilon_{\boldsymbol{p}} \langle n_{\boldsymbol{p}}\rangle. \tag{7}$$

Now we let $V \to 0$ take the limit of continuity and replace sums over $\boldsymbol{p}$ by integrals over $\boldsymbol{p}$, and we obtain the following equation of state

$$\begin{aligned} \frac{PV}{T} &= -\frac{2V}{(2\pi)^2} \int_0^\infty \mathrm{d}p\, p^2 \log(1 - z\mathrm{e}^{-\beta p^2/2m}) - \log(1 - z), \\ N &= \frac{2V}{(2\pi)^2} \int_0^\infty \mathrm{d}p\, p^2 \frac{z\mathrm{e}^{-\beta p^2/2m}}{1 - z\mathrm{e}^{-\beta p^2/2m}} + \frac{z}{1 - z}. \end{aligned} \tag{8}$$

These equations can be written into the equivalent form

$$\frac{PV}{T} = \frac{V}{\lambda^3} g_{5/2}(z) - \log(1 - z), \tag{9}$$

$$N = \frac{V}{\lambda^3} g_{3/2}(z) + \frac{z}{1 - z}, \tag{10}$$

where $\lambda = \sqrt{2\pi/mT}$ is the thermal wavelength and

$$\begin{aligned} g_{5/2}(z) &= -\frac{4}{\sqrt{\pi}} \int_0^\infty \mathrm{d}x x^2 \log(1 - z\mathrm{e}^{-\beta x^2}), \\ g_{3/2}(z) &= z\frac{\partial}{\partial z} g_{5/2}(z). \end{aligned} \tag{11}$$

Moreover, the internal energy is found from the formulas (2) and (4),

$$U = \frac{3}{2} \frac{TV}{\lambda^3} g_{5/2}(z), \tag{12}$$

and as consequence the relation $U = 3/2PV$ is fulfilled.

From equation (5), we see that

$$\langle n_0\rangle = \frac{z}{1 - z}, \tag{13}$$

which is the average occupation number for a single particle with occupation level $\boldsymbol{p} = 0$. Equation (10) can also be written as

$$\lambda^3 \frac{\langle n_0\rangle}{V} = \lambda^3 \frac{N}{V} - g_{3/2}(z). \tag{14}$$

This equation tell us that $\frac{\langle n_0\rangle}{V} > 0$ and therefore the temperature and the specific volume are such that $\lambda^3 \frac{N}{V} > g_{3/2}(z)$. This means that a finite fraction of the particles will be in the ground state with $\boldsymbol{p} = 0$, that is, the Bose gas condensates. In the region of condensation, the fugacity $z \sim 1$ and the function $g(z)$ goes to the Riemann ζ function $g_l(z) \rightharpoonup \zeta(l)$.

The thermodynamical surface which separates the condensation region from the rest of the $P - V - T$ space is given by

$$\lambda_c^3 \frac{N}{V} = g_{3/2}(1) = 2.612, \tag{15}$$

thus λ_c can be interpreted as the value for which the thermal wavelength is of the same order of magnitude as the average interparticle separation. Equation (15) defines the critical temperature for which the Bose Condensate forms. This temperature is given by

$$T_c = \frac{2\pi}{m_\phi^{5/3}} \left(\frac{\rho}{g_{3/2}(1)} \right)^{2/3}, \tag{16}$$

where $\rho = m_\phi N/V$ is the density of the Bose gas. At constant temperature, equation (16) defines a critical density

$$\rho_c = \frac{m_\phi g_{3/2}(z)}{\lambda^3}. \tag{17}$$

Thus, the region of condensation of the Boson gas is determined by $T < T_c$ or $\rho > \rho_c$.

After the Bose gas condensates most of the bosons lie in the ground state, the scalar field starts to oscillate around the minimal of its potential and the scalar field starts to behave as dust (Turner 1983). Thus, after the scalar field decouples from the rest of the matter, the temperature of the BEC goes like

$$T_{\rm BEC} = T_{\rm BEC}^{(0)} \left(\frac{a_0}{a} \right)^2, \tag{18}$$

where $T_{\rm BEC}^{(0)}$ is the actual temperature of the BEC, a is the scale factor of the Universe and $a_0 = 1$ is the value of the scale factor at present.

In the same way, as the BEC behaves as matter, its density goes like $\rho_{\rm BEC} = \rho_{\rm BEC}^{(0)}/a^3$, where $\rho_{\rm BEC}^{(0)}$ is the actual matter content of BEC in the Universe. With this result, equation (16) can also be transformed into

$$T_c = \frac{2\pi}{m_\phi^{5/3}} \left(\frac{\Omega_{\rm BEC}^{(0)} \rho_{\rm crit}}{\zeta(3/2)} \right)^{2/3} \frac{1}{a^2}, \tag{19}$$

$$= 6.2 \times 10^{-31} \frac{\left(\Omega_{\rm BEC}^{(0)} h^2\right)^{2/3}}{(m_\phi/{\rm GeV})^{5/3}} \frac{1}{a^2}\,{\rm GeV}, \tag{20}$$

where $\Omega_{\rm BEC}^{(0)}$ is the actual rate of BEC, $\rho_{\rm crit}$ is the critical density of the Universe, $h \equiv H_0/(100\,{\rm km\,s^{-1}\,Mpc^{-1}})$, H_0 being the actual value of Hubble's parameter.

If the actual standard model of particles could be extended to higher temperatures, we would have to expect that the scalar field which forms the BEC interacts with the rest of the particles to a temperature over some temperature T_s. Because the physics of elemental particles is well known up to temperatures of GeV, we do not expect that any exotic particles as these scalar fields appear under temperatures of TeV. Here, we have two possibilities. The first one is that the scalar field has never had interaction with the rest of the particles and it evolves independently from the rest of the fields, with only a gravitational interaction. In this case, the scalar field condensates at the beginning of the Universe. The second possibility is that in the early Universe the scalar field lived unified with the rest of the particles in a thermal bath and at some moment during its evolution separates from the interaction. If this is the case let us suppose here that the scalar field which forms the BEC decouples from the rest of the matter at a temperature higher than TeV. Beneath this temperature, the scalar field has almost no interaction with the rest of the matter. If we expect that this scalar field forms a BEC, its critical temperature must be lower than the temperature of the scalar field decoupling. This fact gives us an upper bound of the mass m_ϕ of the scalar field

$$m_\phi < 10^{-17}\,{\rm eV}. \tag{21}$$

On the other hand, from numerical simulations (Seidel & Suen 1991) we know that scalar fields form gravitationally bounded objects with a critical mass given by

$$M_{\rm crit} \sim \tilde{m} \frac{m_{\rm pl}^2}{m_\phi}, \tag{22}$$

where $m_{\rm pl}$ is the Planck mass and $\tilde{m}$ is a factor such that $\tilde{m} \approx 0.6$ for both complex scalar fields (boson stars) and real scalar fields (oscillatons). With the value given in (21), the scalar field can form a gravitationally bounded BEC with a critical mass given by

$$M_{\rm crit} > 1.491 \times 10^{64}\,{\rm GeV}, \tag{23}$$

$$= 2.658 \times 10^{40}\,{\rm gr}, \tag{24}$$

$$= 13.36 \times 10^6\,{\rm M}_\odot. \tag{25}$$

This is an interesting result; if there exists a scalar field and it plays any role in the Universe at this moment, it must have a mass lower than the mass given in (21) and must be forming gravitationally bounded BECs with masses around the mass given in Alcubierre et al. (2002).

3 SELF-GRAVITATING BEC

In this section, we give some general features of the gravitational collapse of the BEC; we only pretend to show a generic behaviour of any self-gravitating BEC. The BEC cosmology have been studied by Fukuyama, Masahiro & Tatekawa (2008) and many numerical simulations of this collapse are given in Alcubierre et al. (2002), Guzmán & Ureña-López (2004, 2006) and besides. Guzmán & Ureña-López (2003) found that a BEC in the ground state is very stable under different initial conditions. After the Bose gas condenses the gravitational force makes the gas collapse and form self-gravitating objects. Let us suppose that the halo is spherically symmetric, which could not be too far from the reality. In that case, the space–time metric reads

$${\rm d}s^2 = -{\rm e}^{2\nu}\,{\rm d}t^2 + \frac{{\rm d}r^2}{1-(2MG/r)} + r^2\,{\rm d}\Omega^2, \tag{26}$$

where the function $\nu = \nu(r)$ is essentially the Newtonian potential and $M = M(r)$ is the mass function given by

$$\begin{aligned} M &= 4\pi \int \rho\, r^2\, {\rm d}r, \\ \frac{{\rm d}\nu}{{\rm d}r} &= G \frac{M + 4\pi r^3 P}{r^2\,[1-(2MG/r)]}. \end{aligned} \tag{27}$$

The Einstein field equations reduce to equations (27) and the Oppenheimer–Volkov equation

$$\frac{{\rm d}P}{{\rm d}r} = -G \frac{(P+\rho)(M + 4\pi r^3 P)}{r^2[1-(2MG/r)]}. \tag{28}$$

Let us focus on the case when the gas is far from forming a black hole. In that case, we suppose that $2MG \ll r$ and equation (28) reduces to

$$\frac{{\rm d}P}{{\rm d}r} = -4\pi G\, r\, P(P+\rho). \tag{29}$$

The equation of state can be obtained from the equation $PV = 2/3\,U$, (10) and (12). Combining all equations we obtain that

$$P = \frac{2\pi}{m_\phi^{8/3}} \frac{g_{5/2}(z)}{g_{3/2}(z)^{5/3}} (\rho - \rho_0)^{5/3} \tag{30}$$

$$= \omega(\rho - \rho_0)^{5/3}, \tag{31}$$

where ω is the constant,

$$\omega \equiv \frac{2\pi}{m_\phi^{8/3}} \frac{g_{5/2}(z)}{g_{3/2}(z)^{5/3}}, \tag{32}$$

and $\rho_0 = m_\phi \langle n_0 \rangle / V$ is the mean density of the particles in the ground state. Thus, the Oppenheimer–Volkov equation (28) transforms into

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\frac{12}{5}\pi G r(\rho - \rho_0)\left[\omega(\rho - \rho_0)^{5/3} + \rho\right]. \tag{33}$$

This differential equation can be easily numerically solved. Nevertheless, we have two interesting limits of equation (33). First, suppose that the ω constant is small, such that $P \ll \rho$. This situation occurs for big scalar field masses $m_\phi \sim m_{\mathrm{Planck}}$. In that case, the equation (33) contains an analytical solution given by

$$\rho(r) = \frac{\rho_0}{1 - \{1 - [\rho_0/\rho(0)]\}\,\mathrm{e}^{-\frac{6}{5}\pi G \rho_0 r^2}}, \tag{34}$$

here $\rho(0)$ is the central density of the BEC. Observe that when $\rightharpoonup\infty$, the function $\rho(r) \rightharpoonup \rho_0$. For numerical convenience, we set $\rho(0) = \epsilon\rho_0$ in the plot, ϵ being a constant. The function changes dramatically for different values of ϵ. If $\epsilon > 1$, the density $\rho(r)$ decreases, but if $\epsilon < 1$ the density increases. The behaviour of the density is shown in Fig. 1. This means that if the central density of the BEC is greater than the density of the ground state, we have the upper profile in Fig. 1, but if it is less than it we have the bottom profile.

The second and, for us, more interesting limit of equation (33) is when $P \gg \rho$. This occurs when the scalar field mass is small enough, $m_\phi \ll m_{\mathrm{Planck}}$, as for astrophysical BEC. In this limit, the Oppenheimer–Volkov equation also has an analytical solution given by

$$\rho(r) = \frac{\rho(0) - \rho_0}{\{2\pi G r^2 \omega[\rho(0) - \rho_0]^{5/3} + 1\}^{3/5}} + \rho_0,$$
$$= \left[\frac{p(0)/\omega}{2\pi G r^2 p(0) + 1}\right]^{3/5} + \rho_0, \tag{35}$$

or equivalently $P = 1/[2\pi G r^2 + 1/P(0)]$. In this case, the pressure dominates over the density of the BEC. The pressure acquires a maximum at the origin $r = 0$. Far enough away from the centre of the BEC we can approximate equation (35) with

$$\rho = \left(\frac{1/\omega}{2\pi G r^2}\right)^{3/5} + \rho_0, \tag{36}$$

which implies a space–time metric for the BEC given by

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1 - 2\left(r_0 r^{4/5} + \frac{4}{3}\pi G \rho_0 r^2\right)} - \exp(2\nu)\,\mathrm{d}t^2 + r^2\,\mathrm{d}\Omega^2, \tag{37}$$

where $r_0 \equiv 10/9(4\pi^2/\omega^3)^{1/5}$. Function ν determines the circular velocity (the rotation curves) V_{rot} of test particles around the BEC. Using the geodesic equation of metric (37), one obtains that $V_{\mathrm{rot}}^2 = r g_{tt,r}/(2 g_{tt}) = r\nu'$ (Matos, Guzmán & Núñez 2000). Using equations (27), we can integrate the function ν and obtain the rotation curves. The plot is shown in Fig. 2, where we see that the form of the rotation curves are analogous as expected from those observed in galaxies, especially in low surface brightness (LSB) and dwarf ones (de Blok & Bosma 2002; de Blok, Bosma & McGaugh 2003; Simon et al. 2005); also, SFDM predicts a core density profile that could have some astrophysical advantages (Sánchez-Salcedo, Reyes-Iturbide & Hernandez 2006) over the standard model (cuspy profiles). However, the discussion of the central region of the rotation curves continue. This is the main reason why it is not convenient to try self-gravitating BECs in the Newtonian limit. It remains that the Newton theory can be derived from the Einstein one for slow velocities, weak fields and pressures much smaller than the densities. However, these last conditions are not fulfilled in self-gravitating BEC.

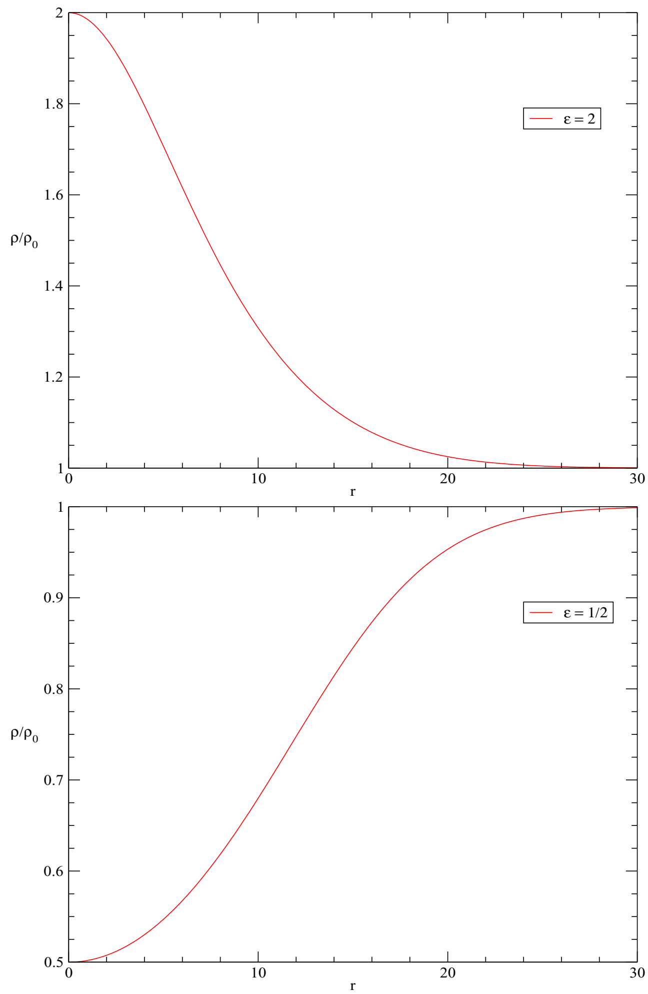

Figure 1. Plot of the $\rho(r)$ function given in equation (34) for $\epsilon > 1$ (top plot) and for $\epsilon < 1$ (down plot). The plot is done in terms of $\rho(r)/\rho_0$. We have set $\epsilon = 2$ and $\epsilon = 1/2$ for each plot, respectively, and $\rho_0 = 0.002$.

From these results and from the simulations given in Guzmán & Ureña-López (2003), it follows a novel paradigm for structure formation that is different from the bottom-up one. In the SFDM paradigm, after the big bang the scalar field expands until it decouples from the rest of the matter. If the scalar field has sufficiently small mass that its critical temperature of condensation is less than the temperature of decoupling, the scalar field forms a BEC. Then the scalar field collapses, forming objects whose final mass is not bigger than the critical mass $m_{\mathrm{Planck}}^2/m_\phi$. These objects contain a density profile very similar to the profile shown in the top panel of Fig. 1. They are very stable under perturbations. It has been proposed

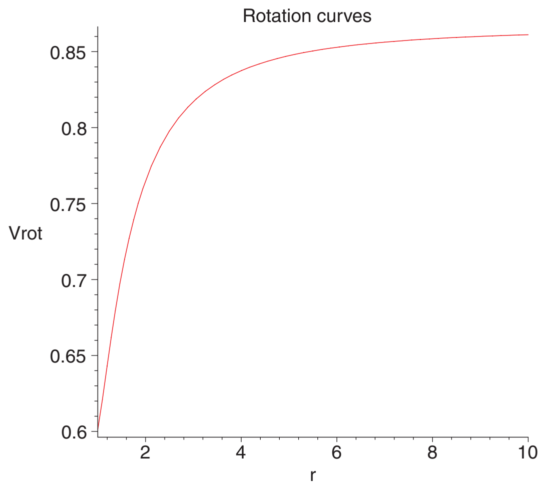

Figure 2. Rotation curve derived from metric (37). The velocity and the coordinate r are in arbitrary units.

that the dark matter in galaxies and clusters is a scalar field with a mass of 10^{-22} eV (Alcubierre et al. 2002). If this were the case, the main difference for the structure formation of this ultralight scalar field with the bottom-up paradigm is that the SFDM objects form just after the collapse of the scalar field and remain unchanged during the rest of the Universe expansion. Furthermore, they can collide together but after the collision the objects remain unaltered, since they behave like solitons (Bernal & Guzmán 2006). This means that in a merging of BECs they pass through each other without any alteration in total mass, as collisionless dark matter. This paradigm implies then that we must be able to see well-formed galaxies with actual masses for very large redshifts, longer than those predicted by the bottom-up paradigm, that is, by CDM. In this sense, some authors (Cimatti et al. 2004) suggest a discrepancy between the observed population of massive spheroidal galaxies at high redshift with the numerical simulations of hierarchical merging in a ΛCDM scenario that underpredict this population. However, the discussion continues because other physical processes, such as feedback, could have important effects in this galaxies.

4 THE COSMOLOGY

In this section, we review the cosmology given by a SFDM model with two different scalar field potentials: $V(\phi) = \frac{1}{2}m^2\phi^2$ and $V(\phi) = V_0[\cosh(\kappa\lambda\phi) - 1]$, where m is the mass of the boson particle, V_0 and λ are free parameters fixed with cosmological data and $\kappa^2 = 8\pi G$. Based on the current observations of 5-yr *WMAP* (*Wilkinson Microwave Anisotropy Probe*) data (Hinshaw et al. 2008) we will consider a universe evolving in a spatially flat Friedmann Lemaître–Robertson–Walker spacetime. We assume that this universe contains a real scalar field (ϕ) as dark matter, radiation (r), neutrinos (ν), baryons (b) and a cosmological constant (Λ) as dark energy.

The total energy density of a homogeneous scalar field is given by

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi),$$

the radiation and baryonic components are represented by perfect fluids with baryotropic equation of state $p_\gamma = (\gamma - 1)\rho_\gamma$, where γ is a constant, $0 \le \gamma \le 2$. For example, for radiation and neutrinos ($\gamma_{r,\nu} = 4/3$), for baryons ($\gamma_b = 1$) and finally for a cosmological constant ($\gamma_\Lambda = 0$).

Thus, the field equations for a universe with these components are given by

$$\begin{aligned} \dot{H} &= -\frac{\kappa^2}{2}(\dot{\phi}^2 + \gamma\rho_\gamma), \\ \ddot{\phi} &+ 3H\dot{\phi} + \partial_\phi V = 0, \\ \dot{\rho}_\gamma &+ 3\gamma H \rho_\gamma = 0, \end{aligned} \tag{38}$$

and the Friedmann equation

$$H^2 = \frac{\kappa^2}{3}\left(\rho_\gamma + \frac{1}{2}\dot{\phi}^2 + V(\phi)\right). \tag{39}$$

In order to analyze the behaviour of the different components of this universe, we are going to use the dynamical system formalism found in Appendix A.

4.1 The ϕ^2 scalar potential

We start our cosmological analysis of SFDM by taking the potential

$$V(\phi) = \frac{1}{2}m^2\phi^2, \tag{40}$$

and developing the standard procedure to transform it into a dynamical system. For doing so, the new variables (A2) for the system of equations (38) read

$$\begin{aligned} x &\equiv \frac{\kappa}{\sqrt{6}}\frac{\dot{\phi}}{H}, \quad u \equiv \frac{\kappa}{\sqrt{6}}\frac{m\phi}{H}, \\ z_\gamma &\equiv \frac{\kappa}{\sqrt{3}}\frac{\sqrt{\rho_\gamma}}{H}. \end{aligned} \tag{41}$$

Using the definitions given in (41), the evolution equations (38) for potential (40) transform into an autonomous system

$$\begin{aligned} x' &= -3x - \frac{m}{H}u + \frac{3}{2}\pi x, \\ u' &= \frac{m}{H}x + \frac{3}{2}\pi u, \\ z'_\gamma &= \frac{3}{2}(\pi - \gamma)\, z_\gamma, \\ -\frac{\dot{H}}{H^2} &= \frac{3}{2}(2x^2 + \gamma z_\gamma^2) \equiv \frac{3}{2}\pi, \end{aligned} \tag{42}$$

where, as in Appendix A, prime denotes a derivative with respect to the e-folding number $N = \ln(a)$. Again the choice of phase-space variables (41) transforms the Friedmann equation into a constraint equation,

$$F \equiv x^2 + u^2 + z_\gamma^2 = 1. \tag{43}$$

Because we are considering an expanding universe, which implies that $H > 0$, and from the variable definitions (41), we can see that $u, z_\gamma \ge 0$. With these variables, the density parameters can be written as

$$\begin{aligned} \Omega_{\rm DM} &= x^2 + u^2, \\ \Omega_\gamma &= z_\gamma^2, \\ \Omega_\Lambda &= l^2, \end{aligned} \tag{44}$$

where we have added explicity a cosmological constant variable $l \equiv z_\Lambda$. Moreover, with the physical constraint $0 \le \Omega \le 1$ and the Friedmann equation $\Omega_{\rm DM} + \Omega_\gamma + \Omega_\Lambda = 1$, the variable space is bounded by

$$0 \le x^2 + u^2 + z_\gamma^2 + l^2 \le 1.$$

On the other hand, observe that the variable space (42) is not a completely autonomous one, because H is an external parameter. In order to close the system, we define a new variable s given by

$$s \;\equiv\; \frac{m}{H}, \tag{45}$$

the dynamical equation (A3d) of which is

$$s' = \frac{3}{2}\pi s.$$

With this new variable, system (42) is now an autonomous one. The whole closed system is

$$x' = -3x - su + \frac{3}{2}\pi x, \tag{46a}$$

$$u' = s\,x + \frac{3}{2}\pi u, \tag{46b}$$

$$z_\gamma' = \frac{3}{2}(\pi - \gamma)\, z_\gamma, \tag{46c}$$

$$l' = \frac{3}{2}\pi l, \tag{46d}$$

$$s' = \frac{3}{2}\pi s. \tag{46e}$$

In order to acquire the geometrical information that dynamical system analysis provides (see Appendix A), we study the stability of (46). To do this, we define the vector $\boldsymbol{x} = (x, u, z_\gamma, l, s)$ and consider a linear perturbation of the form $\boldsymbol{x} \to \boldsymbol{x}_c + \delta\boldsymbol{x}$. The linearized system reduces to $\delta\boldsymbol{x}' = \mathcal{M}\delta\boldsymbol{x}$, where $\mathcal{M}$ is the Jacobian matrix of $\boldsymbol{x}'$ and reads as

$$\mathcal{M} = \begin{pmatrix} \frac{3}{2}\pi - 3 + 6x^2 & -s & 3\gamma x\, z & 0 & -u \\ 6x\, u + s & \frac{3}{2}\pi & 3\gamma\, u\, z & 0 & x \\ 6x\, z & 0 & \frac{3}{2}\pi + 3\gamma z^2 - \gamma & 0 & 0 \\ 6x\, l & 0 & 3\gamma\; l\; z & \frac{3}{2}\pi & 0 \\ 6x\, s & 0 & 3\gamma\, s\, z & 0 & \frac{3}{2}\pi \end{pmatrix}.$$

The equilibrium points $\boldsymbol{x}_c$ of the phase space $\{x,\, u,\, z_\gamma,\, l,\, s\}$, considering only $\gamma = 4/3$, are then

(i) $\{\pm 1, 0, 0, 0, 0\}$ kinetic scalar domination,
(ii) $\{0, 0, 1, 0, 0\}$ radiation domination,
(iii) $\{0, 0, 0, 1, s\}$ cosmological constant domination, and
(iv) $\{0,\, u,\, 0,\, l,\, 0\}$ cosmological constant and potential scalar domination.

Finally, the eigenvalues of the matrix $\mathcal{M}$ valued at the critical points listed above read

(i) $\{6, 3, 3, 3, 3\ -\gamma\}$,
(ii) $\{\frac{3\gamma}{2}, \frac{3\gamma}{2}, \frac{3\gamma}{2}, \frac{7\gamma}{2}, \frac{3}{2}(-2+\gamma)\}$,
(iii) $\{0, 0, \frac{1}{2}(-3 - \sqrt{9 - 4s^2}), \frac{1}{2}(-3 + \sqrt{9 - 4s^2}), -\gamma\}$, and
(iv) $\{-3, 0, 0, 0, -\gamma\}$

As we can see, the radiation domination epoch shows a saddle point; however, in order to reproduce the big bang nucleosynthesis process it is necessary that this kind of matter would have dominated the past of the Universe. In other words, the radiation points should have corresponded to a source point. The domination of dark matter in the past (a source point) and the cosmological constant in the future (an attractor point) are shown in Fig. 3.

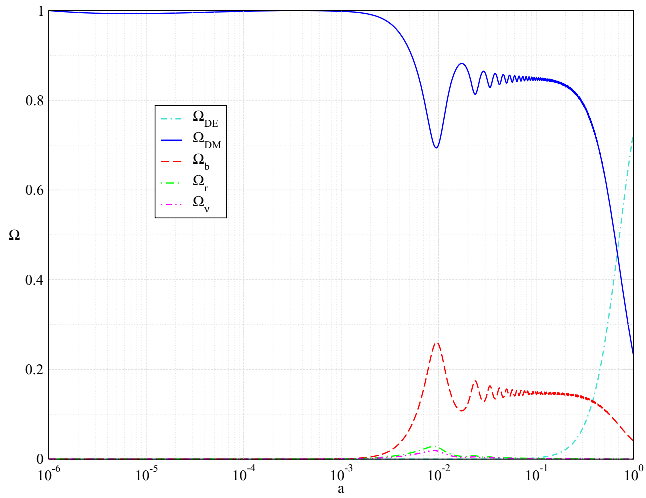

Figure 3. Evolution of the density parameters for the system of equations (46). The plot shows the dark matter domination epoch at early times, a source point. The cosmological constant in the future of the universe is an attractor point.

In the following, we integrate system (46) with the constraint (43), following the procedure shown in Appendix A. In general, this system is very difficult to integrate because it is a non-linear four-dimensional differential system of equations. It is clear that system (46) is a complete system which may or may not fulfill the constraint (43). However, as is shown in Appendix A, system (46) together with constraint (43) is completely integrable. For simplicity, we will take all the perfect fluid components as the equation $z_\gamma' = 3/2(\pi - \gamma)z_\gamma$ with the Friedmann equation $x^2 + u^2 + z_\gamma^2 = 1$.

Thus, we substitute $3/2\pi$ from equation (46e) into the rest of the equations. With this substitution equation (46c) integrates in terms of s as

$$z_\gamma = \sqrt{\Omega_\gamma^{(0)}}\, s\, \exp\left(-\frac{3}{2}\gamma\, N\right), \tag{47}$$

where $\Omega_\gamma^{(0)}$ is an integration constant. We multiply (46a) by $2x$ and (46b) by $2u$ and sum both equations. We obtain

$$(x^2 + u^2)' = -6\,x^2 + 2\ln(s)'(x^2 + u^2). \tag{48}$$

Now, we put constraint (43) and equation (47) into equation (48) to obtain

$$6\,x^2 = 2\,\ln(s)' - 3\gamma\, s^2\, \Omega_\gamma^{(0)} \exp(-3\,\gamma\, N). \tag{49}$$

We substitute (49) and (47) into (46e) to obtain $0 = 0$. Therefore, s is not an independent variable and we cast it into the system as a control variable which parametrizes the decrease of H, a similar result is found by Ureña-López & Reyes-Ibarra (2007). In what follows we will use this important result.

Of course, to guess variable s in order to fulfill constraint (43) is not so easy. In order to avoid this problem, we can consider the observed dynamic for H and model it by the following ansatz

$$H \;\equiv\; \frac{{t_0}^{n-1}}{t^n}, \tag{50}$$

because the behaviour for H at different epochs is well known:

$$H_{\rm dust} = \frac{2}{3t}, \quad H_{\rm rad} = \frac{1}{2t}, \quad H_\Lambda = \sqrt{\frac{\Lambda}{3}}. \tag{51}$$

There exists a restriction in the parameter n. Because is well known that H is a function that is monotonically decreasing, n has to satisfy $n \geq 0$. With the ansatz (50), the dynamical equation for s reads

$$s' = (mt_0)^{\frac{1}{n}-1} n \left(\frac{1}{s}\right)^{\frac{1}{n}-2} = s_0\, s^{-k}, \tag{52}$$

where we have defined $k \equiv 1/n - 2$.

In the following, we investigate if this system can reproduce the observed Universe. We introduce the components of the background universe into the dynamical system described by (46) adding to it baryons (b), radiation (z) and neutrinos (ν). Thus, the system transforms into

$$x' = -3\,x - su + \frac{3}{2}\pi\, x, \tag{53a}$$

$$u' = sx + \frac{3}{2}\pi\, u, \tag{53b}$$

$$b' = \frac{3}{2}\,(\pi - 1)\; b, \tag{53c}$$

$$z' = \frac{3}{2}\left(\pi - \frac{4}{3}\right)\; z, \tag{53d}$$

$$\nu' = \frac{3}{2}\left(\pi - \frac{4}{3}\right)\; \nu, \tag{53e}$$

$$l' = \frac{3}{2}\pi\, l, \tag{53f}$$

$$s' = s_0\, s^{-k}, \tag{53g}$$

with $\pi = 2x^2 + b^2 + \frac{4}{3}z^2 + \frac{4}{3}\nu^2$ and the Friedmann equation reduces to the constraint

$$F = x^2 + u^2 + b^2 + z^2 + \nu^2 + l^2 = 1. \tag{54}$$

Using this ansatz, we can reduce to quadratures the solution of system (53). In order to do this, we observe that

$$\frac{3}{2}\pi = s_0\, s^{-k-1}.$$

Now, using this last identity, equation (53c)–(53f) can be integrated to give

$$z_\gamma = z_0\, [s_0\, (k+1)\, N + s_1]^{1/(k+1)} \mathrm{e}^{-\frac{3}{2}\gamma\, N},$$

for each corresponding value of γ. Finally, equations (53a) and (53b) can be integrated as follows. We divide (53a) by x and (53b) by u and take the difference between both equations. We define $y = x/u$ to obtain

$$y' + 3y + q(N)y^2 = -q(N), \tag{55}$$

where function $q(N) = [s_0(k+1)\, N + s_1]^{1/(k+1)}$. Equation (55) is a Riccati equation which can be reduced to a Bernoulli equation by defining $y = w + y_1$, where y_1 is a known solution of (55). It reduces to

$$w' + (3 + 2\, q\; y_1)\, w + q\, z^2 = 0. \tag{56}$$

Equation (56) can be further reduced by defining $w = 1/w$, we obtain

$$W' - (3 + 2\, q\; y_1)\, W - q = 0, \tag{57}$$

the integral of which is

$$W = \mathrm{e}^{A} \int \mathrm{e}^{-A}\, q\, \mathrm{d}N, \tag{58}$$

with $A = \int (3 + 2\, q\; y_1)\, \mathrm{d}N$. Thus

$$u = u_0\, q \exp\left(\int y\, q\, \mathrm{d}N\right), \tag{59a}$$

$$x = x_0\, q\, \mathrm{e}^{-3\,N} \exp\left(-\int \frac{q}{y}\, \mathrm{d}N\right), \tag{59b}$$

$$z_\gamma = z_0\, q\, \mathrm{e}^{-\frac{3}{2}\gamma\, N}, \tag{59c}$$

$$y = \frac{1}{W} + y_1. \tag{59d}$$

In the particular case where $s_0 = 0$, the integrals can be solved analytically, however this value for s_0 does not have a physical meaning.

On the other hand, we can evaluate the integrals using numerical methods for different values of the free constants. We can obtain a numerical solution for the system using (59) or directly integrating system (53) with an Adams–Bashforth–Moulton (ABM) method and using as initial data the *WMAP* + baryon acoustic oscillations + supernovae (*WMAP*+BAO+SN) recommended values $\Omega^{(0)}_{\Lambda} = 0.721$, $\Omega^{(0)}_{\mathrm{DM}} = 0.233$, $\Omega^{(0)}_{\mathrm{b}} = 0.0454$, $\Omega^{(0)}_{\mathrm{r}} = 0.0004$, $\Omega^{(0)}_{\nu} = 0.0002$; the result is the same.

Figs 4 and 5 show the numerical solutions of the dynamical system (53). In Fig. 4, we set $n \geq 1$ and as examples we show $n = 1, 5$. From these figures, it is clear that the radiation remains subdominant for these values of n. On the other hand, in Fig. 5, where the plots were made for $n = 1/2, 1/5$, the radiation and the neutrinos behave in exactly the same way as in the ΛCDM model so we expect that both of these can reproduce the observed Universe. The first values for n are not able to explain the big bang nucleosynthesis, since radiation never dominates as is required. However, the last values for n can reproduce the radiation-dominated era. Following the radiation-dominated era, ϕ^2 dark matter becomes the component that dominates the evolution and finally the Universe is dominated by the cosmological constant. Fig. 6 shows the constraint F in (54) in order to visualize the integration's error. Observe that $F \approx 1$ at every point in the evolution, indicating that the Friedmann equation is exactly fulfilled all the time; this behaviour is exactly the same for all runs.

4.2 The cosh scalar potential

Now we are going to compare above results with the potential

$$V(\phi) = V_0\, [\cosh(\kappa\lambda\phi) - 1]\,. \tag{60}$$

In order to do so, we define new variables as

$$x \equiv \frac{\kappa}{\sqrt{6}}\frac{\dot{\phi}}{H},$$
$$u \equiv \sqrt{\frac{2\, V_0}{3}}\frac{\kappa}{H}\cosh\left(\frac{1}{2}\kappa\lambda\phi\right),$$
$$v \equiv \sqrt{\frac{2\, V_0}{3}}\frac{\kappa}{H}\sinh\left(\frac{1}{2}\kappa\lambda\phi\right),$$
$$z_\gamma \equiv \frac{\kappa}{\sqrt{3}}\frac{\sqrt{\rho_\gamma}}{H},\quad l \equiv \frac{\kappa}{\sqrt{3}}\frac{\sqrt{\rho_\Lambda}}{H}. \tag{61}$$

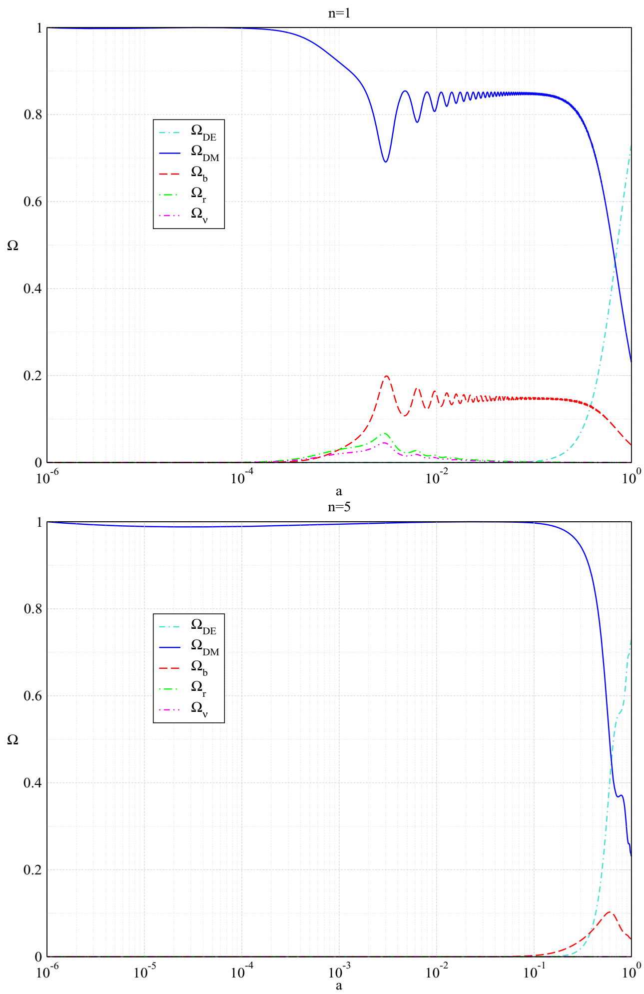

Figure 4. Evolution of the density parameters for the system (53) with $n = 1$ (top panel) and $n = 5$ (bottom panel). These values of n do not reproduce the standard behaviour of ΛCDM.

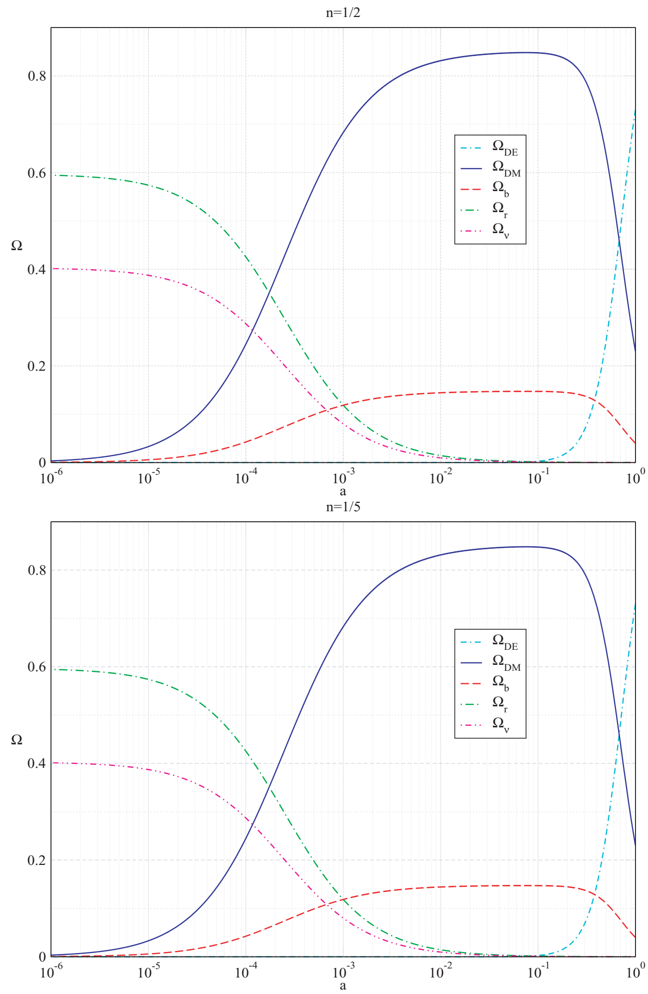

Figure 5. Upper panel: evolution of the density parameters for the system (53) with $n = 1/2$. Lower panel: evolution of the density parameters for the system (53) with $n = 1/5$. SFDM reproduces the standard ΛCDM behaviour in both cases.

Substituting definitions (61) into equations (38), we obtain

$$
\begin{aligned}
x' &= -3x - \lambda v u + \frac{3}{2}\pi x, \\
u' &= \lambda x v + \frac{3}{2}\pi u, \\
v' &= \lambda x u + \frac{3}{2}\pi v, \\
z'_\gamma &= \frac{3}{2}(\pi - \gamma)\, z_\gamma, \\
l' &= \frac{3}{2}\pi l,
\end{aligned}
\tag{62}
$$

where again the prime means derivatives with respect to the e-folding number $N = \ln(a)$ and we also use the function $\pi = 2x^2 + \gamma z^2$. From the definitions (61) it follows the constraints

$$
u^2 - v^2 = \frac{2V_0\kappa^2}{3}\frac{1}{H^2} = \frac{1}{\lambda^2}\frac{m_\phi^2}{H^2}, \tag{63}
$$

and the Friedmann equation (43) written with these variables reads

$$
F = x^2 + u^2 + z^2 + l^2 = 1. \tag{64}
$$

However, equation (64) is actually not a real constraint, since it is inherent in the dynamical equations (62) [see Appendix A

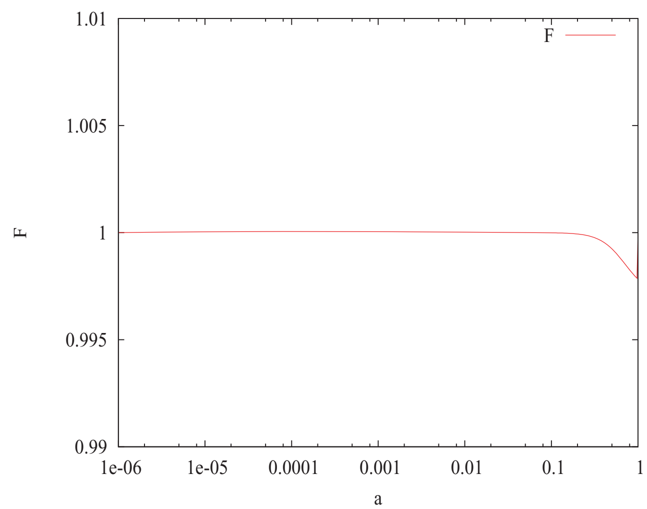

Figure 6. Evolution of the function $F = x^2 + u^2 + b^2 + z^2 + v^2 + l^2$ in (54) for the system (53) with $n = 1$, 5, 1/2 and 1/5. Function F is exactly the same for all values of n in all these cases.

equation (A6)]. Furthermore, constraint (63) is also inherent in the dynamical system; observe that from the second and third equations in (62) it is straightforward to find

$$H' = -\frac{3}{2}\pi H. \tag{65}$$

However, this relation follows directly from the field equations (38). This means that system (62) is compatible with the constraint (63). Using the constraint (63) in the dynamical system (62), we obtain

$$\begin{aligned}
x' &= -3x - u\sqrt{\lambda^2 u^2 + \left(\frac{m}{H}\right)^2} + \frac{3}{2}\pi x,\\
u' &= x\sqrt{\lambda^2 u^2 + \left(\frac{m}{H}\right)^2} + \frac{3}{2}\pi u,\\
z' &= \frac{3}{2}(\pi - \gamma)\, z,\\
l' &= \frac{3}{2}\pi l.
\end{aligned} \tag{66}$$

We notice that the same situation as ϕ^2 potential occurs. Introducing again the variable $s \equiv m/H$ with its dynamical equation

$$s' = (mt_0)^{\frac{1}{n}-1}\, n \left(\frac{1}{s}\right)^{\frac{1}{n}-2}, \tag{67}$$

we obtain

$$\begin{aligned}
x' &= -3x - u\sqrt{\lambda^2 u^2 + s^2} + \frac{3}{2}\pi x,\\
u' &= x\sqrt{\lambda^2 u^2 + s^2} + \frac{3}{2}\pi u,\\
z'_\gamma &= \frac{3}{2}(\pi - \gamma)\, z_\gamma,\\
l' &= \frac{3}{2}\pi l,\\
s' &= s_0\left(\frac{1}{s}\right)^{\frac{1}{n}-2}.
\end{aligned} \tag{68}$$

The density parameters are the same as we have defined in (44). We solve (68) numerically with the same initial conditions as the system of equations (53) and with $\lambda \approx 20$. The solutions are shown in Fig. 7. The plot shows the dynamical evolution for a universe with SFDM with the potential (60); notice that this is equivalent to potential (40).

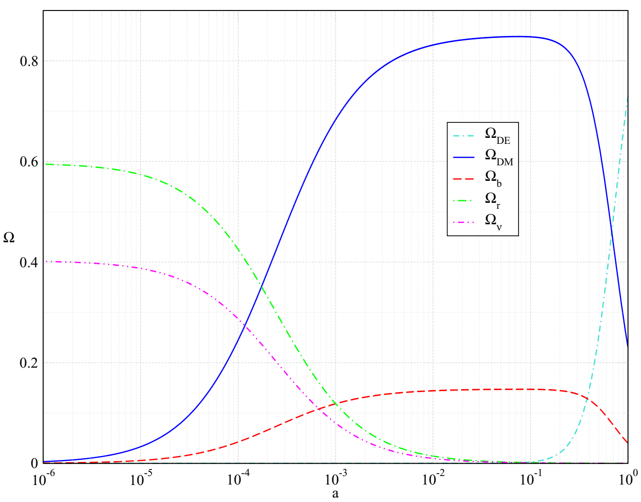

Figure 7. Evolution of the density parameters for the system (68), where the scalar field potential is given by the equation (60).

Finally, we use the same dynamical system formalism for the case of ΛCDM in order to compare it with SFDM. We consider that the background universe is composed by baryons, radiation, neutrinos, CDM and a cosmological constant with an equation of state of a perfect fluid. We solve this system numerically and in general terms the dynamic of both scalar potentials is indistinguishable from the standard model. This is an important goal of this paper.

The next step is to compute the age of the Universe using our model. The age equation can be written as

$$t_o = \int_{N_o}^{N} \frac{1}{H}\,\mathrm{d}N. \tag{69}$$

Using the definition for l from (41) or (61), equation (69) reduces to

$$t_o = \frac{\sqrt{3}}{\kappa\sqrt{\rho_\Lambda}} \int_{N_o}^{N} l\,\mathrm{d}N. \tag{70}$$

We compute (70) and obtain that $t_o \simeq 13.77$ Gyr. This result is in agreement with the cosmological observations from *WMAP*+BAO+SN which estimate $t_o = 13.73 \pm 0.12$ Gyr and therefore $H_o = 70.1 \pm 1.3\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. Furthermore, in Fig. 7, we see that the scale factor of decoupling is $a \sim 10^{-3}$; this means a redshift of $z \sim 1000$. At this redshift, the neutrinos made up ~12 per cent of the Universe. On the other hand, *WMAP* cosmological observations show that when the Universe was only 380 000 yr old, neutrinos permeated the Universe within 10 per cent of its total energy density. Thus, SFDM is in agreement with the measurements of *WMAP*. This result shows that scalar field is a plausible candidate for dark matter because it behaves like CDM.

5 CONCLUSIONS

SFDM has proved to be an alternative model for the dark matter nature of the Universe. We have shown that the scalar field with an ultralight mass condensates very early in the Universe and generically form BECs with density profiles which are very similar to those of the CDM model, but with an almost flat central density profile, as it seems to be in LSB and dwarf galaxies. This fact can be a crucial difference between both models. If the flat central density is not confirmed in galaxies, we can rule out the SFDM model, but if this observation is confirmed it can be a point in favour of the SFDM model. We also show that the $1/2m^2\phi^2$ potential and the $V_0[\cosh(\kappa\lambda\phi) - 1]$ model are in fact the same. They have the same predictions and a control variable which determines the behaviour of the model, given naturally the right expected cosmology and the same cosmology as the CDM model. This implies that the differences between both models, the CDM and SFDM, are in the non-linear regime of perturbations. In this way, they form galaxies and galaxy clusters, especially in the centre of galaxies where the SFDM model predicts a flat density profile. If the existence of supersymmetry is confirmed, the DM supersymmetric particles would be observed by detectors and they would have the right mass, DM density and coupling constant, and therefore the SFDM model can be ruled out. However, if these observations are not confirmed, the SFDM is an excellent alternative candidate for the nature of the DM of the Universe.

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REFERENCES

Alcubierre M., Guzmán F. S., Matos T., Nuñez D., Ureña-López L. A., Wiedelhold P., 2002, Class. Quantum Gravity, 19, 5017
Arbey A., 2006, Phys. Rev. D, 74, 043516
Arbey A., Lesgourgues J., Salati P., 2003, Phys. Rev. D, 68, 023511
Bernal A., Guzmán F.S., 2006, Phys. Rev. D, 74, 103002
Bernal A., Matos T., Núñez D., 2008, Rev. Mex. Astron. Astrofis., 44, 149
Bertacca D., Bartolo N., Matarrese S., 2008, J. Cosmol. Astropart. Phys., 0805, 005
Cimatti et al., 2004, Nat, 430, 184
Coley A. A., 2003, in Astrophys. Space Sci. Library Vol. 291, Dynamical Systems and Cosmology. Kluwer Academic Publishers, Dordrecht
de Blok W. J. G., Bosma A., 2002, A&A, 385, 816
de Blok W. J. G., Bosma A., McGaugh S., 2003, MNRAS, 340, 657
Fukuyama T., Morikawa M., Tatekawa T., 2008, J. Cosmol. Astropart. Phys., 06, 033
Guzmán F. S., 2006, Phys. Rev. D, 73, 021501
Guzmán F. S., Matos T., 2000, Class. Quantum Gravity, 17, L9
Guzmán F. S., Ureña-López L. A., 2003, Phys. Rev. D, 68, 024023
Guzmán F. S., Ureña-López L. A., 2004, Phys. Rev. D, 69, 124033
Guzmán F. S., Ureña-López L. A., 2006, ApJ, 645, 814
Hinshaw G. et al., 2008, ApJS, in press (arXiv:0803.0732)
Ji S. U., Sin S.-J., 1994, Phys. Rev. D, 50, 3655
Kolb E. W., Turner S. T., 1990, The Early Universe, Frontiers in Physics. Addison Wisley, Redwood City, CA
Lee J. W., Koh I. G., 2008, Phys. Rev. D, 53, 2236
Lee J. W., 2008, preprint (arXiv:0801.1442)
Lesgourgues J., Arbey A., Salati P., 2002, New Astron. Rev., 46, 791
Liddle A. R., Ureña-Lopez L. A., 2006, Phys. Rev. Lett., 97, 161301
Liddle A.R., Cédric P., Ureña-López L.A., 2008, Phys. Rev. D, 77, 121301
Madsen J., 1992, Phys. Rev. Lett., 69, 571
Matos T., Guzmán F. S., 1999, F. Astron. Nachr., 320, 97
Matos T., Guzmán F. S., 2000, Ann. Phys. (Leipzig), 9, SI
Matos T., Ureña-López L. A., 2000, Class. Quantum Gravity, 17, 75
Matos T., Guzmán F. S., 2001, Class. Quantum Gravity, 18, 5055
Matos T., Ureña-López L. A., 2001, Phys. Rev. D, 63, 63506
Matos T., Ureña-López L. A., 2002, Phys. Lett. B, 538, 246
Matos T., Guzmán F. S., Núñez D., 2000, Phys. Rev. D, 62, 061301
Padmanabhan T., Choudhury T. R., 2002, Phys. Rev. D, 66, 081301
Peebles P. J. E., 2000a, Phys. Rev. D, 62, 023502
Peebles P. J. E., 2000b, ApJ, 534, L127
Peebles P. J. E., Vilenkin A., 1999, Phys. Rev. D, 60, 103506
Pérez-Lorenzana A., Montesinos M., Matos T., 2008, Phys. Rev. D, 77, 063507
Press W. H., Ryden B. S., Spergel D. N., 1990, Phys. Rev. Lett., 64, 1084
Ruffini R., Bonazzola S., 1969, Phys. Rev., 187, 1767
Sánchez-Salcedo F. J., Reyes-Iturbide J., Hernandez X., 2006, MNRAS, 370, 1829
Seidel E., Suen W., 1991, Phys. Rev. Lett., 66, 1659
Seidel E., Suen W., 1994, Phys. Rev. Lett., 72, 2516
Simon J. D., Bolatto A. D., Leroy A., Blitz L., 2005, ApJ, 621, 757
Sin S. J., 1994, Phys. Rev. D, 50, 3650
Turner M. S., 1983, Phys. Rev. D, 28, 1243
Ureña-López L. A., 2002, Class. Quantum Gravity, 19, 2617
Ureña-López L. A., Reyes-Ibarra M. J., 2007, preprint (arXiv:0709.3996)

APPENDIX A: DYNAMICAL SYSTEM REVIEW

The theory of dynamical systems is used in the study of physical systems that evolve over time. It is assumed that the physical state of the system at an instant of time t is described by an element x of a space phase X, which can be of finite or infinite dimension. The evolution of the system is represented by a differential autonomous equation in X, written symbolically as

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{x}), \qquad \boldsymbol{x} \in X, \tag{A1}$$

where $\boldsymbol{f} : X \to \mathrm{X}$.

The main step to get qualitative information on solutions is studying the flow of the equation in the vicinity of its critical points based on the Hartman–Grobman theorem, namely the study of its stability.

The essential idea is firstly to find the fixed (or critical) points of the equation (A1) which are given by $\boldsymbol{f}(\boldsymbol{x}_{\mathrm{c}}) = 0$, and then linearize the differential equation at each critical point, that is, expand about the points $\boldsymbol{x} = \boldsymbol{x}_{\mathrm{c}} + \delta\boldsymbol{x}$, which yields to

$$\delta\boldsymbol{x}' = \mathcal{M}\delta\boldsymbol{x},$$

where $\mathcal{M}$ is the Jacobian matrix of $\boldsymbol{x}'$. Therefore the general solution for the linear perturbation evolution can be written as

$$\delta\boldsymbol{x}' = \delta\boldsymbol{x}_0 \mathrm{e}^{\mathcal{N}\delta t},$$

where $\mathcal{N}$ is the matrix composed of the eigenvalues m_i associated with $\mathcal{M}$.

The stability of the system (A1) depends on the values of the eigenvalues: if the real part of all eigenvalues is negative, the fixed point is asymptotically stable, that is, an attractor. All eigenvalues with positive real parts make the fixed point asymptotically unstable (commonly called a source or repeller).

On the other hand, a saddle point happens when there exists a combination of stable and unstable points. For an extended review see Coley (2003).

Then we give a procedure for transforming equations (38) and (39), with an arbitrary potential, into a dynamical system. We define the dimensionless variables

$$x \equiv \frac{\kappa}{\sqrt{6}}\frac{\dot{\phi}}{H}, \quad u \equiv \frac{\kappa}{\sqrt{3}}\frac{\sqrt{V}}{H}, \qquad z_\gamma \equiv \frac{\kappa}{\sqrt{3}}\frac{\sqrt{\rho_\gamma}}{H}. \tag{A2}$$

Using above definitions (A2), the evolution equations (38) transform into an autonomous system

$$x' = -3\,x + \frac{3}{2}\pi\,x - \frac{\kappa}{\sqrt{6}\,H^2}V_{,\phi}, \tag{A3a}$$

$$u' = \frac{3}{2}\pi\,u + \frac{\kappa}{\sqrt{6}\,H^2}V_{,\phi}\,\frac{x}{u}, \tag{A3b}$$

$$z_\gamma' = \frac{3}{2}\left(\pi - \gamma\right) z_\gamma, \tag{A3c}$$

$$-\frac{H'}{H} = \frac{3}{2}\left(2x^2 + \gamma z_\gamma^2\right) \equiv \frac{3}{2}\pi. \tag{A3d}$$

This last equation (A3d) can also be written as

$$s' = \frac{3}{2}\pi\,s, \tag{A4}$$

for the variable $s = const./H$, and determines the evolution of the horizon. Here, a prime denotes a derivative with respect to the e-folding number $N = \ln(a)$. The Friedmann equation (43) transforms into a constraint equation

$$F = x^2 + u^2 + z_\gamma^2 = 1. \tag{A5}$$

With these variables, the SFDM density can be written as

$$\Omega_{\rm DM} = x^2 + u^2.$$

Observe that if we derive (A5) with respect to N and substitute system (A3) into this, we obtain

$$F' = 3\,(F - 1)\,\pi, \tag{A6}$$

indicating that constraint (A5) is compatible with system (A3) for all scalar field potentials if the Friedmann equation is fulfilled.

Now we show that system (A3) together with constraint (A5) is completely integrable. To integrate system (A3), first observe that we can substitute $3/2\pi$ from equation (A4) into the rest of the equations. With this substitution equation (A3c) can be integrated in terms of s as

$$z_\gamma = \sqrt{\Omega_\gamma^{(0)}}\, s\, \exp\left(-\frac{3}{2}\gamma\, N\right), \tag{A7}$$

where $\Omega_\gamma^{(0)}$ is an integration constant. Now we multiply (A3a) by $2x$ and (A3b) by $2u$ and sum both equations. We obtain

$$(x^2 + u^2)' = -6\,x^2 + 2\,\ln(s)'(x^2 + u^2). \tag{A8}$$

Now, we put constraint (A5) and equation (A7) into equation (A8) to obtain

$$6\,x^2 = 2\,\ln(s)' - 3\gamma\, s^2\, \Omega_\gamma^{(0)} \exp(-3\,\gamma\, N). \tag{A9}$$

Now we have to integrate equation (A4) with all these results. If we substitute (A9) and (A7) into (A3d) or (A4) we obtain $0 = 0$, which means s is an arbitrary variable which parametrizes the decrease of H and can be cast into the system as a control variable. In other words, equations (A3d) and (A4) are actually identities, and not equations.

Thus, we set the variable s from system (A3) as arbitrary in the equations (A3a), (A3b) and (A3c).

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