ONE DIMENSIONAL SUBSPACES OF EXACT SOLUTIONS OF THE n-DIMENSIONAL EINSTEIN'S EQUATIONS.

by

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ABSTRACT. We describe one method for finding exact solutions of the Einstein's equations in a n-dimensional Riemannian space, when the n-dimensional metric depends on two variables z and $\bar{z}$. In this case, the field equations separate in two systems of non-linear, partial differential equations, one of them taking the form of a Chiral-field equation $(\rho g_{,z} g^{-1})_{,\bar{z}} + (\rho g_{,\bar{z}} g^{-1})_{,z} = 0$. The one-dimensional supspaces of the Chiral equation ($g = g(\lambda)$; $\lambda = \lambda(z, \bar{z})$) led to the n x n matrix system $g_{,\lambda} = Ag$, assuming that λ fulfills the Laplace's equation. We find that the total integration of this system depends only on the characteristic polynomial of the constant matrix A.

INTRODUCTION.

The idea of working in higher dimensions in order to unify all interactions in nature is becoming more interes among the phisicists. The earliest attempt to work in more than four dimensions was the Kaluza-Klein Theory [3], [4]. This theory unified gravitation with electromagnetism taking a five dimensional manifold and assuming that the five metric does not depend on the fifth coordinate. A more interesting approach was made in the 60's and 70's when the Kaluza-Klein idea was generalized to N+4 dimensions unifying gravitation with electroweak and strong interactions [1] String's theory is also a candidate for unifying all interactions in one space of 26 dimensions,

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and the superstring's theory is one in a 10-dimensional space [2]. Even if the standard string and superstring's theory have a flat underground, there are some approaches for studing strings in a curved space [10]. Anyway, it is of great interest for to find exact solutions of the Einstein's equations in a n-dimensional Riemannian space and that is just what we intend to do in this work.

The problem of finding exact solutions of the 4-dimensional Einstein's equation is so dificult, that mathematichians and physicists had spent a lot of time trying to solve it. Its time generalization to N+4 dimensions is obviously much more complicated. Therefore, we have to make some simplifications in orden to be able to use methods developed to find exact solutions in four dimensions. We start assuming that the N+4 - dimensional metric depends only on two coordinates x^1 and x^2. This is not a very strong restriction since many physical solutions, such as the Schwarzschild's or the Kerr's solution, belong to this class. Furthermore we assume that the N+4 - dimensional metric can be written as:

$$ds^2 = f({dx^1}^2 + {dx^2}^2) + g_{\mu\nu}\, dx^\mu dx^\nu \quad \mu, \nu = 3, \ldots, N+4 \qquad (1)$$

where the functions f and $g_{\mu\nu}$ depend only on x^1 and x^2. In [8], it was shown that the N+4 - dimensional Einstein's field equations $R_{ab} = 0$ a,b = 1,..., N+4 with the metric (1) reduce to two systems of non linear partial differential equations:

$$(\ln f)_{,z} = -\frac{1}{z+\bar{z}} + \frac{z+\bar{z}}{4} \operatorname{tr} (g_{,z}\, g^{-1})^2 \qquad (2a)$$

$$(x^1 g_{,z}\, g^{-1})_{,\bar{z}} + (x^1 g_{,\bar{z}}\, g^{-1})_{,z} = 0 \qquad (2b)$$

$$z = x^1 + ix^2 \qquad (g)_{\mu\nu} = g_{\mu\nu}$$

Observe that in order to solve the field equations (2) we have to find first a solution of (2b), then substitute it in (2a). In this work we find exact solutions only of the second system of differential equations namely (2b). We procede in the following form: Assume that the metric g depends only on one parameter $\lambda = \lambda(z,\bar{z})$, i.e. $g_{\mu\nu} = g_{\mu\nu}(\lambda(z, \bar{z}))$, and suppose that λ fulfills the generalized Laplace's equations $(x^1 \lambda_{,z})_{,\bar{z}} + (x^1 \lambda_{,\bar{z}})_{,z} = 0$ [5][8]. Then, the equation (2b) reduce to:

$$g_{,\lambda} = Ag \qquad A \text{ is constant} \qquad (3)$$

g being real and symmetric. One can normalize g in such a way that det g = 1, then g belongs to the Matrix group SL(N,ℝ). It is clear that the transformation $g \rightarrow C\, g\, C^T$ let the equation (2b) unchanged if C is a constant matrix that belongs also to the group SL(N,ℝ). Under this transformation, the matrix A changes to A --> CAC^{-1}. This last relation induces a partition in the set of matrices into equivalent classes' i.e., it is enough to work with a representative matrix for each class. Then, the fist step of our work will be to write down such representatives.

Observe that the group conditions $g = \bar{g}$, det g = 1 and the symmetry condition $g = g^{T}$ are translated to the matrix A in the form:

a) $g=\bar{g} \dashrightarrow A = \bar{A}$

b) detg=1 --->trA = 0

c) $g=g^{T} \dashrightarrow gA^{T} = Ag$ (4)

that means, that we have to find the representative of the classes using the conditions a) and b) in (4). These conditions restrict the representatives of the classes in such a form that we will have a reduced number of matrices to work with.

GETTING THE MATRIX A

We start writing the representatives of the classes of equivalence in a normal form, known as the natural form.

It can be shown that every matrix A with elements on a conmutative field, can be reduced, on this field, to one and only one natural normal form:

$$A = \begin{bmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_s \end{bmatrix}$$

where each A_i, i = 1,..., s , is a square matrix, called a cell, such that, it presents the following form:

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \cdots & & \\ -a_o & -a_1 & -a_2 & & -a_{n-1} \end{bmatrix}$$

Observe that $a_o, a_1, \ldots, a_{n-1}$ are the coefficients of the characteristic polynomial of matrix the A_i:

$$p_i(\lambda)=\lambda^{n}+a_{n-1}\lambda^{n-1}q^{+}\ldots+a_1\lambda+a_o .$$

For to characterize to matrix A, we use the matrix λI-A, this is a polynomial matrix, since its entries are polynomials in λ. We work the similar relation of the natural normal form of matrices, using the invariant factors of its corresponding polynomial matrix.

For the serch of matrices in its natural normal form that represent the classes, we consider the classification of the matrices A according to the degrees of the invariant polynomials (invariant factors) of the matrix A-λI.

Since the invariant factors of A-λI are factors of the characteristic polynomial $p_A(\lambda)$ associated to A, then we can consider the feasible n-decomposition of the polynomial $p_A(\lambda)$. The methodology for finding the matrices A, that we propuse in this paper, consist: first, to build matrices in its natural normal form in correspondence with the dimensions of the cells. We have found a representative of every class of the partition induced by the properties (4). One of all matrices will be similar to the found matrix.

In this way we are classifing the solutions of the equation (3) and building a representative, that is, a general solution to our problem.

In the table are shown the representatives for n = 2, 3,4.

GETTING THE MATRIX g.

Let be the differential equation (3) for some A, as in (4). Then, we have a system of s x n linear differential equations. We remember that pA(λ) is the characteristic polynomial of matrix A, $p_{A_i}(\lambda)$ is the characteristic polynomial of each cell of A and $p_e(\lambda)$ is the characteristic polynomial associated to (3).

We enunciate now some of our results without proof.

Theorem 1.

If A is a natural normal cell itself, then $p_A(\lambda) = p_e(\lambda)$

Definition.

For a square matrix $A=[a_{ij}]$ of order n, an antidiagonal is the set of entries $\alpha_k = \{a_{ij} \mid i + j = k\}$; k = 2,..., 2n.

Theorem 2.

Let A be a natural cell. The elements of the matrix Ag on each α_k, are the same.

Theorem 3.

Let A be a matrix in its natural normal form with s cells. Then

CLASSIFICATION OF MATRICES			
Order n	Cases	Matrices A_1	Invariants
2		$\begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}$	1, λ^2-a
3	3.1	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & 0 \end{bmatrix}$	1, 1, $\lambda^3 - \lambda b - a$
		$\begin{bmatrix} q & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2q^2 & -q \end{bmatrix}$	1, λ-q, $(\lambda+2q)(\lambda-q)$
4	4.1	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & c & 0 \end{bmatrix}$	1, 1, 1, $\lambda^4 - c\lambda^2 - b\lambda - a$
	4.2	$\begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2q^3-qb & b & -q \end{bmatrix}$	1, 1, λ-q, $\lambda^3 + q\lambda^2 - b\lambda - a$ con $a = q(2q^2-b)$
	4.3	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a & 0 \end{bmatrix}$	1, 1, λ^2-a, λ^2-a
	4.4	$\begin{bmatrix} -3a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix}$	1, λ-a, λ-a, $(\lambda-a)(\lambda+3a)$

Table

$g,_\lambda = Ag$ can be descomposed in a system of s x n independent homogeneous linear differential equations. Moreover,

$$p_A(\lambda) = \prod_{i=1}^{s} p_{A_i}(\lambda).$$

If the matrix A is a natural cell, we can develop the differential equation associated to $g,_\lambda = Ag$ for g_{11}. If we solve such differential equation, then we will get g_{11}. After that, we can compute the elements of the first row and the first column of g, differenting g_{11} n times. If we know those elements, then we can know all the elements of the triangular left submatrix of g. To compute the other elements, we need to compute at most n(n-1)/2 derivatives.

If A is a matrix in its natural normal form constituted by cells, then all the elements of g can be known if we solve the differential equations associated to every cell, to make good use of the symmetry of g and Ag. In other words, the equation $g,_\lambda = Ag$ can be expressed as a system of nxs linear homogeneous differential equations, every one of them with some order. If the equation i is of order r_i, then we can get r_i elements of the matrix g after we solve it. If we solve the nxs equations and we get some derivations, we will get the matrix g. Further solutions can be obtained by:

1. Finding other solutions of the Laplace's equations.
2. Using the symmetry of the group $g = Cg_o C^t$, where g_o is a known solution.
3. Using the Table 2.

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