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DIVISION DE GRAVITACION Y FISICA MATEMATICA
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Dear Dr. Tonatiuh Matos
Hugo A. Morales-Técotl
I am pleased to inform you that your paper

"Program for Quantum Gravity using Harmonic Maps"

has been accepted for publication in the Book **"Recent Developments in Gravitation and Mathematical Physics".** It appeared this year published by Since Network Publishing, in CD Rom, ISBN 3-9805735-0-8.

Sincerely Yours,

Dario Nuñez

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By the editorial committee

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Program for Quantum Gravity using Harmonic Maps

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Abstract

The main goal of this work is to present a new alternative quantization program for gravity. The crucial step of the program is the introduction of harmonic maps to express relevant quantities as the Hamiltonian and others. Then the quantization can be carried out using Fock space techniques. As an example we present the quantization of a system with two non-hypersurface-orthogonal Killing vectors.

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General Relativity (GR) is, perturbatively, a non-renormalizable theory of physics and hence at the quantum level the approaches based on perturbation theory to describe other interactions fail in the case of gravity; the only way out are either non-perturbative methods or replace GR by string theory that includes in particular the gravitational interaction. We shall stick here to GR. In this direction the idea of Ashtekar to study quantum gravity in a canonical framework is perhaps the most promising [1].

In quantum mechanics one associates waves to particles, and such waves can be thought of as made out of harmonic functions. On the other hand harmonic functions are present even at the classical level for fields. For example, Newton's gravitational law $\Delta\Phi = 4\pi G\rho$ has as vacuum solutions a series of harmonic functions. We have the same situation with electromagnetism, the Maxwell field equations $\Box A = J$ have as vacuum solutions wave functions, *i.e.*, again harmonic maps. We know that the so called graviton is a prediction of the linearized Einstein field equations, the solutions of these vacuum equations being harmonic maps. The question whether the solutions of the full Einstein equations are harmonic maps or not, *i.e.*, whether $G_{\mu\nu} = 8\pi G T_{\mu\nu} \Rightarrow D\lambda^i = 0$, for some functions λ^i being D the harmonic operator, is an open one. Nevertheless this is the case for many examples. If the space-time contains two commuting Killing vectors, the Einstein equations reduce to a harmonic equation, even in the presence of electromagnetic and scalar fields [2]. Then the question arises of whether it is possible to carry out the quantization of the gravitational field in terms of these harmonic functions. We do not have a final answer to this question, but we will show that harmonic maps play a significant role in such a formidable task. We show this through the example of gravity with two non-hypersurface-orthogonal Killing vectors.

We start with the Papapetrou form of the line element

$$ds^2 = \frac{1}{f}\left[e^{2k}(dx^2 - dt^2) + W^2\, d\varphi^2\right] + f\,(dz + A\, d\varphi)^2$$

where A, f, k and W are functions of x and t only. Let us start with the parametrization [2]

$$f = \frac{e^{2\lambda}}{a^2\, e^{4\lambda} + b^2}, \quad A_{,u} = 4ab\, W\, \lambda_{,u} \quad A_{,v} = -4ab\, W\, \lambda_{,v} \tag{1}$$

where $u = x - t$, $v = x + t$ and the constant parameters a, b, c and d are restricted to $a^2\, d - b^2\, c = -a\, b$. Metric (1) contains two Killing vectors, ∂_φ and ∂_z, they are hypersurface-horthogonal if $a = 0$. In the case $a = 0$, the quantization of Lagrangian (1) has been carried out (for a review of this using the Einstein-Rosen trick see for example [3] and recently using Ashtekar variables see [4]), here we give an example for a arbitrary. We shall treat λ, k and W as the independent dynamical variables. It is straightforward to calculate the ADM Lagrangian, we obtain

$$\mathcal{L}_{ADM} = 2[W(\dot{\lambda}^2 - \lambda'^2) + W'k' - \dot{W}\dot{k} - W''] \tag{2}$$

where dot means derivative with respect to t and prime with respect to x. The field equations derived from the ADM Lagrangian can be obtained by variation with respect to the dynamical variables λ, k and W. We arrive at

$$
\begin{array}{ll}
a) & \dfrac{\delta \mathcal{L}_{ADM}}{\delta k} = 0, \;\Rightarrow\; \ddot{W} - W'' = 0 \\
b) & \dfrac{\delta \mathcal{L}_{ADM}}{\delta \lambda} = 0, \;\Rightarrow\; (W\dot{\lambda}\dot{)} - (W\lambda')' = 0 \\
c) & \dfrac{\delta \mathcal{L}_{ADM}}{\delta W} = 0, \;\Rightarrow\; k'' - \ddot{k} = \dot{\lambda}^2 - \lambda'^2
\end{array}
\tag{3}
$$

The first two equations in (3) express the fact that W and λ are harmonic maps. The third equation in (3) is a consistency equation for k, it tells us that once W and λ are known then k is an integrable function. Now we can calculate the Hamiltonian. In order to do so, we calculate the momenta in terms of the "velocities". We obtain

$$
\begin{aligned}
P_k &= \frac{\delta \mathcal{L}_{ADM}}{\delta \dot{k}} = -2\dot{W} \\
P_\lambda &= \frac{\delta \mathcal{L}_{ADM}}{\delta \dot{\lambda}} = 4W\dot{\lambda} \\
P_W &= \frac{\delta \mathcal{L}_{ADM}}{\delta \dot{W}} = -2\dot{k}
\end{aligned}
\tag{4}
$$

Observe that there is no further constrains for the Lagrangian apart from the corresponding Hamiltonian one

$$
\mathcal{H}_{ADM} = \frac{1}{8W} P_\lambda^2 - \frac{1}{2} P_k \, P_W + 2(W\lambda'^2 - k'W' + W'') \tag{5}
$$

The general solution of (3,a) is $W = U(u) + V(v)$, where $U(u)$ and $V(v)$ are arbitrary functions. In terms of U and V, the function λ can be integrated yielding $\lambda = l(U(u) - V(v))$. From the Einstein equation it is easy to show that the field equations for the k function are given by

$$
\begin{aligned}
k_{,u} &= \frac{1}{2\,W_{,u}} (W_{,uu} + 2W\,(\lambda_{,u})^2) \\
k_{,v} &= \frac{1}{2\,W_{,v}} (W_{,vv} + 2W\,(\lambda_{,v})^2)
\end{aligned}
\tag{6}
$$

Then the solution of the field equations (3) in term of the arbitrary functions U and V is

$$
\begin{aligned}
W &= U(u) + V(v) \\
\lambda &= l(U(u) - V(v)) \\
k &= \frac{1}{2}[\ln(dU\, dV) + l\, W^2]
\end{aligned}
\tag{7}
$$

where dU and dV are the differentials of $U(u)$ and $V(v)$ respectively.

Next we quantize. The first two equations of (3) tell us that this can be achieved using Fock space. However, the solution for k (7) has a non-linear term in W, therefore the commutation relations may contain non-linear terms in the commutators algebra. Indeed if we develop the canonical variables W, λ and k in terms of cration and annihilation operators we get

$$
\begin{aligned}
W &= \int \frac{d\kappa}{(2\pi)2\kappa_0}\left[a_\kappa e^{-i\kappa\mathbf{x}} + a_\kappa^\dagger e^{i\kappa\mathbf{x}}\right] = \int \frac{d\kappa}{(2\pi)2\kappa_0}\omega \qquad (8)\\
\lambda &= \int \frac{d\kappa}{(2\pi)2\kappa_0}\left[b_\kappa e^{-i\kappa\mathbf{x}} + b_\kappa^\dagger e^{i\kappa\mathbf{x}}\right]\\
k &= \int \frac{d\kappa}{(2\pi)2\kappa_0}\left[c_\kappa e^{-i\kappa\mathbf{x}} + c_\kappa^\dagger e^{i\kappa\mathbf{x}}\right]
\end{aligned}
$$

where $\kappa = (\kappa_0, \kappa_1)$ and $\mathbf{x} = (x, t)$. Using the canonical commutation relations for the dynamical variables and its momenta

$$
\begin{aligned}
[W(x,t), P_W(x',t)] &= i\delta(x-x')\\
[\lambda(x,t), P_\lambda(x',t)] &= i\delta(x-x') \qquad (9)\\
[k(x,t), P_k(x',t)] &= i\delta(x-x')
\end{aligned}
$$

and zero otherwise, the commutation algebra for the creation and annihilation operators turns out

$$
[a_\kappa, c_{\kappa'}^\dagger] = (2\pi)2\kappa_0\delta(\kappa-\kappa')
$$

$$
[b_\kappa, b_{\kappa'}^\dagger] = \tfrac{1}{W}(2\pi)2\kappa_0\delta(\kappa-\kappa') \qquad (10)
$$

with all the other commutation relations vanishing. The presence of the term $1/W$ in the second commutation relation is due to the non-linear term of W in (7). In terms of these operators, the Hamiltonian can be written as

$$
\begin{aligned}
H_{ADM} &= 2\int [W(\dot\lambda^2 + \lambda'^2) - (\dot k\dot W + k'W') + W'']dx\\
&= 2\int \frac{d\kappa}{(2\pi)2\kappa_0}\left\{\kappa_0[\omega(b_\kappa b_\kappa^\dagger + b_\kappa^\dagger b_\kappa) - (c_\kappa a_\kappa^\dagger + c_\kappa^\dagger a_\kappa)] - \kappa^2\,\omega\right\} \qquad (11)
\end{aligned}
$$

The difference with linear field quantization is that here we have the presence of W in the energy (11). This means that the number operator is made out of three operators.

We know that some other gravitational models can be put in terms of harmonic maps. The program of harmonic map quantization for gravity might be then pushed reducing the problem of quantization to that of making sense of composite operators built up from the harmonic maps. The latter being quantized in Fock space. Work is in progress to deal with the aforementioned models.

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