Wormholes, Star-trek: Reality and Science Fiction

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Abstract. Wormholes are a speculative derivation of the Einstein's equations since the pioneer works of Albert Einstein and Nathan Rosen in 1935. The cosmology recently opened a door for the possibility of the existence of wormholes, because 73% of the matter of the Universe is made of dark energy, which could collapse to a star with an inner wormhole. If this could be possible, then the star-trek would be a reality.

INTRODUCTION

One of the most important factors for evolution of people is doubtless the communication. European nations evolve so strong because of the existence of the Mediterranean, an easily sailable sea which permits the communication between people and many regions of the Earth. This is a part of the explanation why humanity evolves so fast; the communication between people is now very efficient. Could you imagine the development of our specie if we could establish contact with beings from other planet, or from other region of the galaxy or from other galaxy? The consequences of this contact would be the beginning of a new era of the human beings. This is the reason why for many people is so important to study in some way the traveling trough starts and to visit our Universe. The main problem for this travel was given in the Einstein's work of special relativity, the maximal speed a space shift can reach is the speed of light. With this speed, in our frame, any space shift will need hundreds or thousands of years to reach any interesting region. And this is supposing that we are able to reach this speed. This travels are then impossible. The idea of the wormholes is that this "topological" objects are able to communicate to regions of the space-time, even if they are very far away. In principle, a space shift could travel trough this objects around the Universe without violating any physic law in a reasonable time. This is the reason why this speculations are so interesting for the physicists, even when we know that this objects are still science fiction, it is worth to keep looking for the possibility of their existence.

As we mention in the abstract of this work, the wormhole (WH) solutions of the Einstein equations were started by Einstein himself, and Nathan Rosen [1]. In this pioneer work they were interested in giving a field representation of particles. The idea was further developed by Ellis, [2] and others, where they tryed to model "bridges" between two regions of the space-time. The idea consists in the following. First we write the Schwarzschild metric

$$ds^2 = -(1-\frac{2M}{r})dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2 d\Omega^2 \quad (1)$$

where M is the mass of the black hole and $d\Omega^2 = d\theta^2 + \sin^2(\theta)d\varphi$. Observe that the radial coordinate r is such that $0 < r < \infty$. If we perform the coordinate transformation $l^2 = r - 2M$, we arrive at

$$ds^2 = -\frac{1}{1+\frac{2M}{l^2}}dt^2 + 4(l^2+2M)dl^2 + (l^2+2M)^2 d\Omega^2 \quad (2)$$

where now the coordinate l runs from $-\infty < l < \infty$. In these coordinates, the factor in front of $d\Omega^2$ is never zero. Let us analyze this metric. First, we make a photo of the black hole, i.e., we make $t = const$. This metric has spherical symmetry which implies that the black hole looks exactly the same for any θ. We can watch the black hole from the equator, this means we make $\theta = \pi/2$. Under this conditions, the metric can be rewritten as

$$ds^2 = 4(l^2+2M)dl^2 + (l^2+2M)^2 d\varphi^2 \quad (3)$$

Now, we can compare this metric with the three dimensional cylindrically symmetric metric

$$ds^2 = d\rho^2 + dz^2 + \rho^2\, d\varphi^2 = \left(1+(z_{,\rho})^2\right) d\rho^2 + \rho^2\, d\varphi^2 \quad (4)$$

where we have assumed that $z = z(\rho)$. We can compare (3) and (4), making $\rho = l^2 + 2M$ and $\left(1+(z_{,\rho})^2\right) = (l^2+2M)/l^2 = \rho/(\rho-2M)$. Integrating z we obtain $z = \sqrt{8M\,|\rho-2M|}$. If we go around the φ coordinate, we obtain the figure Fig1. This is an Einstein bridge communicating two spaces.

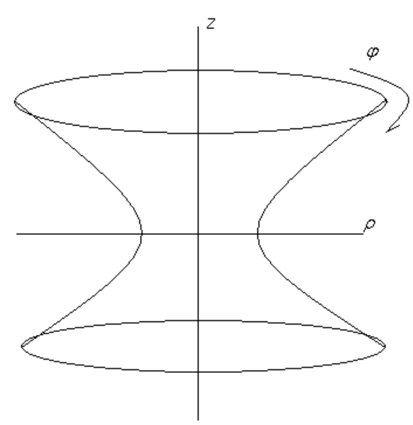

FIGURE 1. The plot of the function $z(\rho)$ for the Schwarzschild metric. The coordinate φ goes around the figure.

This can be interpreted as follows. Observe that the radius of the sphere around the singularity of the black hole never closes. This is so because the area of this sphere is $A(l) = 4\pi(l^2 + 2M)^2$, but the minimum of this area is $A(0) = 4\pi(2M)^2$, which is never zero. There, in the minimum area region, the radius of the sphere is $\rho_{min} = 2M$. However, this metric has a serious problem. The gravitational potential of this metric is given by $e^{2u} = 1 - M/r$, where e^{2u} is the factor in front of the $-dt^2$ term, and u might be interpreted in these coordinates as the "Newtonian potential". Close to the horizon, this interpretation fails, but close to the horizon the gravitational forces can be enormous. A deeper analysis of these forces requires the study of the geodesic deviations of this metric. Nevertheless, if an astronaut would go trough the black hole, we would expect that the mass of the black hole should not be so big, in order that the tidal forces of the black hole do not destroy the astronaut. Suppose the black hole has a mass similar to the Earth's one. Then the mass parameter is $M = 8.8$mm (in units where $G = c = 1$) and the radius is too small for an astronaut to go trough. If the radius is of order of kilometers, then the tidal forces will destroy the astronaut. The first work which has attracted a lot of attention for the study of the wormholes was presented in a series of papers by Morris and Thorne [3]. In these papers, they propose the idea of considering such solutions as actual connections between two separated regions of the Universe. Perhaps, this is the first time these solutions evolved from a sort of science-fiction scenario to a solid scientific topic, even at the point of considering the actual possibility of their existence. The idea of a wormhole is just to create a short way across the Universe, like a worm through an apple. It turned out that the solution proposed by Ellis [2] actually could be interpreted as the identification, or union, of two different regions, no matter how far apart they were or even if those regions were in the same space-time. You could still identify two regions. Ellis obtained a solution which allows one to go from one region to another by means of this identification.

In the eighties, Carl Sagan wrote the novel *Contact*, where the astronomer Ellie Arroway traveled through a wormhole to visit other civilizations. One of the beings of the other civilization adopted the form of her father in order to make easy the contact between them. Nevertheless, Sagan wrote a science-fiction novel and he wanted to be sure that there was not a violation of any physical law. Some years after the publication of *Contact*, Sagan asked to physicist how it could be possible to travel across the Universe without violating Lorentz invariance or any other physical law. Kip Thorne and his students gave an answer to the question. They found the following metric for a wormhole

$$ds^2 = -dt^2 + \frac{dr^2}{1 - \frac{b^2}{r^2}} + r^2 d\Omega^2 \tag{5}$$

As in the black hole case, the coordinate transformation $r^2 = l^2 + b^2$ produces

$$ds^2 = -dt^2 + dl^2 + (l^2 + b^2) d\Omega^2 \tag{6}$$

This metric has the feature that the tidal forces are small and the radius of the throat is b, which can be as big as you want. This is then a wormhole for transportation. Nevertheless, they found that such solutions require an "exotic" type of matter able to violate the energy conditions which we usually impose on any typical kind of matter, (see [4] for a detailed review of this subject). The solutions exist but they need to be generated by matter which apparently does not exist. Actually one needs something very peculiar to warp space-time or to make holes in it. The energy momentum tensor implies that the matter density of this wormhole is negative. The source of this wormhole is a scalar field, such that $\phi \sim \arctan(l/b)$, but with negative kinetic part. This feature was a serious drawback for the possibility of their existence in nature, so WH's remained in the realm of fiction.

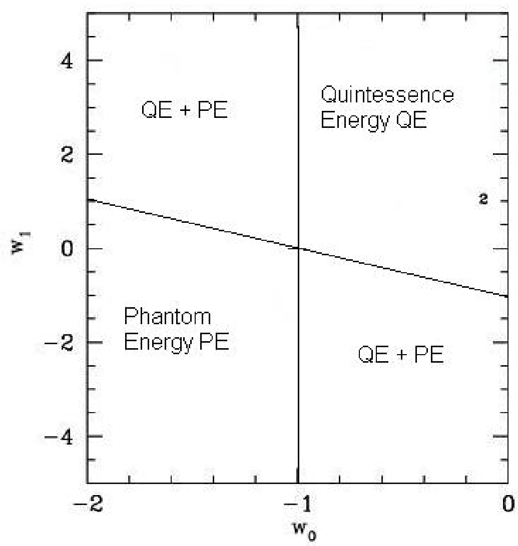

FIGURE 2. Plot of the different regions for the dark energy state equation in order to fit the several observational constraints.

Nevertheless, the situation changed radically in the last years. In 1998 it was shown that the Universe is in a stage of accelerated expansion. This implies the existence of an unknown sort of matter, a kind of anti-gravitational or repulsive matter. The bigest surprise is that this "new substance" should dominate the matter contents of the Universe: it represents more than 73% of the whole matter of the Universe and it is called *dark energy*. This new type of matter forms the overwhelming majority of matter in the Universe and seems to be found everywhere [5]. More recently, some works have also discussed the plausibility of energy-condition violations at the classical level of the dark energy (see for example [6]). This study consists in the following. If the dark energy of the Universe is a perfect fluid, it should be a state equation like $p = \omega\rho$. The most popular supposition for the factor ω is that it is lineal in the scale factor, which means

$$\omega = A_0 + A_1\, a = \omega_0 + \omega_1 \frac{z}{1+z} \tag{7}$$

where $a = 1/(1+z)$ is the scale factor of the Universe and gives the rate of how much the Universe is expanding, z is the redshift of the object we are observing. This ansatz is very popular because it summarizes the different hypothesis of the dark energy with simple values. For example, if the dark energy is the cosmological constant, the values for ω are $\omega_0 = -1$, $\omega_1 = 0$, etc. The most interesting hypothesis on the nature of the dark energy is that it is a scalar field or a Phantom field, that is, a scalar field for which the kinetic term has opposite sign. In Figure 2 we see the different values for the constants of the different hypothesis of the dark energy.

After the discovery of the dark energy, there is an agreement in the scientific community that matter which violates some of the energy conditions may exist. Thus, the idea that the WH's can be rejected because of the type of the required matter is not as tenable as it was once. In Figure 3 it is shown the results from the observations on Ia supernova (SNIa) and gamma ray burst (GRB) as standard candles. We see that the observations let a great space for the phantom energy, which can cause the existence of WH's.

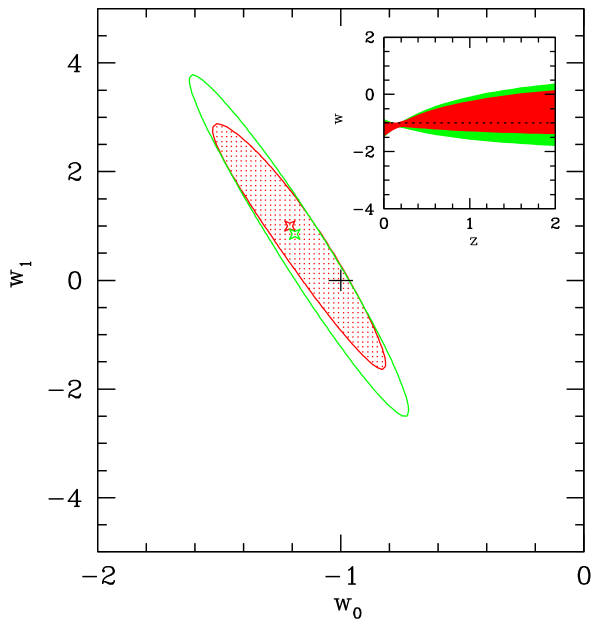

FIGURE 3. Plot of the results for the different observations, taking into account the SNIa and the GRB together. This figure is courtesy of the authors of reference [7]

A mayor problem faced by WH solutions is their stability. One expect that WH should be traversable, that is, an astronaut should be able to go from one side of the throat to the other in a finite time as measured by the astronaut and by another observer far away from him, and without feeling large tidal forces. The stability problem of the "bridges" has been studied since the $60's$ by Penrose [8] in connection with the stability of Cauchy horizons. However, the stability of the throat of a WH was just recently studied numerically by Shinkay and Hayward [9]. They shown that the WH, as proposed by Thorne [3], when perturbed by a scalar field with a stress-energy tensor with the usual sign, may collapse to a black hole and the throat should be closed. In the same way, when the perturbation is due to a scalar field of the same type as that generating the WH, the throat grows exponentially, showing that the solution is highly unstable. It is in this sense that the possibility of the existence of traversable wormholes is again limited. To solve this problem, it is intuitively clear that a rotating solution would have a higher possibility of being stable. We expect that stationary axial symmetric solutions, more general than the one proposed by Thorne, could be stable. Then, the next step is to construct a rotating scalar field wormhole [12]. The idea is to use solution generation techniques developed in the late $80's$ for the Einstein's equations (see [11]), for which it is possible, in the chiral formulation [13], to derive the Kerr solution starting from the Schwarzschild one. In [12] these techniques were applied on the WH proposed by Ellis and Thorne. The final result is the following ansatz for the line element:

$$\begin{aligned} ds^2 &= -f\,(dt + a\cos\theta\, d\varphi)^2 \\ &+ \frac{1}{f}\left[dl^2 + \left(l^2 - 2\,l\,l_1 + {l_0}^2\right)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right], \end{aligned} \tag{8}$$

where $f = f(l)$ is an unknown function to be determined by the field equations, l_0, l_1 are constant parameters with units of distance such that $l_0^2 > l_1^2$, $l_0 \neq 0$, and a is the rotational parameter, i.e., the angular momentum per unit mass (we use the speed of light $c = 1$). In these coordinates the distance l covers the complete manifold, going from minus to plus infinity. Notice that, modulo f, there is already the throat or bridge feature of the WH's in such a line element, as the coefficient of the angular variables is again never zero.

The formation process of a WH is still an open question. We suppose that some scalar field fluctuation collapses in such a way that it forms a rotating scalar field configuration. This configuration has three regions; the interior, where the rotation is non-zero and two exterior regions, one on each side of the throat, where the rotation stops. The inner boundaries of this configuration are defined where the rotation vanishes. The interior field is the source of the WH. In [12] it has been conjectured that the rotation will keep the throat from being unstable. We follow reference [12] for the construction of the solutions and its analysis.

The Einstein equations with an stress-energy tensor describing an opposite sign massless scalar field, ϕ (see [14]), are given by

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}\,\phi_{,\mu}\,\phi_{,\nu}, \tag{9}$$

with $R_{\mu\nu}$ the Ricci tensor. Using the ansatz (8), the components ${}^t_t$ and the sum of ${}^t_t$ and ${}^r_r$ in equation (9) reduce to

$$\left(\left(l^2 - 2\,l_1\,l + {l_0}^2\right)\frac{f'}{f}\right)' + \frac{a^2 f^2}{l^2 - 2\,l_1\,l + {l_0}^2} = 0, \tag{10}$$

$$\left(\frac{f'}{f}\right)^2 + \frac{4\left({l_0}^2 - l_1^2\right) + a^2 f^2}{\left(l^2 - 2\,l_1\,l + {l_0}^2\right)^2} - \frac{16\pi G}{c^4}\,\phi'^2 = 0, \tag{11}$$

where prime stands for the derivative with respect to l. Recall that the Klein Gordon equation is a consequence of the Einstein's ones, so we only need to solve (10) and (11). Even though this is a solution for the complete space-time, we can see that it as describing the internal parts of a rotating wormhole, from the throat to some distance on each mouth, and match it with an external, asymptotically flat solution, such as Kerr or the static wormhole presented above. In what follows we present this last matching for the radial part of the solution, showing that it can be a smooth one. Then, the matched function for the rotating WH solution reads

$$f=\begin{cases}\exp(\lambda-\frac{\pi}{2}) & \text{if} \quad l>l_+\\ 2\,\frac{\phi_0\sqrt{D\left({l_0}^2-{l_1}^2\right)}\,e^{\phi_0\left(\lambda-\frac{\pi}{2}\right)}}{a^2+De^{2\phi_0\left(\lambda-\frac{\pi}{2}\right)}} & \text{if} \quad l_-\leq l\leq l_+\\ \phi_1\exp(\lambda+\frac{\pi}{2}) & \text{if} \quad l<l_-\end{cases} \tag{12}$$

where l_- and l_+ respectively are the matching points on the left and right hand side (see Fig.4). It can be seen that the interior solution matches with the rhs exterior once provided that the parameter D becomes:

$$D=2\,\phi_0\sqrt{\left({l_0}^2-{l_1}^2\right)\left(\left({l_0}^2-{l_1}^2\right){\phi_0}^2-\frac{a^2}{E^2}\right)}+2\,\left({l_0}^2-{l_1}^2\right){\phi_0}^2-\frac{a^2}{E^2} \tag{13}$$

where the constant E is determined by the radio l_+ where the two solutions match, it is given by:

$$E=\exp\left[\phi_0\left(\arctan\left(\frac{l_+-l_1}{\sqrt{l_0^2-l_1^2}}\right)-\frac{\pi}{2}\right)\right]. \tag{14}$$

In order to have a real solution everywhere we impose the constraint $4M^2=\left({l_0}^2-{l_1}^2\right){\phi_0}^2>\frac{a^2}{E^2}$. On the other hand, the matching of the interior solution is smooth with the lhs exterior one, if the constant ϕ_1 is chosen such that

$$\phi_1=2\,\frac{\phi_0\sqrt{D({l_0}^2-{l_1}^2)}\,e^{2\phi_0\lambda_-}}{a^2+De^{2\phi_0\left(\lambda_--\frac{\pi}{2}\right)}}, \tag{15}$$

where λ_- is evaluated at some $l=l_-$ on the lhs. In Fig. 4 we see the plot of f. Observe that the matching on the rhs could be smooth if l_+ is sufficiently large and the rotation parameter a is sufficiently small. On the contrary, for small l_+ or/and big rotation parameter the matching on the rhs is continuous but not necessarily smooth. On the lhs and for the scalar field the matching is always smooth. Following the procedure above described to obtain the shape of the throat, we obtain a figure similar to Fig. 1 as for the non-rotating case.

Let us write the solution in terms of the mass parameter M. We start with the constant D, it reads

$$D=M^2\,\left(4\sqrt{4-\frac{J^2}{E^2}}+8-\frac{J^2}{E^2}\right)=M^2\,d^2 \tag{16}$$

where $J=a/M$. Observe that d does not depend on the mass parameter $-M=mG/c^2$, where m is the total mass of the scalar field star. Furthermore, if we want a traversable

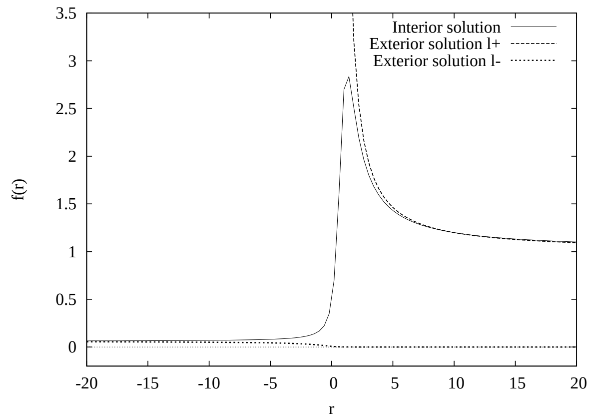

FIGURE 4. Plot of the function f for the exterior and interior solutions, observe that for these values the matching of the solutions on the rhs is smooth, while the matching is always smooth on the lhs. The values of the parameters are $l_1 = 0.5, l_+ = 10, \phi_0 = 2, l_0 = 1, a = 0.6$ and we plot for radius from -20 to 20.

wormhole, we expect a gravitational field similar to the Earth's one, this means $m \sim$ Earth's mass. For this value the mass parameter is $-M \sim 0.01$ meters. But we want that a spaceship can go trough the throat of the wormhole, we can suppose that $l_0 \sim 10$ meters, then, $l_1 \sim 10$ meters as well, depending on the value of ϕ_0. Thus, the interior solution reads

$$f_{int} = 4\,\frac{-M\sqrt{D}e^{\phi_0\left(\lambda-\frac{\pi}{2}\right)}}{a^2+De^{2\phi_0\left(\lambda-\frac{\pi}{2}\right)}} = 4\,\frac{d\,e^{\phi_0\left(\lambda-\frac{\pi}{2}\right)}}{J^2+d^2\,e^{2\phi_0\left(\lambda-\frac{\pi}{2}\right)}} \tag{17}$$

where the function λ is given by

$$\lambda = \arctan\left(-\phi_0\frac{l}{2M}\right) \tag{18}$$

Observe that $l_{Sch} = -2M$ is the Schwarzschild radius. The scalar field star which produce the wormhole could be thousands of meters long. Compare with the mass parameter M of orders of millimeters, the matching could be considered as if it were at infinity. In that case the parameter $E = 1$. Thus the solution has only 3 parameters, M, J and ϕ_0. In this unites $|J| \leq 2$. Then the parameter d is bounded to $2 \leq d \leq 4$. Nevertheless, the matching condition on the left hand side (in $l_\_$) does not depend on the mass parameter M, it is given by

$$\phi_1 = 4\,\frac{d\,e^{2\phi_0\lambda_-}}{J^2+d^2e^{2\phi_0\left(\lambda_- -\frac{\pi}{2}\right)}}. \tag{19}$$

where $\lambda_- = \lambda(l_\_)$. We can take $l_\_$ such that $\lambda_- \simeq -\pi/2$. Thus, on the left hand side the metric at minus infinity can be written as

$$ds^2_{\text{far}} = -\phi_1 dt_+^2 + \frac{1}{\phi_1}\left(dl_+^2 + l_+^2 d\Omega^2\right) = -dt_-^2 + dl_-^2 + l_-^2 d\Omega^2, \tag{20}$$

where now $\Delta t_{-} = \sqrt{\phi_1}\Delta t_{+}$ and $\Delta l_{-} = \frac{1}{\sqrt{\phi_1}}\Delta l_{+}$.

The constant ϕ_1 can be very big. Amazingly the major values for ϕ_1 are obtained for small rotation, $J << 1$. It can be seen that ϕ_1 has a maximum value when ϕ_0 is

$$\phi_{0max} = -\frac{1}{2\pi}\ln\left(\frac{J^2E^2}{4\sqrt{4E^2-J^2}\,E+8E^2-J^2}\right) \tag{21}$$

Thus, an observer on the left hand side space (on l_{-}) will feel that the time goes fast for small changes in Δt_{+} and will measure small changes of space for big changes on Δl_{+}. It is left to find a phantom field star with small rotation and scalar charge given by

$$\sqrt{\frac{8\pi G}{c^4}}q_\phi = 2M\sqrt{\frac{2}{\phi_0^2}+\frac{1}{2}} \tag{22}$$

with a scalar charge given by $\phi_0 = \phi_{0max}$. For example, if the phantom scalar star has a rotation parameter like $J = 10^{-10}$, then $\phi_1 \sim 1.4\times10^5$ when the scalar charge is $q_\phi \sim 0.3M\ m_{Plank}$. For a star with an Earth's mass, this charge is equivalent to $q_\phi \sim 0.003\ m_{Plank}$ per meter. These results are very similar to those propose by Carl Sagan in his science fiction story. A spaceship can travel trough the throat in a small time across very, very big distances. Suppose that the astronaut on the spaceship travels 18 hours = 6.48×10^4 seconds long, like in the Carl Sagan's Ellie Arroway did. An observer on the rhs space-time will measure $\Delta t_{+} = \frac{1}{\sqrt{\phi_1}}\Delta t_{-} = 205$ seconds, this means 3.4 minutes. On the other side, if Arroway travels only some meters, on the Earth people will measure some kilometers. Furthermore, an observer in our side of the space-time will measure a speed

$$\frac{\Delta l_{+}}{\Delta t_{+}} = \phi_1\frac{\Delta l_{-}}{\Delta t_{-}} \tag{23}$$

where $\Delta l_{-}/\Delta t_{-}$ is the speed of the Arroway proper spaceship. For the example we are studying here, this means that we see a speed $\sim 10^5$ times bigger than the spaceship proper speed. For example, if the spaceship can flight ~ 100 km per second, as a normal earthlings spaceship can do, the local observer will detect a speed $\sim 10^7$ km per second, this is ~ 100 times the speed of light. This velocity is enough for traveling to other stars.

Even when the wormholes still belong to the science fiction arena, we can say that the existence of the dark energy and the Casimir effect have open the door for the possibility of traveling across the Universe.

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