Primordial Perturbations Produced by a Self Interacting Scalar Field in the Braneworld: The Dynamical Systems Perspective.

Miguel A. García Aspeitia[a*], Juan Aldebarán Magaña[b*], Tonatiuh Matos [a*] and Pablo A. Rodriguez[a*]

[a]Departamento de Física, Centro de Investigación y de Estudios Avanzados del IPN, A.P. 14-740, 07000 México D.F., México. and

[b]Instituto de Astronomía,Universidad Nacional Autónoma de México, AP 70-543, 04510 D.F., México

In this work we explore the primordial perturbations by the slow-roll inflation produced by the simplest chaotic inflation model driven by a scalar field with potential $V_\Phi = \frac{1}{2}m_\Phi^2\Phi^2$ in a hidden brane and it is analyzed through a dynamical system to explore the consecuences in the evolution of the visible brane (our Universe). We use the most accepted constraints of the five dimensional Planck mass endorsed by the current experimental data in our universe (visible brane) to fit the initial conditions of Φ and $\dot{\Phi}$ of the inflation in the hidden brane.

INTRODUCTION

Braneworld theory is a promising alternative to observe effects in the dynamic of the high energy universe and it's an important window to understand the effects of quantum gravity [12] and their effects in higher dimensions. Exist different braneworld models trying to explain the real topology of our universe; some of these are proposed by Arkani-Hamed, Dimopoulos and Dvali (ADD) [19], Horava and Witten (M-theory) [18], Dvali, Gabadadze and Porrati (DGP) [22] and Randall, Sundrum (RSI-II) [20], [21]. To probe the differents model it's neccesary constraint the 5D Planck mass and compare with the model that most fit with this observations. For this reason, as exmaple, in the ADD model, exist different constraints of tha extra dimension Planck mass produced by experiments as particle colliders, astrophysical and cosmological observations. In particle colliders it is possible obtain information in low probability and high energy processe by

$$e^+e^- \to \gamma G, \quad e^+e^- \to ZG, \quad Z \to Gff^-, \tag{1}$$

where G is the graviton. In the first processe (electron-antielectron), the colliders obtain information because the photon energy is missing in a $e^+e^- \to \nu\nu^-\gamma(\gamma)$ background process [16]. Most information about the particle collider experiments can be obtained in [16] and [17]. In the Cosmological case, the Cosmic Microwave Background Radiation (CMB) constrain the extra dimension Planck mass as $M_n \gtrsim 65 - 750TeV,\ \ 4 - 32TeV$ and the existence of Kaluza-Klein gravitons (KK) increases the amount of matter in the Universe and help with a more fast cooling [16]. In this particular case, we assume two branes [10], [20] with different matter contents with a Anti D'Sitter-Schwarzchild (AdS-S) five dimensional bulk and the Z_2-symmetry between the interior and exterior region of each brane is imposed.

In the beginning (before of the collison) each brane have the next features:

---

* Part of the Instituto Avanzado de Cosmología (IAC) collaboration http://www.iac.edu.mx/

CP1256, *Gravitational Physics: Testing Gravity from Submillimeter to Cosmic Scale*
edited by H. A. Morales-Técotl, L. A. Ureña-López, R. Linares-Romero, and H. H. García-Compeán

1. In the hidden brane exist a scalar field with a chaotic quadratic potential as the responsible of the inflation. It's important to mention that this potential is consistent with the data for $V \sim \Phi^p$ with $p = 2$ at least in the cosmological observations (in the visible brane). We assume a perturbed FLRW braneworld and we obtain primordial perturbations produced by the scalar field and analyzed through a dynamical system point of view.
2. The observable brane is filled with an a primordial scalar field, where it's mass can be related with the mass of the inflation through RS [20] prescription at least locally (in a Minkowski frame). The mass of this primordial scalar field varies as the hidden brane approaching to the collision with the visible brane.

After the collision, both branes move together with a constant distance [10] and the hidden brane it's permeated with a scalar field with not inflationary properties while the visible brane it's permeated with a SF with a ultra light mass $m_{sfdm} \sim 10^{-22} eV$ [8] and behaves like a scalar field dark matter (SFDM) [8], [11]. The other fields predicted by the standar model emerge of the collision and with the SFDM create the current conditions of our universe.

THE MODIFIED EINSTEIN EQUATIONS IN THE BRANE.

We assume that the five dimensional Einstein equation is valid in the next way

$$G^{(5)}_{AB} + \Lambda^{(5)} g_{AB} = k^2_{(5)} T^{(5)}_{AB}, \tag{2}$$

where $k_{(5)}$ is the five dimensional gravitational constant and $\Lambda^{(5)}$ is the five dimensional cosmological constant. The relations between the five dimensional gravitational constant and the five dimensional Planck mass is [3]

$$k^2_{(5)} = 8\pi G_{(5)} = \left(\frac{8\pi}{M^3_{(5)}} \right), \tag{3}$$

where we use natural units $(\hbar = 1, c = 1)$, $G_{(5)}$ is the five dimensional Newtonian constant and $M_{(5)}$ is the five dimensional Planck mass. In the brane worlds, the energy momentum-tension on the brane $T_{\mu\nu}$ and the brane tension λ cause a discontinuity in the extrinsic curvature and this is given by the Israel-Darmoise equation [4]

$$[K_{\mu\nu}]^+_- = -k^2_{(5)} \{ \frac{1}{3}(\lambda - T) g_{\mu\nu} + T_{\mu\nu} \}, \tag{4}$$

where $T = g^{\mu\nu} T_{\mu\nu}$, $K_{\mu\nu} = g^A_\mu g^B_\nu \bigtriangledown_A n_B$, n^A is the unit normal vector to the brane. Then, the modified Einstein equation from the view of the brane.

$$G_{\mu\nu} + \Lambda_{(4)} g_{\mu\nu} = k^2_{(4)} T_{\mu\nu} + k^4_{(5)} \Pi_{\mu\nu} - \xi_{\mu\nu}, \tag{5}$$

where

$$\Lambda_{(4)} = \frac{\Lambda_{(5)}}{2} + \left(\frac{k^4_{(5)}}{12} \right) \lambda^2, \tag{6}$$

$$k^2_{(4)} = 8\pi G_N = \left(\frac{k^4_{(5)}}{6} \right) \lambda, \tag{7}$$

$$\Pi_{\mu\nu} = -\frac{1}{4}T_{\mu\alpha}T^{\alpha}_{\nu} + \frac{1}{12}TT_{\mu\nu} + \frac{1}{24}(3T_{\alpha\beta}T^{\alpha\beta} - T^2)g_{\mu\nu}, \tag{8}$$

$$\xi_{\mu\nu} \equiv C^{(5)E}_{AFB} n_E n^F g^A_\mu g^B_\nu, \tag{9}$$

and the conservations law equations can be obtained by $T^{\mu}_{\nu;\mu} = 0$. The previous modified Einstein equation represent the behavior of the gravity in the brane with a five dimensional frame of the brane theory. The contributions of the brane theory in the Einstenian gravity is the quadratic part of the energy momentum tensor (8) and the non local effects produced by the Weyl tensor (9). This terms will play a role in the search of a new physics produced by the extra dimension.

COSMOLOGICAL PERTURBATIONS.

The perturbed metric can be written by

$$g_{00} = -a(\tau)^2(1 + 2\Psi(\tau, \vec{x})), \tag{10}$$

$$g_{ij} = a(\tau)^2\delta_{ij}(1 - 2\phi(\tau, \vec{x})), \tag{11}$$

where $a(\tau)$ is the scale factor, $\Psi(\tau, \vec{x})$ correspond to the Newtonian potential and $\phi(\tau, \vec{x})$ correspond to the spatial curvature perturbation. In the other hand, the perturbed energy-momentum tensor for the matter on the brane can be written by

$$T^0_0 = -(\rho + \delta\rho), \tag{12}$$

$$T^j_i = (P + \delta P)\delta^j_i + \delta\pi^j_i, \tag{13}$$

$$T^j_0 = (\rho + P)v_i = -T^0_i, \tag{14}$$

where ρ and P is the background energy density and presure respectively, $\delta\pi^j_i = \delta\pi^j_{;i} - \frac{1}{3}\delta^j_i\delta\pi^k_k$ is the tracefree anisotropic stress perturbation. The perturbed quadratic energy-momentum tensor is [1]

$$\Pi^0_0 = -\frac{\rho}{12}(\rho + 2\delta\rho), \tag{15}$$

$$\Pi^0_i = \frac{\rho}{6}(\rho + P)v_i, \tag{16}$$

$$\Pi^j_i = \frac{\rho}{12}((2P + \rho + 2(1 + \frac{P}{\rho})\delta\rho + 2\delta P)\delta^j_i - (1 + \frac{3P}{\rho})\delta\pi^j_i). \tag{17}$$

And finally we write the perturbed Weyl tensor ξ_i^j by

$$-\xi_0^0 = -k_{(4)}^2(\rho_\xi + \delta\rho_\xi), \tag{18}$$

$$-\xi_i^0 = k_{(4)}^2\delta q_{\xi;i}, \tag{19}$$

$$-\xi_i^j = k_{(4)}^2((P_\xi + \delta P_\xi)\delta_i^j + \delta\pi_{\xi i}^j), \tag{20}$$

where $\delta q_\xi = (\rho_\xi + P_\xi)v_\xi$. Two of the 5-dimensional Einstein equations are equivalent to the conservation equations and can be writen in the next way

$$\dot{\delta\rho} + 3\frac{\dot{a}}{a}(\delta\rho + \delta P) - 3\dot{\phi}(\rho + P) + \bigtriangledown^2\delta q = 0, \tag{21}$$

and

$$\dot{\delta q} + 4\frac{\dot{a}}{a}\delta q + \partial_i(\delta P) + (\delta\rho + \delta P)\Psi_i = 0, \tag{22}$$

It is possible obtain a relation between the conformal and cosmological time as $\frac{d}{d\tau} = a\frac{d}{dt}$.

THE SCALAR FIELD AND PERTURBED EINSTEIN FIELD EQUATIONS.

In this section, we writte the perturbations of a general scalar field (SFDM in the visible and inflation SF in the hidden brane). Then the energy-momentum tensor associated with the scalar field is

$$T_{ij} = \Phi_{,i}\Phi_{,j} - \frac{1}{2}g_{ij}(g^{\alpha\beta}\Phi_{,\alpha}\Phi_{,\beta} + 2V(\Phi)), \tag{23}$$

then if we perturb the scalar field as $\Phi(\tau,\vec{x}) = \Phi^{(0)}(\tau) + \delta\Phi(\tau,\vec{x})$ we obtain the perturbed energy-momentum tensor

$$\delta T_0^0 = -a(\tau)^{-2}(\dot{\Phi}^{(0)}\delta\dot{\Phi} - \Psi\dot{\Phi}^{(0)2}) - V_{,\Phi}\delta\Phi = -\delta\rho, \tag{24}$$

$$\delta T_i^j = a(\tau)^{-2}(\dot{\Phi}^{(0)}\delta\dot{\Phi} - \Psi\dot{\Phi}^{(0)2})\delta_i^j - V_{,\Phi}\delta\Phi\delta_i^j = \delta P, \tag{25}$$

$$\delta T_i^0 = -a(\tau)^{-2}(\dot{\Phi}^{(0)}\delta\Phi_{,i}), \tag{26}$$

where $\delta\pi_i^j = 0$ because is a scalar field and in this case we assume that no local effects produced by the Weyl tensor are negligible and does not have effects in the role of the Universe evolution. In the other hand the quadratic energy-momentum tensor can be written by

$$\delta\Pi_0^0 = -\frac{1}{12}(\dot{\Phi}^{(0)2} + 2a^2V^{(0)})(a(\tau)^{-2}(\dot{\Phi}^{(0)}\delta\dot{\Phi} - \Psi\dot{\Phi}^{(0)2}) - V_{,\Phi}\delta\Phi), \quad (27)$$

$$\delta\Pi_i^0 = -\frac{1}{12}(\dot{\Phi}^{(0)2} + 2a^2V^{(0)})(a(\tau)^{-2}(\dot{\Phi}^{(0)}\delta\Phi_{,i})), \quad (28)$$

$$\delta\Pi_i^j = \frac{1}{12}(\frac{3}{4}\dot{\Phi}^{(0)4} - a^4V^{(0)2} + a^2\dot{\Phi}^{(0)2}V^{(0)} + 2\dot{\Phi}^{(0)2}\delta\rho + (\dot{\Phi}^{(0)2} + 2a^2V^{(0)})\delta P)\delta_i^j, \quad (29)$$

and therefore the projected Weyl Tensor is

$$\delta\xi_\mu^\nu = 0. \quad (30)$$

To obtain the perturbed Klein-Gordon equation [6], [7] we use the equation (21)

$$\delta\ddot{\Phi} + 2\frac{\dot{a}}{a}\delta\dot{\Phi} - \dot{\Psi}\dot{\Phi} - 3\dot{\Phi}\dot{\phi} + a^2V_{,\Phi\Phi}\delta\Phi + 2a^2\Psi V_{,\Phi} - \bigtriangledown^2\delta\Phi = 0. \quad (31)$$

In the other hand, the perutbed Einstein field equations can be written by

$$\delta G_\mu^\nu + \Lambda_{(4)}\delta_\mu^\nu = k_{(4)}^2\delta T_\mu^\nu + k_{(5)}^4\delta\Pi_\mu^\nu. \quad (32)$$

Using the results of the last equations and the equation (32) we can write the next field equations in the cosmological time

$$6H(\phi_{,0} + H\Psi) - \frac{2}{a^2}\bigtriangledown^2\phi - \Lambda_{(4)} = -(k_{(4)}^2 + \frac{k_{(5)}^4}{12}[\Phi_{,0}^{(0)2} + 2V^{(0)}])(\Phi_{,0}^{(0)}\delta\Phi_{,0} - \Psi\Phi_{,0}^{(0)2} + V_{,\Phi}\delta\Phi), \quad (33)$$

$$2(H\Psi + \phi_{,0})_{,i} - a\Lambda_{(4)} = (k_{(4)}^2 + \frac{k_{(5)}^4}{12}[\Phi_{,0}^{(0)2} + 2V^{(0)}])\Phi_{,0}^{(0)}\delta\Phi_{,i}, \quad (34)$$

$$\begin{aligned} 2[\phi_{,00} &+ H(\Psi_{,0} + 2\phi_{,0}) + (2\dot{H} + H^2)\Psi - \frac{1}{3a^2}\bigtriangledown^2(\phi - \Psi)] \\ &+ \Lambda_{(4)} = k_{(4)}^2[\Phi_{,0}^{(0)}\delta\Phi_{,0} - \Psi\Phi_{,0}^{(0)2} - V_{,\Phi}\delta\Phi] \\ &+ \frac{k_{(5)}^4}{12}[\frac{3}{4}\Phi_{,0}^{(0)4} - V^{(0)2} + \Phi_{,0}^{(0)2}V^{(0)} \\ &+ 2\Phi_{,0}^{(0)2}[\Phi_{,0}^{(0)}\delta\Phi_{,0} - \Psi\Phi_{,0}^{(0)2} + V_{,\Phi}\delta\Phi] \\ &+ (\Phi_{,0}^{(0)2} + 2V^{(0)})[\Phi_{,0}^{(0)}\delta\Phi_{,0} \\ &- \Psi\Phi_{,0}^{(0)2} - V_{,\Phi}\delta\Phi]], \end{aligned} \quad (35)$$

$$-\frac{2}{3a^2}(\phi-\Psi)^j_i+\Lambda_{(4)}\delta^j_i = k^2_{(4)}\delta T^j_i+k^4_{(5)}\delta\Pi^j_i \quad (i\neq j)\ , \tag{36}$$

and the respectively Klein-Gordon equation (31) in the cosmological time

$$\begin{aligned}\delta\Phi_{,00} &+ 2H\delta\Phi_{,0}-\Psi_{,0}\Phi_{,0}-3\Phi_{,0}\phi_{,0}+V_{,\Phi\Phi}\delta\Phi+2\ \Psi V_{,\Phi}\\ &- \frac{1}{a^2}\bigtriangledown^2\delta\Phi=0,\end{aligned} \tag{37}$$

where H is the Hubble cosmological constant.

PRIMORDIAL PERTURBATIONS CORRECTIONS IN THE BRANE.

In the next part it is more useful do it the calculations in the Fourier space. Therefore it is neccesary define the Fourier component of $\delta\Phi(\tau,x^i)$ by

$$\delta\Phi(\tau,x^i)=\frac{1}{(2\pi)^3}\int d^3\kappa\delta\Phi(\kappa^i)\exp(i\kappa_i x^i), \tag{38}$$

where k^i is the comoving wave number [6]. Using (38) we can rewrite the Einstein equations in the Fourier space by

$$\begin{aligned}-\alpha\Phi^{(0)}_{,0}\delta\Phi(\kappa^i)_{,0} &= \alpha(3H\Phi^{(0)}_{,0}\delta\Phi(\kappa^i)-\Psi\Phi^{(0)2}_{,0}\\ &+ V_{,\Phi}\delta\Phi(\kappa^i))+\frac{2}{a^2}\kappa^2\phi,\end{aligned} \tag{39}$$

$$2\phi_{,0}=-2H\Psi+\alpha\Phi^{(0)}_{,0}\delta\Phi(k^i), \tag{40}$$

rearranging the equation (35) it is possible to write in the next form

$$\begin{aligned}2[\phi_{,00} &+ H(\Psi_{,0}+2\phi_{,0})+(2\dot{H}+H^2)\Psi+\frac{1}{3a^2}\kappa^2(\phi-\Psi)]\\ &= \frac{1}{2}\alpha[\frac{3}{2}\Phi^{(0)2}_{,0}-V^{(0)}+2(\Phi^{(0)}_{,0}\delta\Phi(\kappa^i)_{,0}\\ &- \Psi\Phi^{(0)2}_{,0})\left(2+\frac{\Phi^{(0)2}_{,0}-2V^{(0)}}{\Phi^{(0)2}_{,0}+2V^{(0)}}\right)\\ &- V_{,\Phi}\delta\Phi(\kappa^i)],\end{aligned} \tag{41}$$

and

$$-\frac{2}{3a^2}(\phi-\Psi)^j_i=k^4_{(5)}\delta\Pi^j_i,(i\neq j), \tag{42}$$

where

$$\alpha = \left(\frac{k^4_{(5)}}{12}\right)\left(\Phi^{(0)2}_{,0} + 2V^{(0)}\right), \tag{43}$$

and the Klein-Gordon equation in Fourier Space can be written by

$$\begin{aligned} \delta\Phi(\kappa^i)_{,00} \; &+ \; 2H\delta\Phi(\kappa^i)_{,0} + (\frac{\kappa^2}{a^2} + V_{,\Phi\Phi})\delta\Phi(\kappa^i) \\ &= \; \Psi_{,0}\Phi_{,0} + 3\;\Phi_{,0}\phi_{,0} - 2\Psi V_{,\Phi}, \end{aligned} \tag{44}$$

DYNAMICAL SYSTEM WITH THE CHAOTIC INFLATION MODEL.

To obtain the physical meaning of the previos system (39-44), it is neccesary obtain the solution. Sience does not exist analytic solutions of the previous equations, it is neccesary transform the equations in a dynamical system and obtain a numerical solution with a Runge-Kutta IV method.

Now, we are in the capability of choose an appropiate adimensional variables in the next way

$$\begin{aligned} &x = \frac{\vartheta^2}{12}\frac{\Phi^2_{,0}}{H}\,, u = \frac{\vartheta^2 V^{(0)}}{6H}\,, j = \frac{\vartheta^2}{6H}\Phi_{,0}\sqrt{V}, \\ &z_1 = \vartheta\delta\Phi, l_1 = \Psi\,, U = -\vartheta\frac{V_{,\Phi}}{H^2}\,, x_1 = \phi\,, \\ &s = \sqrt{2}\frac{m}{H}, y = \vartheta\frac{\Phi_{,0}}{H}, x_2 = \frac{\phi_{,0}}{H}\,, \frac{\dot{H}}{H^2} = -\frac{3}{2}\Pi\,, \\ &l_2 = \frac{\Psi_{,0}}{H}\,, z_2 = \vartheta\frac{\delta\Phi_{,0}}{H}\,, m^2 = V_{,\Phi\Phi}\,, q = \frac{c}{H}, \end{aligned} \tag{45}$$

being $\vartheta^2 = k^2_{(5)}$ and c acts like an appropiate normalization constant. It is possible write the next autonomus dynamical system with respect to $n = lna$ by

$$z_1' = yl_1 - (3y - U)\frac{z_1}{y} - \frac{2\kappa^2}{a^2}\frac{l_1}{y}\frac{q^2}{(x+u)}, \tag{46}$$

$$l_1' = \frac{x+u}{2q}yz_1 - l_1, \tag{47}$$

$$\begin{aligned} l_2' \; &= \; 3\left(\frac{\Pi}{2} - 1\right)l_2 + \frac{x+u}{4q}[\left(\frac{3}{2}y^2 - 6qu\right) \\ &+ \; 2y(z_2 - l_1 y)\left(2 + \frac{x-u}{x+u}\right) + Uz_1] \\ &+ \; (3\;\Pi - 1)l_1, \end{aligned} \tag{48}$$

$$z_2' = z_2\left(\frac{3}{2}\Pi - 2\right) + 2Ul_1 + 4yl_2 - \left(\frac{\kappa^2}{a^2} + m^2\right) q^2 z_1, \quad (49)$$

being $'$ the differentiation with respect to the $e-$foldings $(\frac{d}{dt} = H\frac{d}{dn})$. In this case we use the condition $\phi = \Psi \Rightarrow l_1 = x_1, l_2 = x_2$.

ACKNOWLEDGMENT

This work was partially supported by CONACyT México, under grant 49865-F, 216536/219673. MAGA want to acknowledge the enlightening conversation with Abdel Pérez Lorenzana, Ivan Rodriguez Montoya and Abril Suarez. Instituto Avanzado de Cosmologia (IAC) collaboration.

[1] D. Langloise, R. Maartens, M. Sasaki, D. Wands. [arXiv:hep-th/0012044v1] 6 Dec 2000
[2] T. Shiromizu, K. Maeda, M. Sasaki. DAMPT University of Cambridge [arXiv:gr-qc/9910076v3] 17 Jan 2000.
[3] R. Maartens. [arXiv:gr-qc/0312059v2] 29 Apr 2004.
[4] P. McFadden. PhD thesis University of Cambridge [arXiv:hep-th/0612008v2] 2 Dec 2006
[5] C. P. Ma, E. Bertschinger. [arXiv:astro-ph/9506072v1] 11 Jun 1995
[6] I. Huston, K. Malik. [arXiv:astro-ph.CO/0907.2917v1] 16 Jul 2009
[7] K. Malik, D. Wands. Physics Reports 475 (2009) 1-51Elsevier
[8] I. Rodriguez, Juan A. Magña, T Matos. [arXiv:astro-phCO/0908.0054v1] 1 Aug 2009. To be Published
[9] H. Kodama, M. Sasaki. Progress of Theoretical Physics Supplement No. 78, 1984
[10] Miguel Ángel García Aspeitia and Tonatiuh Matos [arXiv:gr-qc/0906.3278v1] 17 Jun 2009. To be Published
[11] T. Matos, L Ureña Lopez, M. Alcubierre, R. Becerril, F. Guzman and D. Núñez. Lect. Notes Phys. 646, 401-420 (2004). Springer-Verlag Berlin Heidelberg.
[12] Jens Lyng Petersen. [arXiv:hep-th/9902131v2] 19 Feb. 1999.
[13] Kazuya Koyama. [arXiv:astro-ph/0303108v2] 11 Dec 2003
[14] Kazuya Koyama. [arXiv:astro-ph/0601220v1] 10 Jan 2006
[15] Syksy Rasanen. [arXiv:astro-ph/0208282v2] 12 Sep 2002
[16] Marco Cavaglia. International Journal of Modern Physics A. World Scientific Publishing Company.
[17] Abdel Pérez Lorenzana [arXuv:hep-ph/0503177v2] 30 May 2005
[18] E. Witten, Nucl. Phys. B471 (1996) 135; P. Horava and E. Witten, Nucl. Phys. B460 (1996) 506; idem, B475 (1996) 94.
[19] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B429 (1998) 263; I. Antoniadis, et al., Phys. Lett. B436 (1998) 257; I. Antoniadis, S. Dimopoulos, G. Dvali, Nucl. Phys. B516 (1998) 70.
[20] L. Randall and R. Sundrum, Phys. Rev. Lett 83 (1999) 3370.
[21] L. Randall and R. Sundrum, Phys. Rev. Lett 83 (1999) 4690.
[22] G. Dvali, G. Gabadadze, and M. Porrati, Phys. Lett. B 484, 112 (2000)