A Hydrodynamic Model of Galactic Halos

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Abstract The problem about the existence of dark matter in the universe is nowadays an open problem in cosmology. In this work we will present how we can build a hydrodynamic model in order to study dark matter halos of galaxies. The theoretical general idea is to start with the Einstein-Hilbert Lagrangian in which we have added a scalar field coupled minimally to the geometry. Then, by making variations of the corresponding action we come up with the Einstein field equations for the geometry and a Klein-Gordon equation for the scalar field. This set of coupled partial differential equations is non-linear. If we assume that dark matter halos can be described in the weak field limit we obtain a set of equations known as Schrödinger-Poisson equations. This set of equations can be written in the form of Euler equations for a fluid by making a Madelung transformation, where the self-interaction potential of the fluid is present. Also, there appears a quantum-like potential which depends non-linearly on the density of the fluid. We present results on the Jeans stability of the fluid model for dark matter and show how the physical parameters of the model can be determined, in particular, we compute the mass of the scalar field.

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1 Introduction

In the last decades we have witnessed a revolution of the human knowledge due to the appeareance of two fundamental problems in modern cosmology, both related to the structure formation in the universe. One is the dark matter problem and the other is the dark energy problem. The former was made evident from several observations, in particular, from observations of the rotation curve of spiral galaxies where the visible matter content is not enough to give the observed rotation curves of the stars or of the gas particles in the exterior regions of galaxies. Also, observations tell us that the universe appears to be expanding at an increasing rate. One way to explain this behavior is to introduce a constant in the Einstein field equations, the so called cosmological constant, that act in the opposite way to gravity. This constant has an associated density and in the general case is called dark energy, cosmological constant being one possibility.

A very successful model of dark matter is called cold dark matter (CDM) where the halos of galaxies or clusters of galaxies are composed of non-collisionless particles that at the moment of recombination were non-relativistic. The predictions of the CDM model is in concordance with observations at large scales ($\sim$ Mpc). However, CDM fails at small scales ($\sim$ kpc), for example, it predicts that the density profile in the origin of the dark matter halo diverges as $1/r$ and observations of galaxies show that the density is almost flat at the origin. This problem is called the cuspy problem. In addition, the CDM does not answer the fundamental question: What is dark matter? There are several candidates to explain the dark matter in the universe. One possibility is that dark matter is composed of one kind of supersymmetric particles called WIMP's (Weakly Interacting Massive Particles). But this particles have not been detected so far.

In this work we are proposing a hydrodynamical model to explain the dark matter component of the universe. The model considers that a scalar field can be the dark matter in the universe. Then by introducing this field in the corresponding terms in the Einstein field equations and making the weak field limit, valid to study halos of galaxies we come up with the equation that this field satisfy, this equation together with the Poisson equation that result from Einstein equations for the geometry in the newtonian limit, are called Schrödinger-Possion equations and they govern the scalar field and the standard newtonian potential with the source given in terms of the scalar field. This is our main set of equation that we have to solve to model dark matter halos of galaxies or cluster of galaxies. In order to obtain the hydrodynamical model, we transform this set of equations into an Euler system type of equations where the "fluid" fields are a density and a velocity field which are functions of the scalar field. Then, we linearize this set of equations to study the Jeans' instability of this scalar fluid which serves us to obtain the mass of the scalar field.

This paper is organized as follows: In section 2 we present the theoretical framework that start by writing a general Einstein-Hilbert action. In section 3 we present the hydrodynamical model. In section 4 we study the Jeans' instability and based on this analysis we compute the mass of the scalar field. Finally, in section 5 we give our conclusions.

2 Theoretical framework

Our hydrodynamical dark matter model is constructed starting with the Einstein-Hilbert action, where we have a complex scalar field, ϕ, coupled non-minimally with the geometry,

$$S[g_{\mu\nu},\phi] = \int d^4x \frac{1}{\kappa}\sqrt{-g}\mathscr{R} + \int d^4x \mathscr{L}_m\,, \tag{1}$$

where κ is a constant used to recover the Newtonian limit, $g = \det(g_{\mu\nu})$ is the determinant of the space-time metric $g_{\mu\nu}$, $\mathscr{R}$ is the Ricci curvature scalar. The first term in equation (1) gives the usual Einstein field equations of general relativity. The second term in the equation above is the matter term responsible to curve the space-time. In our case we will take the lagrangian density associated to the complex scalar field given by

$$\mathscr{L}_m = -\frac{1}{2}\sqrt{-g}\left[\gamma^2\nabla^\mu\phi\nabla_\mu\phi^* + m^2c^2\phi\phi^*\right]. \tag{2}$$

where ∇_μ is the covariant derivative compatible with the space-time metric, γ is a constant, m is the scalar field mass and c is the speed of light. The first term in this lagrangian is the kinetic term and the second is the contribution of the scalar field potential given by $V(\phi) = m^2c^2\phi\phi^*$. Applying a variational principle we make the variations of (1) with respect to the metric $g_{\mu\nu}$ we obtain the field equations for the space-time geometry,

$$\mathscr{G}_{\mu\nu} \equiv \mathscr{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathscr{R} = \frac{\kappa}{2}T_{\mu\nu}\,, \tag{3}$$

where $\mathscr{G}_{\mu\nu}$ is the Einstein tensor, $\mathscr{R}_{\mu\nu}$ is the Ricci tensor, and $T_{\mu\nu}$ is the energy-momentum tensor given by

$$T_{\mu\nu} = \gamma^2\nabla_{(\mu}\phi\nabla_{\nu)}\phi^* - \frac{1}{2}g_{\mu\nu}\left[\gamma^2\nabla^\alpha\phi\nabla_\alpha\phi^* + m^2c^2\phi\phi^*\right]. \tag{4}$$

where $\nabla_{(\mu}\phi\nabla_{\nu)}\phi^* = (\nabla_\mu\phi\nabla_\nu\phi^* + \nabla_\nu\phi\nabla_\mu\phi^*)/2$. We should mention that the Einstein tensor and the energy-momentum tensor are symmetric, i.e., $\mathscr{G}_{\mu\nu} = \mathscr{G}_{\nu\mu}$ y $T_{\mu\nu} = T_{\nu\mu}$.

Next, in order to obtain the equation of motion of the scalar field ϕ, we make the variation of the action (1) with respect to ϕ^*,

$$\delta S[\phi^*] = \frac{1}{2}\int d^4x\sqrt{-g}\left(\gamma^2 g^{\mu\nu}\nabla_\mu\phi\,\delta\nabla_\nu\phi^* + m^2c^2\phi\,\delta\phi^*\right). \tag{5}$$

Considering that the condition $\delta\partial_\nu\phi^* = \partial_\nu\delta\phi^*$ is valid and integrating by parts we obtain,

$$\delta S[\phi^*] = -\int d^4x\nabla_\nu\left(\gamma^2\sqrt{-g}g^{\mu\nu}\nabla_\mu\phi\delta\phi^*\right)$$

$$+\int d^4x\delta\phi^*\sqrt{-g}\left(\gamma^2\nabla_\mu(g^{\mu\nu}\nabla_\nu\phi)-m^2c^2\phi\right)=0\,, \tag{6}$$

where the first integral is a boundary term and from the second integral we obtain the equation of motion for the scalar field,

$$\nabla_\mu\nabla^\mu\phi-\left(\frac{mc}{\gamma}\right)^2\phi=0\,. \tag{7}$$

This equation is known as the Klein-Gordon equation. When we make the variation of the action (1) with respect to ϕ we obtain the respective equation for ϕ^*, which is equivalent to compute the complex conjugate of (7).

Now, given that, in general relativity, we associate to the curvature of the space-time a gravitational field, then weak gravitational fields correspond to approximately flat space-times. In such a case we consider the following metric,

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu} \tag{8}$$

with $|h_{\mu\nu}|\ll 1$ y $\eta_{\mu\nu}=\mathrm{diag}(-1,1,1,1)$ the Minkowski metric. We are saying that the space-time metric $g_{\mu\nu}$ is the flat space-time metric plus a perturbation. After a lengthly algebraic computation we arrive to the Einstein equations,

$$\partial_\alpha\partial^\alpha\bar{h}_{\beta\nu}=-\kappa T_{\beta\nu}\,, \tag{9}$$

where $\partial_\alpha\partial^\alpha$ is the D'Alambertian operator in the flat space-time, i.e., is the operator in the wave equation. This equations are the field equations for a weak gravitational field in its standard form. This expression is known as the weak field limit.

A particularly important application of the weak field limit is the Newtonian limit of general relativity. This limit not only corresponds to weak fields, it also corresponds to small velocities of the sources, which implies that the energy density T^{00} is much bigger than the stress density. In such a case we can consider only the $\bar{h}^{00}$ component and ignore the rest of the terms [6]. For a detailed analysis of Newtonian approximation of general relativity see [7]. We have used such a reference to present the results that follows. The field equations reduce to

$$\partial_\alpha\partial^\alpha\bar{h}^{00}=-\frac{16\pi G}{c^2}m^2\phi\phi^*\,, \tag{10}$$

where we have defined $\kappa=32\pi G/c^4$ and $T_{00}\approx\frac{1}{2}m^2c^2\phi\phi^*$. Besides, this implies that for small velocities, the time derivative in the D'alambertian operator is much smaller than the spatial derivatives, then the equation reduces to

$$\nabla^2\bar{h}^{00}=-\frac{16\pi G}{c^2}m^2\phi\phi^*\,, \tag{11}$$

where the operator ∇^2 is the laplacian in three dimensions. Comparing this result with the field equation of Newton $\nabla^2U=4\pi G\rho$ we conclude that in this limit $\bar{h}^{00}=-4U/c^2$ and $\bar{h}^{i0}=\bar{h}^{ij}=0$ (for $i\neq j$, $i,j=1,2,3$) and $m^2\phi\phi^*$ corresponds to the source. Returning to the definition of $\bar{h}_{\alpha\beta}$ in terms of $h_{\alpha\beta}$, we have that $h^{00} = h^{ii} = -2U/c^2$. The final expression of the newtonian limit of the Einstein field equations will be presented in (18) due to we still have to see how ϕ reduces in this limit. In the next section we will show this.

The metric of space-time in the newtonian approximation is

$$ds^2 = -(c^2 + 2U)dt^2 + (1 - 2U/c^2)d\mathbf{r} \cdot d\mathbf{r}\,, \tag{12}$$

where $d\mathbf{r} \cdot d\mathbf{r} = dx^2 + dy^2 + dz^2$. We have to mention that the series expansion parameter is the dimensionless quantity $U/c^2 \ll 1$, i. e., the weak field approximation.

2.1 *Schrödinger type of equation*

In this section we will show that the newtonian approximation to the Klein-Gordon equation will give us an equation for the scalar field similar to the Schrödinger equation. For simplicity in the computations we will use a unit system where $c = \gamma = 1$, but at the end of this section we will recover the SI units. Substituting (12) in (7)

$$(1 - 2U)\partial_t^2 \phi - (1 + 2U)\nabla^2 \phi + m^2 \phi \approx 0\,. \tag{13}$$

As an ansatz to make the newtonian limit of the Klein-Gordon equation let us take [1] [5]

$$\phi = \psi(\mathbf{x}, t)e^{-imt}\,. \tag{14}$$

Substituting in (13) and using that $U \ll 1$, we obtain

$$-\frac{1}{2m}\ddot{\psi} + i\dot{\psi} + \frac{1}{2m}\nabla^2 \psi - mU\psi = 0 \tag{15}$$

where $\ddot{\psi} = \partial_t^2 \psi$ y $\dot{\psi} = \partial_t \psi$. For a slowly time varying field we have that $\ddot{\psi} \approx 0$, then

$$i\dot{\psi} = -\frac{1}{2m}\nabla^2 \psi + mU\psi\,, \tag{16}$$

We return to SI units by making $m \to mc^2/\gamma$, $\nabla^2 \to c^2\nabla^2$ and $U \to U/c^2$ and obtain

$$i\gamma\dot{\psi} = -\frac{\gamma^2}{2m}\nabla^2 \psi + mU\psi\,, \tag{17}$$

This is an equation that have the form of the Schrödinger equation of quantum mechanics and we will call it *Schrödinger type equation*.

The reason why we said that equation (17) is of Schrödinger type and not that the equation is the Schrödinger equation of the quantum mechanics is simple, the field

[1] Another way of doing the newtonian approximation of the Klein-Gordon equation can be seen in reference [7].

ϕ in (7) is of classical nature and so is ψ. However, the methods of solution that we use to solve the standard Schröndiger equation can be applied in our case.

Using the ansatz (14) the Einstein field equations (11) reduce to the *Poisson equation*

$$\nabla^2 U = 4\pi G\rho \,, \tag{18}$$

with U the gravitational potential due to the scalar field where its associated source is $\rho = m^2\psi\psi^*$. The coupled set of equations given by (17) and (18) are known as *Schrödinger-Poisson system*. Studies of this system can be seen in [1, 3, 2].

3 Scalar field hydrodynamical model

We take as a motivation the hydrodynamical formulation of the Schrödinger equation of quantum mechanics as was given by Madelung and de Broglie [4], and de Broglie and Bohm [8], we will express (17) in a representation of hydrodynamic type. In this Madelung formulation the Schrödinger equation, which is a complex linear differential equation, is replaced by two coupled partial differential equations for the corresponding density and velocity fields.

Appliying the Madelung transformation[4]

$$\psi = R(\mathbf{r},t)e^{iS(\mathbf{r},t)} \,, \tag{19}$$

where R y S are real functions, in the Schrödinger equation

$$i\gamma\frac{\partial\psi}{\partial t} = -\frac{\gamma^2}{2m}\nabla^2\psi + mU(\mathbf{r},t)\psi \,, \tag{20}$$

we obtain

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0 \,, \tag{21}$$

$$\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla\left(U - \frac{\gamma^2}{2m^2}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right) . \tag{22}$$

where we have defined the field variables

$$\mathbf{v} = \frac{\gamma}{m}\nabla S \,, \tag{23}$$

$$\rho = m^2R^2 \,. \tag{24}$$

The equations (21) and (22) are the hydrodynamic formulation of the Schrödinger type equation for the field. These equations indicate that the time evolution of the field $\psi(\mathbf{r},t)$ is equivalent to flux of a “fluid” of density $\rho(\mathbf{r},t)$ where its particles of mass m are moving with a velocity $\mathbf{v}(\mathbf{r},t)$ under the action of a force derived from an external potential $U(\mathbf{r},t)$ plus an additional force due to a potential similar to the one whose origin is of quantum nature, $Q(\mathbf{r},t)$, that depends on the density of the fluid [4, 9, 10] defined as

$$Q(\mathbf{r},t) \equiv -\frac{\gamma^2}{2m^2}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\,. \tag{25}$$

From (19) we see that the scalar field is invariant under a change of phase $S' = S + 2\pi n$, and therefore,

$$\oint m\mathbf{v}\cdot d\mathbf{r} = \gamma\oint \nabla S\cdot d\mathbf{r} = \gamma\oint dS = 2\pi\gamma n\,, \tag{26}$$

with $n = 0, \pm 1, \pm 2, \ldots$. This result is a *compatibility condition* between the hydrodynamic equations (21) and (22) and the Schrödinger type equation, that is, to a solution $(\rho, \mathbf{v})$ there corrosponds a unique well defined ψ.

We should mention that the "fluid" described by (21) and (22) has an essential difference with respect to a ordinary fluid: In a rotational movement, $\mathbf{v}$ decreases when the distance to the center decreases and vice versa. This is due to the condition of compatibility (26).

As we have seen above, the dynamics of the scalar field is given by a system of equations of hydrodynamic type, this system of equations is similar to the one that describes a non-viscous fluid. Then, the set of equations in this formulation known as Schröndinger-Poisson is

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0 \tag{27}$$

$$\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla\left(U - \frac{\gamma^2}{2m^2}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right) \tag{28}$$

$$\nabla^2 U = 4\pi G\rho\,. \tag{29}$$

This system gives the dynamics of a self-gravitating "fluid" under the action of a potential of quantum type. By analyzing the structure of the equations we realize that we do not have a pressure term then we may think that we are dealing with a constant "equation of state". In the literature when $p = 0$ is said that the system is of dust type, and although is not realist, it is a good approximation to model some real special systems.

The dust equation of state can be used to model a collection of cold collisionless particles as in cosmology. In the astrophysical context it is useful to describe the structure of the rotation of rings or discs of particles. Also it can be used to study the collapse of shell of matter initially at rest, known as the Oppenheimer-Snyder. Given that there is no pressure, the movement of the fluid elements is according to the space-time geodesics. In an scheme of particles, we can consider the dust as a set of particles that have exactly the same local velocity (known as a lamellar flux), in such a way that there is no random component [11].

However, given that the force of quantum type is opposing to the self-gravitational force we can think that it is due to an effective pressure of the fluid an express it as

$$-\nabla Q = \frac{1}{\rho}\nabla p_q(\rho)\,, \tag{30}$$

where $p_q \neq 0$ plays the role of an equation of state.

4 Jeans' instability

In this section we present the stability analysis of the Schrödinger-Poisson system in its hydrodynamic form. This analysis is based on the study of the Jeans' instability for the gravitational collapse of protostellar gas.

James Jeans showed that for a homogeneous and isotropic fluid, small fluctuations in the density and velocity fields can be generated [12]. In particular, he shows that the fluctuations in the density can grow if the effects of instabilities due to pressure are smaller than gravity caused by a fluctuation of density. There is no surprise in the existence of such an effect because gravity is an attractive force. So the pressure forces are very small, a region with bigger density than the background is expected to attract more matter that its surroundings, then that region becomes even denser. If the system is denser more matter will attract, resulting that the fluctuation of density will collapse giving an object bound gravitationally. The criterion consists simply in to check if the typical length of a fluctuation is larger than the Jeans' length of the fluid.

Before we compute the Jeans' length let us make an order of magnitude estimate in order to understand its physical meaning. Let us consider that in a given instant there is an spherical homogeneous fluctuation of radius λ with positive density $\rho_1 > 0$ and mass M in a medium of mean density ρ_0. The fluctuation will grow if the self-gravitating force per unit mass, F_g, exceed the opposing force per unit mass due to the pressure F_p

$$F_g \approx \frac{GM}{\lambda^2} \approx \frac{G\rho\lambda^3}{\lambda^2} > F_p \approx \frac{p\lambda^2}{\rho\lambda^3} \approx \frac{v_s^2}{\lambda}\,, \tag{31}$$

where v_s is the speed of sound. This relationship implies that the fluctuation will grow if $\lambda > v_s(G\rho)^{-1/2}$. This establishes the existence of the Jeans' length $\lambda_J \approx v_s(G\rho)^{-1/2}$.

4.1 Hydrodynamic linear equations

To study the Jeans' instability it is necessary to linearize the hydrodynamic system of equations

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0\,, \tag{32}$$

$$\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla\left(U - \frac{\gamma^2}{2m^2}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right)\,, \tag{33}$$

$$\nabla^2 U = 4\pi G\rho\,. \tag{34}$$

This system of equations admits the static solution with $\rho = \rho_0$, $\mathbf{v} = 0$ and $\nabla U = 0$. However, if $\rho \neq 0$, the gravitational potential should vary spatially, i. e., an homogeneous distribution of matter can not be stationary and should be globally expanding or contracting. The incompatibility of an static universe with the cosmological principle seems to be in Newtonian gravity but the same effect is also the reason that the Einstein universe is unstable. Anyhow, when we consider an expanding universe the Jeans' results do not change qualitatively. Even though that this theory is inconsistent, can be interpreted to give correct results. Let us look for a solution of the hydrodynamic system (32)-(34) that represents a small perturbation to static solution

$$\rho \approx \rho_0 + \varepsilon\rho_1\,, \qquad \mathbf{v} \approx \varepsilon\mathbf{v}_1\,, \qquad U \approx U_0 + \varepsilon U_1\,, \tag{35}$$

where the perturbed variables ρ_1, $\mathbf{v}_1$ and U_1 are functions of space and time and $\varepsilon \ll 1$. Substituting (35) in (32)-(34) and keeping only the terms up to first order in ε, we obtain the linear set

$$\partial_t\rho_1 + \rho_0\nabla\cdot\mathbf{v}_1 = 0\,, \tag{36}$$

$$\partial_t\mathbf{v}_1 = \nabla U_1 + \frac{\gamma^2}{4m^2}\nabla\left[\nabla^2(\rho_1/\rho_0)\right]\,, \tag{37}$$

$$\nabla^2 U_1 = 4\pi G\rho_1\,. \tag{38}$$

4.2 Jeans' stability analysis

Let us study the solutions to the linear system (36)-(38) looking for solutions in the form of plane waves

$$\delta u_i = \delta_{i0}e^{i(\mathbf{k}\cdot\mathbf{r}+\omega t)}\,, \tag{39}$$

where the perturbations δu_i with $i = 1,2,3$ correspond to ρ_1, $\mathbf{v}_1$ and U_1, and the corresponding amplitudes δ_{0i} to D, $\mathbf{V}$ and $\mathscr{U}$, respectively. Vector $\mathbf{r}$ is the position vector, $\mathbf{k}$ is the wave vector and ω is the angular frequency of oscillation, which is in general a complex number. Using (39) in (36)-(38) and defining $\delta_0 \equiv D/\rho_0$ we obtain

$$\omega\delta_0 + \mathbf{k}\cdot\mathbf{V} = 0 \tag{40}$$

$$\omega\mathbf{V} + \mathscr{U}\mathbf{k} + \frac{\gamma^2}{4m^2}\delta_0 k^2\mathbf{k} = 0\,, \tag{41}$$

$$k^2\mathscr{U} + 4\pi G\rho_0\delta_0 = 0\,. \tag{42}$$

We very briefly consider the solutions with $\omega = 0$, i. e., the ones that do not depend on time. It is evident from (40), that the wave vector $\mathbf{k}$ is perpendicular to the velocity $\mathbf{v}_1$ and it is satisfied that $\nabla\times\mathbf{v}_1 \neq 0$. Putting together these expressions we obtain

$$k = \left(\frac{16\pi G\rho_0 m^2}{\gamma^2}\right)^{1/4}\,. \tag{43}$$

Now, let us find the dependent solutions, $\omega \neq 0$. We derive (36) with respect to t and consider that $\mathbf{v}_1$ is a function such that $\partial_t \nabla \cdot \mathbf{v}_1 = \nabla \cdot (\partial_t \mathbf{v}_1)$, and then we substitute the expressions (37) and (38) to obtain

$$\partial_t^2 \rho_1 - 4\pi G \rho_0 \rho_1 + \frac{\gamma^2}{4m^2} \nabla^2 \nabla^2 \rho_1 = 0\,. \tag{44}$$

Substituting (39) in (44) we arrive to the dispersion relationship

$$\omega^2 = \frac{\gamma^2}{4m^2} k^4 - 4\pi G \rho_0\,. \tag{45}$$

This relationship is analogous to the one Jeans found [12], the difference is in the quantum potential which makes that the magnitude of the wave vector has a 4th power. For stable solution, we ask that $\omega > 0$, that implies

$$k > k_J \equiv \left(\frac{16\pi G \rho_0 m^2}{\gamma^2} \right)^{1/4}, \tag{46}$$

where we have defined the Jeans' wave vector k_J. This results gives us that the Jeans's wave length, $\lambda_J = 2\pi/k_J$, is given by

$$\lambda_J = \left(\frac{\pi^3 \gamma^2}{G \rho_0 m^2} \right)^{1/4}. \tag{47}$$

Then, for wave lengths larger than the Jeans's wave length we have that perturbations increase or decrease exponentially. Using the Jeans' length definition, we rewrite the dispersion relationship as

$$\omega = \pm \frac{\gamma}{2m} k^2 \left[1 - \left(\frac{\lambda}{\lambda_J} \right)^4 \right]^{1/2}. \tag{48}$$

We recall that the speed of sound in the Jeans' theory is defined as $v_s^2 \equiv dp/d\rho$. But this is not our case, so we define a quantity that we will call the speed of sound as

$$\tilde{v}_s \equiv \frac{\gamma}{2m} k\,. \tag{49}$$

From (39), (40)-(42) and (49) we obtain

$$\frac{\rho_1}{\rho_0} = \delta_0 e^{i(\mathbf{k}\cdot\mathbf{r} \pm |\omega| t)}, \tag{50}$$

$$\mathbf{v}_1 = \mp \frac{\mathbf{k}}{k} \tilde{v}_s \delta_0 \left[1 - \left(\frac{\lambda}{\lambda_J} \right)^4 \right]^{1/2} e^{i(\mathbf{k}\cdot\mathbf{r} \pm |\omega| t)}, \tag{51}$$

$$U_1 = -\delta_0 \tilde{v}_s^2 \left(\frac{\lambda}{\lambda_J} \right)^4 e^{i(\mathbf{k}\cdot\mathbf{r} \pm |\omega| t)}. \tag{52}$$

When $\lambda > \lambda_J$ the frequency ω is imaginary

$$\omega = \pm i(4\pi G\rho_0)^{1/2}\left[1-\left(\frac{\lambda_J}{\lambda}\right)^4\right]^{1/2}. \tag{53}$$

In this case the above expressions become

$$\frac{\rho_1}{\rho_0} = \delta_0 e^{i\mathbf{k}\cdot\mathbf{r}\pm|\omega|t}, \tag{54}$$

$$\mathbf{v}_1 = \mp i\frac{\mathbf{k}}{k^2}\delta_0(4\pi G\rho_0)^{1/2}\left[1-\left(\frac{\lambda_J}{\lambda}\right)^4\right]^{1/2} e^{i\mathbf{k}\cdot\mathbf{r}\pm|\omega|t}, \tag{55}$$

$$U_1 = -\delta_0\tilde{v}_s^2\left(\frac{\lambda}{\lambda_J}\right)^4 e^{i\mathbf{k}\cdot\mathbf{r}\pm|\omega|t}. \tag{56}$$

They represent wave solutions, whose amplitudes increase or decrease in time. The characteristic scale time for the evolution of these amplitudes are defined as

$$\tau \equiv |\omega|^{-1} = (4\pi G\rho_0)^{-1/2}\left[1-\left(\frac{\lambda_J}{\lambda}\right)^4\right]^{-1/2}. \tag{57}$$

For scales $\lambda \gg \lambda_J$, the characteristic time τ is the same as the collapse time for free fall (the collapse time of a system under the action of its own gravity in the absence of opposing forces), $\tau_{ff} \approx (G\rho_0)^{-1/2}$, but when $\lambda \longrightarrow \lambda_J$, this time diverges.

We define the Jeans's mass as the mass that it is contained in a sphere of radius $\lambda_J/2$ and mean density ρ_0, i. e.,

$$M_J = \frac{1}{6}\pi\rho_0\left(\frac{\pi^3\gamma^2}{G\rho_0 m^2}\right)^{3/4}. \tag{58}$$

The Jeans' mass is the minimum mass a perturbation needs in order to grow.

In a work by Sikivie and Yang [13] the cosmological case of a model of axions as a dark matter model is considered. The conclusion of this work is that the Bose-Einstein condensate of axions and CDM are not distinguishable to all scales of observational interest. However, in the nonlinear regime of structure formation, this two theories differ in the quantum potential.

4.3 Scalar field mass estimation

From equation (47) we obtain the mass of the scalar field

$$m = \left(\frac{\pi^3 \gamma^2}{G\rho_0 \lambda_J^4} \right)^{1/2} , \tag{59}$$

We can make an estimation of the order of magnitude as follows

$$\gamma \sim 10^{-34}\,\mathrm{J\cdot s} \to \gamma^2 \sim 10^{-68}\,\mathrm{J^2\cdot s^2}\,, \tag{60}$$

$$G \sim 10^{-11}\,\mathrm{m^3/Kg\cdot s^2} \tag{61}$$

According to the standard theory of structure formation, a value for the Jeans' length λ_J can be fixed for the epoch in which matter and radiation were equal, when $a \simeq 1/3200$. In that time, $\rho \simeq 3\times 10^{10}\mathrm{M}_\odot a^{-3}/\mathrm{Mpc}^3$ [14], and the Jeans' length $\sim \mathscr{O}(10^2)$ pc [15]. Here $\mathrm{M}_\odot = 1.9891\times 10^{30}$Kg is the mass of the sun. Then, taking the density $\rho_0 \sim 10^{-14}\mathrm{Kg/m^3}$ and $\lambda_J \sim 10^2$pc we have

$$m^2 \sim \frac{\gamma^2}{G\rho_0\lambda_J^4} \tag{62}$$

$$\sim 10^{-114}\mathrm{Kg}^2 \tag{63}$$

$$m \sim 10^{-57}\mathrm{Kg} \tag{64}$$

In units of $e\mathrm{V/c^2}$ the mass is

$$m \sim \frac{10^{-57}\mathrm{Kg}10^{16}\mathrm{m^2/s^2}}{\mathrm{c}^2} = 10^{-41}\mathrm{J/c^2} = 10^{-22}e\mathrm{V/c^2}\,. \tag{65}$$

Therefore, with the computation of the Jeans' stability, we obtain that the mass of the scalar field is $\sim 10^{-22}e\mathrm{V/c^2}$, a result also obtained in [16, 17]. From the Jeans' mass definition we finally obtain

$$M_J \sim 10^{10}\mathrm{M}_\odot\,. \tag{66}$$

5 Conclusions

We have shown how a hydrodynamic model can be build to study the dark matter component of the universe. Starting with the Einstein-Hilbert action where a term which contains a complex scalar field is added we make the newtonian limit to arrive to a set of equations known as Schrödinger-Poisson equations. If in addition, we use the Madelung transformation this set of equations can be further transformed to a set of partial differential equations similar to the Euler equations of the hydrodynamics. Then, we used the standard techniques known to analyze astrophysical fluids such as the Jeans' instability method which we have used to determine the mass of the scalar field. The value we have obtained for the mass is consistent with values obtained using other methods [16, 17].

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