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Phase Transition from the Symmetry Breaking of Charged Klein–Gordon Fields

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Abstract. We analyze the phase transition associated with the $U(1)$ symmetry breaking of the complex Klein–Gordon (KG) equation with a Mexican–hat scalar field potential up to one loop in perturbations immersed in a thermal bath. We show that the KG equation reduces to a Gross–Pitaevskii like–equation (GP), which also contains the entire information of the phase transition. Indeed, the thermal bath contributions, together with the corresponding $U(1)$ local symmetry, allow us to interpret the resulting GP equation as a charged and finite temperature version of the system. Finally, we obtain the hydrodynamics and consequently, the corresponding thermodynamics, and show that breakdown of the $U(1)$ local symmetry of the KG field into the new version of the GP equation corresponds, under certain circumstances, to a phase transition of the gas into a condensate, superfluid, and/or superconductor.

Keywords: Bose-Einstein condensates, Symmetry breaking, Phase transitions
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INTRODUCTION

Since its observation with the help of magnetic traps [1], the phenomenon of Bose–Einstein condensation has spurred an enormous amount of works on the theoretical and experimental realms associated with this topic. The principal interest in the study on Bose–Einstein condensation is its interdisciplinary nature. From the thermodynamic point of view, this phenomenon can be interpreted as a phase transition, and from the quantum mechanical point of view as a matter wave coherence arising from overlapping de Broglie waves of the atoms, in which many of them condense to the grown state of the system. In quantum field theory, this phenomenon is related to the spontaneous breaking of a gauge symmetry [2]. Symmetry breaking is one of the most essential concepts in particle theory and has been extensively used in the study of the behavior of particle interactions in many theories [3]. The concept with the accompanying wave function describing the condensate, was first introduced in explaining superconductivity and super fluidity [4]. Phase transitions are changes of state, related with changes of symmetries of the system. The analysis of Symmetry breaking mechanisms have turn out to be very helpful in the study of phenomena associated to phase transitions in almost all areas of physics. Bose-Einstein Condensation is one topic of interest that uses in an extensive way the Symmetry breaking mechanisms [2], its phase transition associated with the condensation of atoms in the state of lowest energy and is the consequence of quantum, statistical and thermodynamical effects.

On the other hand, the results from finite temperature quantum field theory [5, 6] raise important challenges about their possible manifestation in condensed matter systems. By investigating the massive Klein–Gordon equation (KG), in Refs. [7, 8] we were able to show, that the Klein–Gordon equation can simulate a condensed matter system. In Refs. [7, 8] it was proved that the Klein–Gordon equation with a self interacting scalar field (SF) in a thermal bath reduces to the Gross–Pitaevskii equation in the no-relativistic limit, provided that the temperature of the thermal bath is zero. Thus, the Klein–Gordon equation reduces to a generalized relativistic, Gross–Pitaevskii (GP) equation at finite temperature. But a question remains open. The Klein–Gordon equation with a self interacting scalar field potential defines a symmetry breaking temperature, at which the system experiments a phase transition. However, this phase transition does not necessarily means a condensation of the particles of the system.

In the present work we study the complex Klein-Gordon (KG) equation with a Mexican hat scalar field potential in a thermal bath. The idea is very simple, the KG equation up to one loop in perturbations is able to explain the phase

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transition of a scalar field, like the Higgs field, close to the moment of the phase transition, when the KG equation breaks the $U(1)$ symmetry of the corresponding Lagrangian. On the other hand, the GP equation is able to explain the behavior of a BEC at zero temperature. In this work we wonder if the KG equation is able to generalize the GP equation in a region close to the phase transition of the Bose gas into a BEC, because the KG equation contains the information of the temperature in the scalar field potential. In base of this idea, we rewrite the complex KG equation into a GP like–equation an interpret it as a GP one at finite temperature. In order to see if this equation can explain the phase transition of a Bose gas into a BEC we write its corresponding thermodynamics, deriving a corresponding first law for the Bose gases. The only difference we find here with respect to the traditional first law of the thermodynamics is a therm where the quantum mechanical character of the KG equation is present. On the other hand, a phase transition does no necessarily means gas condensation. Therefore we obtain the temperature and the conditions of condensation of this Bose gas. At the end of the work we qualitatively see under which conditions the Bose gas becomes superfluid or/and superconductor.

GAUGE SYMMETRY BREAKING

We start with a model having a local U(1) symmetry given by the lagrangian,

$$\mathscr{L} = -\left(\nabla_\mu \Phi^* + \mathrm{i}eA_\mu \Phi^*\right)\left(\nabla^\mu \Phi - \mathrm{i}eA^\mu \Phi\right) - V(|\Phi|) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{1}$$

where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ is the Maxwell tensor and the scalar field potential V is the easiest case of a double-well interacting Mexican-hat potential for a complex SF $\Phi(\vec{x},t)$, interacting inside a thermal bath in a reservoir that can have interaction with its surroundings up to one loop of correction, that goes as

$$V(\Phi) = -\hat{m}^2\Phi\Phi^* + \frac{\hat{\lambda}}{2}(\Phi\Phi^*)^2 + \frac{\hat{\lambda}}{4}k_B^2T^2\Phi\Phi^* - \frac{\pi^2 k_B^4}{90\hbar^2c^2}T^4, \tag{2}$$

where $\hat{m}^2 = m^2c^2/\hbar^2$ is the scalar particle mass, $\hat{\lambda} = \lambda/(\hbar^2c^2)$ is the parameter describing the interaction, k_B is Boltzmann's constant, $\hbar$ is Planck's constant, c is the speed of light and T is the temperature of the thermal bath, this result includes both quantum and thermal contributions.

The dynamics of a SF is governed by the Klein-Gordon (KG) equation, it is the equation of motion of a field composed of spinless particles. To confine the BEC experimentally one needs to add an external field controlled by hand in order to cause the condensation, like a laser for example. Thus, in this case we will add an external field ϕ that will interact with the SF to first order, such that the KG equation can be expressed as follows

$$\Box_E^2\Phi - \frac{dV}{d\Phi^*} - 2\hat{m}^2\phi\Phi = 0. \tag{3}$$

For a charged field the D'Alambertian operator is given by

$$\Box_E^2 \equiv \left(\nabla_\mu - \mathrm{i}eA_\mu\right)\left(\nabla^\mu - \mathrm{i}eA^\mu\right), \tag{4}$$

where A_μ is the electromagnetic four potential. Observe that we can rewrite the D'Alambertian as

$$\Box_E^2 = (\nabla - 2\mathrm{i}\hat{e}A)\cdot\nabla - (\frac{1}{c}\frac{\partial}{\partial t} - 2\mathrm{i}e\varphi)\frac{1}{c}\frac{\partial}{\partial t} - \mathrm{i}e\nabla_\mu A^\mu - e^2A_\mu A^\mu, \tag{5}$$

where we have written the electromagnetic four potential as $A_\mu = (A, \varphi)$. In what follows we will use the Lorentz gauge $\nabla_\mu A^\mu = 0$. It is convenient to consider the total potential V adding the external one and the term $e^2A_\mu A^\mu = e^2A^2$ to the potential, such that

$$V_T(\Phi) = -\frac{m^2c^2}{\hbar^2}\Phi\Phi^* + \frac{\lambda}{4\hbar^2c^2}k_B^2T^2\Phi\Phi^* - \frac{e^2}{\hbar^2}A^2\Phi\Phi^* + \frac{\lambda}{2\hbar^2c^2}(\Phi\Phi^*)^2 - \frac{\pi^2k_B^4}{90\hbar^2c^2}T^4 + \frac{2m^2c^2}{\hbar^2}\phi\Phi\Phi^*. \tag{6}$$

Thus, the KG equation can be re–written as

$$\Box^2\Phi - \frac{dV_T}{d\Phi^*} = 0, \tag{7}$$

where now $\Box^2 = \nabla_\mu \nabla^\mu$.

The Maxwell equations also read

$$\nabla^\mu F_{\mu\nu} = -j_\nu = ie(\Phi^* \nabla_\nu \Phi - \Phi \nabla_\nu \Phi^*) + 2e^2 \Phi \Phi^* A_\nu. \tag{8}$$

We can define an effective mass by

$$m_{eff} c = \sqrt{m^2 c^2 + e^2 A^2}. \tag{9}$$

For the V_T potential (6), the critical temperature T_c^{SB} where the minimum of the potential $\Phi = 0$ becomes a maximum and at which the symmetry is broken is given by

$$k_B T_c^{SB} = \frac{2c^2}{\sqrt{\lambda}} \sqrt{m_{eff}^2 + 2m^2 \phi}. \tag{10}$$

The scalar potential (2) has a minimum in $\Phi = 0$ when the temperature $T > T_c^{SB}$. If $T < T_c^{SB}$, the point $\Phi = 0$ becomes a maximum and potential (2) has two minima in

$$\begin{aligned} R_{min} &= \pm \sqrt{\frac{1}{\lambda}(m^2 c^4 + e^2 c^2 A^2 - \frac{\lambda}{4} k_B^2 T^2 + 2m^2 c^4 \phi)} \\ &= \pm \frac{k_B}{2} \sqrt{(T_c^{SB})^2 - T^2}, \end{aligned} \tag{11}$$

being $\Phi = Re^{i\Theta}$.

In the maximum $\Phi = 0$ the second derivative of the potential V_T with respect to the SF reads

$$\begin{aligned} V_{T,\Phi\Phi} &= -\left(\hat{m}^2 + e^2 A^2 - \frac{\hat{\lambda}}{4} k_B^2 T^2 + 2\hat{m}^2 \phi \right) \\ &= -\frac{\hat{\lambda}}{4} k_B^2 \left((T_c^{SB})^2 - T^2 \right) \\ &= -\frac{(T_c^{SB})^2 - T^2}{(T_c^{SB})^2} \left(\hat{m}_{eff}^2 + 2\hat{m}^2 \phi \right). \end{aligned} \tag{12}$$

THE GENERALIZED GROSS-PITAEVSKII EQUATION

Now for the SF we perform the transformation

$$\Phi = \Psi \, \mathrm{e}^{-\mathrm{i}\hat{m}ct},$$

In terms of the complex function Ψ, the KG equation (3) now reads,

$$\mathrm{i}\hbar c \dot{\Psi} + \frac{\hbar^2}{2m} \Box_E^2 \Psi - \frac{\lambda}{2mc^2} |\Psi|^2 \Psi - mc^2(\phi - 1)\Psi + ec\varphi \Psi - \frac{\lambda k_B^2 T^2}{8mc^2} \Psi = 0, \tag{13}$$

where we have kept just the equation for the Ψ part, the complex conjugate can be described in the same way. The notation used is; $\dot{} = 1/c\,\partial/\partial t$ and $\kappa^2 |\Psi|^2 = \kappa^2 \Psi \Psi^* = n$, where κ is the scale of the system, which is to be determined by a experiment. (13) is the KG equation (3) or (9) rewritten in terms of the function Ψ and temperature T. This equation is an exact equation defining the field $\Psi(\mathbf{x}, t)$, where ϕ defines the external potential acting on the system and the terms in λ represent the interaction potential within the system. We will consider equation (13) as a generalization of the Gross-Pitaevskii equation for finite temperatures and relativistic particles. This is because at second order, when $T = 0$ and in the non-relativistic limit, $\Box^2 \to \nabla^2$, eq. (13) becomes the Gross-Pitaevskii equation for Bose-Einstein Condensates (BEC), provided that $\lambda = 8\pi\hbar^2 c^2 a$, being a the s-wave scattering length [32]. The static limit of equation (13) is known as the Ginzburg-Landau equation.

Here is important to remark that if we assume the external potential ϕ and the additional field A^2 time independent and proportional to $\sim r^s$, with $s \geq 1$, it is possible to express the critical temperature (10) near to the center of the system ($r \sim 0$) approximately as $k_B T_c^{SB} \sim \frac{2mc^2}{\sqrt{\lambda}}$. Notice also that, in order to interpret equation (13) as a Gross–Pitaevskii like–equation, we have to relate the interaction parameter λ with the scattering length a, as $\lambda = 8\pi\hbar^2 c^2 a \kappa^2$. By using this fact, the critical temperature (10) can be estimated up to $T_c^{SB} \sim 5\frac{Joules^{-1}m^{-3/2}}{\kappa}\left(\frac{m}{gr}\right)\sqrt{\frac{cm}{a}} \times 10^{62}\,K$ [8]. For instance, in the case of ^{87}Rb, we obtain $T_c^{SB} \sim \frac{1}{\kappa}10^{42}\,K$, which depends on the value of the *scale* κ. On the other hand, it can be shown that $n = \kappa^2 k_B\,(T_c^{SB})^2/4(1-(T/T_c^{SB})^2) \sim \kappa^2 k_B\,(T_c^{SB})^2/4 = m^2c^2/(8\pi\hbar^2\,a)$, notice that the density n does not depend on the value of κ. Thus, for the ^{87}Rb we have $n \sim 10^{36}/\mathrm{cm}^3$. These facts suggest that in order to obtain a measure of the corresponding symmetry breaking temperature in typical systems, we will need very dense systems together with large values of the *scale* κ.

THE HYDRODYNAMICAL VERSION

In what follows we transform the generalized Gross-Pitaevskii equation (13) into its analogous hydrodynamical version, [10, 11], for this purpose the ensemble wave function Ψ will be represented in terms of a modulus n and a phase S as,

$$\kappa\Psi = \sqrt{n}\,\mathrm{e}^{\mathrm{i}S}. \tag{14}$$

where the phase $S(\vec{x},t)$ is taken as a real function. As usual this phase will define the velocity. Here we will interpret $n(\vec{x},t) = \rho/M_T$ as the rate between the number density of particles in the condensed state, $\rho = mn_0 = mN_0/L^3$, being N_0 the number of particles in condensed state and M_T the total mass of the particles in the system, being both, S and n, functions of time and position. The concept of SB is often used as a sufficient condition for BEC. The assumption that the ground state can be macroscopically occupied, is nothing but Einstein's criterion for condensation.

So, from this interpretation we have that when the KG equation oscillates around the $\Phi = 0$ minimum, the number of particles in the ground state is zero, $n = 0 = \rho$. Below the critical temperature T_c^{SB}, close to the second minimum, $\Phi^2_{min} = k_B^2((T_c^{SB})^2 - T^2)/4$, the density will oscillate around $n = \kappa^2 k_B^2((T_c^{SB})^2 - T^2)/4$ as can be seen by equation (10). In order to see this more clear, we perform the Madelain transformation (14) in the generalized Gross-Pitaevskii equation (13).

From (13) and (14), separating real and imaginary parts we obtain

$$c\,\dot{n} + \frac{\hbar}{m}n\left[\Box^2 S - e\,(\nabla\cdot A - \dot{\varphi})\right] + \frac{\hbar}{m}\left((\nabla S - eA)\cdot\nabla n - (\dot{S} - e\varphi)\dot{n}\right) \quad = \quad 0, \tag{15a}$$

$$\frac{\hbar c}{m}(\dot{S} - e\varphi) + \frac{\lambda}{2m^2c^2\kappa^2}n + c^2(\phi - 1) + \frac{\lambda}{8m^2c^2}k_B^2T^2 + \frac{\hbar^2}{m^2}\left(\frac{\Box^2\sqrt{n}}{\sqrt{n}}\right) + \frac{\hbar^2}{2m^2}\left((\nabla S - eA)^2 - (\dot{S} - e\varphi)^2\right) \quad = \quad 0. \tag{15b}$$

Taking the gradient of (15b) and using the definitions of the fluxes

$$j \quad = \quad \frac{2en}{\kappa^2}(\nabla S - eA) \tag{16a}$$

$$j \quad = \quad \frac{2en}{\kappa^2}(\dot{S} - e\varphi) \tag{16b}$$

$$j_\mu \quad = \quad (j, j - \frac{2en}{\kappa^2}\hat{m}) \tag{16c}$$

and the velocity

$$v \equiv \frac{\hbar}{m}(\nabla S - eA) \tag{17}$$

equations (15) can be rewritten as

$$\dot{n} + \nabla\cdot(nv) - \frac{\hbar\kappa^2}{2mec}\dot{j} = 0, \tag{18a}$$

$$\begin{aligned}\dot{v} + (v\cdot\nabla)v - \frac{\hbar}{m}e\,(cE + v\times B) = -c^2\nabla\phi - \frac{\lambda}{m^2c^2\kappa^2}\nabla n - \frac{\hbar^2}{m^2}\nabla\left(\frac{\nabla^2\sqrt{n}}{\sqrt{n}}\right) \quad + \\ \frac{\hbar^2}{2m^2}\nabla(\dot{S} - e\varphi)^2 + \frac{\hbar^2}{m^2}\nabla\left(\frac{\partial_t^2\sqrt{n}}{\sqrt{n}}\right) \quad - \quad \frac{\lambda k_B^2}{4m^2}T\nabla T\end{aligned} \tag{18b}$$

where $E = -1/c\partial A/\partial t + \nabla\cdot\varphi$ and $B = \nabla\times A$ respectively are the electric and the magnetic field vectors. Notice that in (18b) $\hbar$ enters on the right-hand side through the term containing the gradient of n. This term is usually called the 'quantum pressure' and is a direct consequence of the Heisenberg uncertainty principle, it reveals the importance of quantum effects in interacting gases. Multiplying by n, (18b) can be written as:

$$n\dot{v} + n(v\cdot\nabla)v = nF_E + nF_\phi - \nabla p + nF_Q + \nabla\sigma, \tag{19}$$

where $F_E = \frac{e}{m}(cE + v\times B)$ is the electromagnetic force, $F_\phi = -\nabla\phi$ is the force associated to the external potential ϕ, p can be seen as the pressure of the SF gas that satisfies the equation of state $p = wn^2$, ∇p are forces due to the gradients of pressure and $\omega = \lambda/(2m^2c^2\kappa^2)$ is an interaction parameter, $F_Q = -\nabla U_Q$ is the quantum force associated to the quantum potential, [12, 32],

$$U_Q = \frac{\hbar^2}{m^2}\left(\frac{\nabla^2\sqrt{n}}{\sqrt{n}}\right), \tag{20}$$

and $\nabla\sigma$ is defined as

$$\begin{aligned}\nabla\sigma = \frac{\hbar^2}{2m^2}n\nabla(\dot{S} - e\varphi)^2 \quad &- \quad \frac{1}{4}\frac{\lambda}{m^2}k_B^2 nT\nabla T \\ -\zeta\nabla(\ln n\dot{)} \quad &+ \quad \frac{\hbar^2 n}{2m^2}\nabla\left(\frac{\ddot{n}}{n}\right),\end{aligned} \tag{21}$$

where the coefficient ζ is given by

$$\zeta = \frac{\hbar^2}{2m^2}\left[-\nabla\cdot(nv) + \frac{\hbar\kappa^2}{2mec}\dot{j}\right],$$

and the term $\nabla(\ln n\dot{)}$ can be written as

$$\nabla(\ln n\dot{)} = -\nabla(\nabla\cdot v) - \nabla[\nabla(\ln n)\cdot v] + \frac{\hbar\kappa^2}{2mec}\nabla\left(\frac{1}{n}\dot{j}\right).$$

System (18) is the hydrodynamical representation to equation (13) and is completely equivalent to it.

THE NEWTONIAN LIMIT

Neglecting second order time derivatives and products of time derivatives we can simplify system (18). In this limit we arrive to the non-relativistic system of equations (18)

$$\dot{n} + \nabla\cdot(nv) = 0, \tag{22a}$$

$$\begin{aligned} n\dot{v} + n(v\cdot\nabla)v &= nF_E + nF_\phi - \nabla p \\ &+ nF_Q + \nabla\sigma. \end{aligned} \tag{22b}$$

Equation (22a) is the continuity equation, and (22b) is the equation for the momentum. Observe that this last one contains forces due to the external potential, to the gradient of the pressure, viscous forces due to the interactions of the condensate and forces due to the quantum nature of the equations. Quantity $\nabla(\ln n\dot{)}$ plays a very important roll, in this limit it reads

$$\nabla(\ln n\dot{)} = -\nabla(\nabla\cdot v) - \nabla[\nabla(\ln n)\cdot v].$$

Thus

$$\nabla\sigma = -\frac{1}{4}\frac{\lambda}{m}k_B^2 nT\nabla T - \zeta\left[\nabla(\nabla\cdot v) + \nabla[\nabla(\ln n)\cdot v]\right], \tag{23}$$

where now we have

$$\zeta = -\frac{\hbar^2}{2m^2}\nabla\cdot(nv),$$

We interpret the function $\nabla\sigma$ as the viscosity of the system, it contains terms which are gradients of the temperature and of the divergence of the velocity and density (dissipative contributions). The measurement of the temperature dependence in this thermodynamical quantity at the phase transitions might reveal important information about the behavior of the gas due to particle interaction.

THE THERMODYNAMICS

In what follows we will derive the thermodynamical equations from the hydrodynamical representation. We can derive a conservation equation for a function α, starting with the following relation

$$(n\alpha)^{\cdot} = n\dot{\alpha} + \alpha\dot{n} \tag{24}$$

where α can take the values of ϕ and U_Q, both of them fulfil equation (24). Using the continuity equation (22a) in (24) we obtain

$$(n\alpha)^{\cdot} + \nabla\cdot(nv\alpha) = -nv\cdot F_{\alpha} + n\dot{\alpha}.$$

Nevertheless, this procedure is not possible for σ because in general we do not know it explicitly, only in some cases it might be possible to integrate it.

Observe how the quantum potential U_Q also fulfills the following relation

$$n\dot{U}_Q + \nabla\cdot(nv_{\rho}) = 0, \tag{25}$$

which follows by direct calculation, and where we have defined the velocity density v_{ρ} by

$$v_{\rho} = \frac{\hbar^2}{4m^2}(\nabla \ln n)^{\cdot},$$

which can be interpreted as a velocity flux due to the potential U_Q. Using the continuity equation (22a), equation (25) can be rewritten as

$$(nU_Q)^{\cdot} + \nabla\cdot(nU_Q v + J_{\rho}) + nv\cdot F_Q = 0 \tag{26}$$

where we have defined the quantum density flux

$$J_{\rho} = nv_{\rho}.$$

Equation (26) is another expression for the continuity equation of the quantum potential U_Q.

As we know, in general (for non-relativistic systems), the total energy density of the system ε is the sum of the kinetic, potential and internal energies [13], in this case we have an extra term U_Q due to the quantum potential

$$\varepsilon = \frac{1}{2}nv^2 + n\phi + nu + nU_Q + \psi_E \tag{27}$$

being u the inner energy of the system and

$$\psi_E = \frac{e}{m}(\varphi - v\cdot A) \tag{28}$$

the electromagnetic energy potential, defined in terms of the vector potential A and the electric potential φ. Observe that ψ_E fulfills the continuity equation

$$(n\psi_E)^{\cdot} + \nabla\cdot(nv\psi_E + j_B) = n\dot{\psi}_E - nv\cdot F_E \tag{29}$$

being j_B given by the continuity equation of the vector potential A

$$\frac{\partial A}{\partial t} + (v\cdot\nabla)A = -(A\cdot\nabla)v + \frac{m}{e}j_B, \tag{30}$$

Then from (27) we have that u will satisfy the equation

$$(nu)^{\cdot} + \nabla\cdot J_u - \nabla\cdot J_{\rho} + n\dot{\phi} = -p\nabla\cdot v, \tag{31}$$

being J_u the energy current, given by a energy flux and a heat flux, J_q,

$$J_u = nuv + J_q + J_B - pv,$$

where $\nabla\cdot J_q = v\cdot(\nabla\sigma)$, and $\nabla\cdot J_B = v\cdot(nj_B)$ expressions that as we can see is related in a direct way to the velocity and gradients of temperature in the condensate, and is the one that shows in an explicit way the temperature dependence of the thermodynamical equations. With these definitions at hand we have

$$(nu)^{\cdot} + \nabla\cdot(nvu + J_q + J_B - pv - J_{\rho}) + n\dot{\phi} = -p\nabla\cdot v. \tag{32}$$

In order to find the thermodynamical quantities of the system in equilibrium (taking p as constant on a volume L), we restrict the system to the regime where the auto-interacting potential is constant in time, with this conditions at hand for (32) we have

$$(nu)^{\cdot}+\nabla\cdot(nvu+J_q+J_B-pv-J_\rho)=-p\nabla\cdot v \tag{33}$$

From (33) we can have a straightforward interpretation of the terms involved in the phase transition. As always the first term will represent the change in the internal energy of the system, $-p\nabla\cdot v$ is the work done by the pressure and $\nabla\cdot v$ is related to the change in the volume, J_q contains terms related to the heat generated by gradients of the temperature ∇T and dissipative forces due to viscous forces $\sim\nabla(\nabla\cdot v)$ and finally but most important we have an extra term, $\nabla\cdot J_\rho$, due to gradients of the quantum potential (20).

Integrating this resulting expression on a close region, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int nu\,\mathrm{d}V+\oint(J_q+J_B+pv)\cdot n\,\mathrm{d}S \quad - \quad \oint J_\rho\cdot n\,\mathrm{d}S=-p\frac{\mathrm{d}}{\mathrm{d}t}\int\mathrm{d}V.$$

Equation (33) is the continuity equation for the internal energy of the system and as usual, from here we have an expression that would describe the thermodynamics of the system in an analogous way as does the first law of thermodynamics, in this case for the KG equation or a BEC. This reads

$$\mathrm{d}U=\hat{\mathrm{d}}Q+\hat{\mathrm{d}}Q_B+\hat{\mathrm{d}}A_Q-p\mathrm{d}V \tag{34}$$

where $U=\int nu\,\mathrm{d}V$ is the internal energy of the system, [31], and as we can see, its change is due to a combination of heat Q added to the system and work done on the system (pressure dependent), and

$$\frac{\hat{\mathrm{d}}A_Q}{\mathrm{d}t}=\frac{\hbar^2}{4m^2}\oint n(\nabla\ln n)^{\cdot}\cdot n\ \mathrm{d}S=\oint nv_\rho\cdot n\,\mathrm{d}S,$$

is the corresponding quantum heat flux due to the quantum nature of the KG equation. The second term on the right hand side of equation (34) would make the crucial difference between a classical and a quantum first law of thermodynamics.

Analogously, for the magnetic heat we have

$$\frac{\hat{\mathrm{d}}Q_B}{\mathrm{d}t} \quad = \quad \int\nabla\cdot J_B\,dV=\int v\cdot(nj_B)\,dV=\frac{m}{e}\int n\left(\frac{\partial A}{\partial t}+(v\cdot\nabla)A+(A\cdot\nabla)v\right)\cdot v\,dV, \tag{35}$$

where the vector potential A fulfills the Maxwell equations, in terms of the fluxes (16) it reads

$$F^{\mu\nu}{}_{,\nu}=-j^\mu, \tag{36}$$

where as usual $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$. In terms of the vector and the electric potential, the Maxwell equations are given by

$$\Box A \quad = \quad -j, \tag{37a}$$

$$\Box\varphi \quad = \quad -j-\frac{2en}{\kappa^2}\hat{m}, \tag{37b}$$

where we have used the Lorentz gauge. Observe that the fluxes contain the information of the velocity of the fluid and of the electromagnetic term as well. This point will be important for the superconductivity.

THE CONDENSATION TEMPERATURE (NON–RELATIVISTIC)

In this section let us calculate the condensation temperature in the non–relativistic regime associated with the aforementioned system within the semiclassical approximation [15, 31, 32]. The analysis of a Bose–Einstein condensates in the ideal case, weakly interacting, and with a finite number of particles, trapped in different potentials [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32] (and references therein) show that the main properties associated with the condensate, and in particular the condensation temperature, depends strongly on the characteristics of the trapping potential in question, the number of spatial dimensions, and the associated single–particle energy spectrum.

Inserting plane waves in the KG equation (9) [14], neglecting the term proportional to T^4 in (6), assuming that the temperature is sufficiently small, allows us to obtain the single–particle dispersion relation between energy and momentum (we considered here the low velocities limit)

$$E_p \simeq \frac{p^2}{2m} + \frac{\lambda}{2mc^2}|\Phi|^2 + \frac{\lambda}{4mc^2}(k_BT)^2 + mc^2\phi + e\varphi - \frac{e}{mc}A\cdot p. \tag{38}$$

Now we set $\lambda = 8\pi\hbar^2c^2a\kappa^2$ and $\phi = \alpha r^2$ in (38), where $\alpha = 1/2(\omega_0/c)^2$ and ω_0 is a characteristic frequency. We obtain the semiclassical energy spectrum in the Hartree–Fock approximation for a bosonic gas trapped in an isotropic harmonic oscillator [15, 31, 32], corrected with two extra terms due to the contributions of the thermal bath and to the electromagnetic field.

For simplicity we set $A = 0$, and a dependence of the form $\varphi \sim r^2$ for the electric potential, clearly this can be generalized to other situations [36]. Additionally, if our system is in static thermal equilibrium, the corresponding density is only function of the spatial coordinates, i .e., $n(\vec{r},t) \approx n(\vec{r})$ [15], and consequently $|\Phi|^2 \equiv \kappa^{-2}n(\vec{r})$.

The number of particles in the 3–dimensional space obeys the normalization condition [15, 32]

$$N = \int d^3\vec{r}\; n(\vec{r}), \tag{39}$$

where the spatial density $n(\vec{r})$ reads

$$n(\vec{r}) = \frac{1}{(2\pi\hbar)^3}\int d^3\vec{p}\; n(\vec{r},\vec{p}), \tag{40}$$

being $n(\vec{r},\vec{p})$ the Bose–Einstein distribution function given by

$$n(\vec{r},\vec{p}) = \frac{1}{e^{\beta(E_p-\mu)}-1}. \tag{41}$$

Integrating (40) over the momentum space allows to obtain the spatial density associated with our system

$$n(\vec{r}) = \left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2} g_{3/2}(Z), \tag{42}$$

where $Z = \exp[\beta(\mu - \frac{\lambda\kappa^{-2}}{2mc^2}n(\vec{r}) - \frac{\lambda(k_BT)^2}{4mc^2} - mc^2\phi - e\varphi)]$. The function $g_\nu(z)$ is the so–called Bose–Einstein function defined by [29]

$$g_\nu(z) = \frac{1}{\Gamma(\nu)}\int_0^\infty \frac{x^{\nu-1}dx}{z^{-1}e^x-1}. \tag{43}$$

being $\Gamma(\nu)$ the Gamma function. In order to calculate the condensation temperature we expand expression (42) up to first order in the coupling constant λ. Using the properties of the Bose–Einstein functions [29], we obtain

$$n(\vec{r}) \approx n_0(\vec{r}) - \lambda g_{3/2}(z(\vec{r}))\left[\frac{\Lambda^{-6}\kappa^{-2}}{2mc^2\kappa_BT}g_{1/2}(z(\vec{r})) + \Lambda^{-3}\frac{\kappa_BT}{4mc^2}\frac{g_{1/2}(z(\vec{r}))}{g_{3/2}(z(\vec{r}))}\right], \tag{44}$$

where $n_0(\vec{r}) = \Lambda^{-3}g_{3/2}(z(\vec{r}))$, is the density for the case $\lambda = 0$, being $\Lambda = (2\pi\hbar^2/m\kappa_BT)^{1/2}$ the de Broglie thermal wavelength, and $z(\vec{r}) = \exp(\beta(\mu - \alpha mc^2r^2 - e\varphi))$. When $\varphi \sim r^2$ and with the help of the normalization condition (39), we obtain the corresponding number of particles

$$N \simeq \left(\frac{m}{2\Omega\hbar^2}\right)^{3/2}(k_BT)^3 g_3(e^{\beta\mu}) - \frac{\lambda\kappa^{-2}m^2(k_BT)^{7/2}}{16\pi^{3/2}c^2\hbar^6\Omega^{3/2}}G_{3/2}(e^{\beta\mu}) - \frac{\lambda}{4c^2}\left(\frac{m^{1/3}}{2\Omega\hbar^2}\right)^{3/2}(k_BT)^4 g_2(e^{\beta\mu}), \tag{45}$$

where

$$G_{3/2}(e^{\beta\mu}) = \sum_{i,j=1}^{\infty}\frac{e^{(i+j)\beta\mu}}{i^{1/2}j^{3/2}(i+j)^{3/2}}, \tag{46}$$

being $\Omega = m(\alpha c^2 + const \times e)$. When φ is only position dependent, we notice immediately from expression (44) that the correction in the number of particles can be associated with an effective external potential, and therefore Ω can be related to an effective frequency. If we set $\lambda = 0$ in equation (45), we recover the expression for the number of particles in the non–interacting case. In the thermodynamic limit for the non–interacting case, $\lambda = 0$, the value of the chemical potential at the condensation temperature is $\mu = 0$ [32]. If we further assume that above the condensation temperature the number of particles in the ground state is negligible, this allows us to obtain an expression for the condensation temperature T_0 given by

$$k_B T_0 = \Big(\frac{2\Omega\hbar^2}{m}\Big)^{1/2}\Big(\frac{N}{\zeta(3)}\Big)^{1/3}. \tag{47}$$

In order to obtain the shift in the condensation temperature caused by λ and the thermal bath, we expand expression (45) up to first order in $T = T_0$, $\mu = 0$, $\lambda = 0$. Additionally, at the condensation temperature, the chemical potential within the semiclassical approximation can be expressed as $\mu_c = \frac{\lambda\kappa^{-2}}{2mc^2} n(\vec{r} = 0)$, as it is suggested in expression (42), thus

$$\mu_c \approx \frac{\lambda\kappa^{-2} m^{1/2}(\kappa_B T_c)^{3/2}\zeta(3/2)}{2(2\pi)^{3/2}c^2\hbar^3} - \lambda^{3/2}\frac{\sqrt{2}\pi\kappa^{-2}(\kappa_B T_c)^2}{(2\pi c^2\hbar^2)^{3/2}}, \tag{48}$$

where we have used that $g_{3/2}(e^{-\delta}) \approx \zeta(3/2) - |\Gamma(-1/2)|\delta^{1/2}$, when $\delta \to 0$ [29]. Expression (48) basically corresponds to the definition of the chemical potential at the condensation temperature in the usual case [15, 32], except for the extra term contribution due to the thermal bath. Using these results, we finally obtain the shift in the condensation temperature caused by λ and the thermal bath, in function of the number of particles

$$\frac{T_c - T_0}{T_0} \equiv \frac{\Delta T_c}{T_0} = -\lambda\frac{m^{1/2}}{\kappa^2\hbar^3 c^2}\chi_1\Theta N^{1/6} + \lambda\chi_2\Theta^2 N^{1/3}, \tag{49}$$

where

$$\chi_1 = \frac{1}{3\zeta(3)}\left(\frac{\zeta(3/2)\zeta(2)}{2(2\pi)^{3/2}} - G_{3/2}(1)\right),$$

$$\chi_2 = \frac{1}{3\zeta(3)}\left(\frac{1}{4mc^2} + \frac{(2\lambda)^{1/2}\zeta(2)\pi}{(2\pi)^{3/2}\kappa^2\hbar^3 c^3}\right),$$

together with $\Theta = \big(2\Omega\hbar^2/m\big)^{1/4}$ and T_0 defined in (47).

The second right hand side term in the shift (49) is the contribution due to the thermal bath and the field φ. Notice that if we set $\varphi = 0$, we recover the result given in reference [8].

Setting $\alpha = 1/2(\omega_0/c)^2$ and $\lambda = 8\pi\hbar^2 c^2\kappa^2 a$ in (49), we recover the condensation temperature for a bosonic gas trapped in an isotropic harmonic oscillator, corrected by the contributions of the thermal bath and the external field φ. For instance, in the case of ^{87}Rb, with $a \sim 10^{-9}m$, $N \sim 10^6$, and $\Omega \sim \mathrm{m}\times(10Hz)^2$, from the second right hand side term in (49) we obtain the correction $7.9\times 10^{-78}\kappa^2 + 7.5\times 10^{-38}\kappa$. In other words, in order to obtain relevant corrections over the usual result in typical laboratory conditions, the *scale* κ must be very large and the external field φ must be very weak, at least near to the center of the system. In reference [8] we obtain a bound for the *scale* κ leading to $\kappa \lesssim 10^{38}$. Thus, with the experimental data given above, we obtain for the first right hand side term of expression (49) the usual shift $\sim 10^{-2}$, as expected [15, 32]. Observe that this result is an approximation because we have set $A = 0$.

THE PHASE TRANSITION

From hereafter we study the transition between the $\Phi = 0$ state to the minimum $\Phi_{min} = k_B\sqrt{T_c^2 - T^2}/2$ with low energy one with $T < T_c$.

During the time when $T >> T_c$ there are not scalar particles in the ground state. We will suppose that the scalar particles decouple from the rest of the matter at some moment, such that the total density here on remains constant. Below the critical temperature $T < T_c$, close to the local minimum the density oscillates around the value $n = k_B^2\kappa^2(T_c^2 - T^2)/4$. We study the case when the function S in (14) has a simple expression, $S = s_0 t$, with $s_0 << mc/\hbar$ in the non-relativistic limit. This implies that the velocity $v = 0$, also, if there does not exist an external force in the system then, $F_\phi = 0$. In this case the viscosity (dissipative term) of the BEC might in fact contain the whole information of the phase transition. From equation (23) we observe that the viscosity $\nabla\sigma$ contains only a therm with the anisotropies of the temperature. That means that when the temperature of the system isotropies the fluid becomes a superfluid. Furthermore, from the expression for flux (16a) we observe that the vectorial flux contains only a term with the vector potential A. Thus, the flux expression (16a) becomes the London equation,

$$j = \frac{2ne^2}{\kappa^2} A,$$

indicating that the system becomes superconductor. The Maxwell equation (37a) becomes

$$\Box A = \frac{2ne^2}{\kappa^2} A,$$

which is the Proca equation, indicating that the photon acquires a mass $2ne^2/\kappa^2 c$. Obviously, playing with the conditions of the system we can find situations with superconductivity or superfluidity in different situations.

Finally, to illustrate the previous exposition, we give the following example. Suppose that in the system there are only condensed and excited particles of the same specie. Thus $n = N_0/(N_{ex} + N_0)$ (total number of particles), where N_0 is the number of condensed particles and N_{ex} the number of exited particles. Combining the equations obtained in the theoretical framework lead to the number of particles in the condensate in a four dimensional space,

$$N_0 = \frac{N\kappa^2}{k_B^2 T_c^2} \left[1 - \left(\frac{T}{T_c} \right)^2 \right],$$

being $N = N_0 + N_{ex}$, and remembering we are considering space and time, note that in this case the exponent 2 in the critical temperature appears naturally. As always this expression shows the dependence of the condensate fraction N_0/N as a smooth function of temperature from $T \gtrsim T_c$ down to $T = 0K$. In this case, the finite temperature terms are obtained from the one loop corrections of the SF density, and are down to be in complete agreement with the standard theory, [33, 34, 35].

Observe that only a fraction $N/(k_B^2 \kappa^2 T_c^2)$ of the scalar particles reaches the ground state at $T = 0$, this value can only be determined experimentally and fits the value of the scale κ. So in principle we are able to mimic the result that in the presence of interactions we have $N_0 < N$ even at $T = 0$. The main idea we want to point out here is that these phenomena might be equivalent for a BEC on earth as for the cosmos, and this might follow the previous equations exactly, so this function might be tested in the laboratory. As the actually known observational evidence favors an open universe in what concerns BEC, we therefore need experimental tests to measure any new results that can be drown from the previous set of thermodynamical equations. If confirmed, the phase transition of a BEC can be explained using quantum field theory in a straightforward way.

CONCLUSIONS

In this work we studied the phase transition of a boson gas with zero spin represented by the Klein-Gordon equation with a Lagrangian containing a $U(1)$ symmetry, with mass m and self-interaction λ, given by a mexican hat scalar field potential, immersed in a thermal bath at temperature T, close to the critical temperature of symmetry breaking, up to one loop in perturbations theory. We rewrite the KG equation and interpret it as a generalized Gross-Pitaevskii one at finite temperature, with viscosity and dissipative terms. We show that the transition from the phase with the $U(1)$ symmetry to the phase with this symmetry broken can be interpreted as a phase transition from the gas state to the condensation state of the Bose gas. We obtained the condensation temperature as well. By rewriting the generalized GP equation in terms of hydrodynamic quantities we were able to derive the thermodynamic of the phase transition and find that the first law contains a new term that is a direct consequence of the quantum character of KG equation. It remains to see whether this generalization of the GP equation can describe the transition of a Bose gas into a BEC state in the laboratory. In other words, we propose that the superfluid and/or superconductor like-behavior in a BEC can be measured experimentally in a laboratory, in order to compare the results given here with realistic systems.

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