Exact rotating wormholes with ghost matter

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In this proceedings we review a method for generating rotating wormhole solutions of the Einstein equations with ghost (phantom) matter. One of the solutions contains a ring singularity only on the south hemisphere. We analyze the geodesic motion around this solution and show that there is no geodesics touching the ring singularity of the wormhole in any way. This leads us to the famous Penrose's cosmic censorship conjecture.

Keywords: Wormholes; ghost fields; cosmic censorship.

1. Introduction

This work pretends to give a method to study the Einstein equations with phantom energy as source. One of the main problems of Einstein-phantom solutions is that all the well know ones are unstable[1,2]. In[3] it was conjectured that rotation or the presence of magnetic field could stabilise a phantom star. Here we give a general method, based in old generation methods of exact solutions of the Einstein field equations[4,5], to obtain exact solutions of the Einstein equations with phantom matter source.

Starting with the Lagrangian (compare for example with[6,7])

$$\mathcal{L} = \sqrt{-g}\,(-R - 2\,(\nabla\,\phi)^2 + e^{-2\,\alpha\,\phi}\,F^2), \tag{1}$$

where g is the determinant of the metric tensor, R is the scalar curvature, ϕ the Phantom field, F the Maxwell one and ∇ is the covariant derivative. The constant α is a free parameter which governs the strength of the coupling of the Phantom to the Maxwell field. When $\alpha = 0$, the action reduces to the uncoupled Einstein-Maxwell phantom theory. When $\alpha \neq 1$, the action is a theory of phantom field with a dilaton like coupling with electromagnetism. We will consider this theory for all values of $\alpha \neq 0$.

The field equations derived from Lagrangian (1) are give by (see also[6])

$$\begin{aligned} \nabla_\mu(e^{-2\alpha\phi}F^{\mu\nu}) &= 0;\\ \nabla^2\phi - \frac{\alpha}{2}e^{-2\alpha\phi}F^2 &= 0;\\ -2\nabla_\mu\phi\nabla_\nu\phi + 2e^{-2\alpha\phi}(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{2}g_{\mu\nu}e^{-2\alpha\phi}F^2) &= R_{\mu\nu}. \end{aligned} \tag{2}$$

Considering only isolated axial symmetric and stationary phantom stars, we analyse spacetimes characterised by two Killing vector fields X and Y and introduce coordinates t and φ which are chosen such that $X = \frac{\partial}{\partial t}$ and $Y = \frac{\partial}{\partial\varphi}$. The corresponding line element can then be expressed as[8]

$$ds^2 = -f\,(dt - \omega\,d\varphi)^2 + f^{-1}[e^{2k}(d\rho^2 + dz^2) + \rho^2 d\varphi^2], \qquad (3)$$

where f, ω, and k are functions of ρ and z only. The electromagnetic potential has the form $A_\mu = (A_0, 0, 0, A_3)$, and again A_0, A_3, and the Phantom field, ϕ are functions of ρ and z only. The main goal of this work is to give solutions of the Einstein field equations using metric (3).

2. Functional Space Formulation

Using the functional geodesic formulation for Lagrangian (1) (compare with[6] and[7]), the method is fully explained in[9]. And, introducing the operator $D = (\partial_\rho, \partial_z)$, taking out a total divergence term and eliminating the terms with $D\,k$ by means of a Legendre transformation, one can obtain that the original Lagrangian, given by (1), can be rewritten as

$$\mathcal{L} = \frac{\rho}{2\,f^2}\,Df^2 - \frac{f^2}{2\,\rho}\,D\omega^2 - \frac{2\,\rho}{\alpha^2\,\kappa^2}\,D\,\kappa^2 + \frac{2\,f\,\kappa^2}{\rho}\,[(\omega\,DA_0 + DA_3)^2 - \frac{\rho^2}{f^2}\,DA_0^2], \quad (4)$$

where $\kappa^2 = e^{-2\,\alpha\,\phi}$. Considering only space-times with $\alpha \neq 0$. Taking the Euler-Lagrange equations, obtained directly from extremizing the action for such Lagrangian[a] $D(\frac{\partial\mathcal{L}}{\partial D\,Z^a}) - (\frac{\partial\mathcal{L}}{\partial Z^a}) = 0$, with $Z^a = (f, \omega, A_0, A_3, \kappa)$, and defining a differential operator $\tilde{D} = (-\partial_z, \partial_\rho)$, it follows that $D\,\tilde{D} = 0$ for any analytic function. One can conclude from the second Maxwell equation, the existence of a potential[4] χ, such that $\tilde{D}\,\chi = \frac{2\,f\,\kappa^2}{\rho}\,(\omega\,DA_0 + DA_3)$. Using χ the second Einstein's equation can be rewritten as $D(\frac{f^2}{\rho}\,D\,\omega + \psi\,\tilde{D}\,\chi) = 0$, with $\psi = 2\,A_0$, so that there exists another potential[4] ϵ, defined by $\tilde{D}\,\epsilon = \frac{f^2}{\rho}\,D\,\omega + \psi\,\tilde{D}\,\chi$. The use of these potentials χ and ϵ, will be helpful in the procedure of defining harmonic functions. The set of field equations in terms of the potentials χ and ϵ can be obtained from the Lagrangian

$$\mathcal{L} = \frac{\rho}{2\,f^2}\,[D\,f^2 + (D\,\epsilon - \psi\,D\,\chi)^2] - \frac{2\,\rho}{\alpha^2\,\kappa^2}\,D\,\kappa^2 - \frac{\rho\,\kappa^2}{2\,f}\,(D\psi^2 + \frac{1}{\kappa^4}\,D\,\chi^2). \qquad (5)$$

We have to do an important remark. The new Lagrangian (5) cannot be obtained if the transformation is made directly on the original Lagrangian, given by (1) or (4). This fact implies that the transformations defined by χ, and ϵ must have a degeneracy. Lagrangian (5) can be seen as obtained from the line element dS^2 of a potential space, in other worlds, $\mathcal{L} \to dS^2 = G_{AB}\,d\Psi^A\,d\Psi^B$, with $\Psi^A = (f, \epsilon, \chi, \psi, \kappa)$, so that the equations of motion obtained from variations of this Lagrangian with respect to the coordinates, Ψ^A, can be thought as geodesics in such potential space

$$dS^2 = \frac{1}{2\,f^2}\,[d\,f^2 + (d\,\epsilon - \psi\,d\,\chi)^2] - \frac{2}{\alpha^2\,\kappa^2}\,d\,\kappa^2 - \frac{\kappa^2}{2\,f}\,(d\psi^2 + \frac{1}{\kappa^4}\,d\,\chi^2). \qquad (6)$$

[a] The Lagrangian (4) sometimes is viewed as describing the line element in the potential space[4], thus the motion equations can be seen as geodesics in this potential space.

As usual[4], metric (6) defines a Riemannian potential space with constant scalar curvature, $R = -12 + \alpha^2$. All the covariant derivatives of the Riemann tensor are proportional to $\alpha^2 + 3$, (or zero for $\alpha = 0$). This fact is very important and let us decide which method we can use to solve the field equations. In what follows a method will be given for solving the (6) field equations with $\alpha \neq 0$, where the formalism of the chiral equations can no be applied. The functional geodesic formulation consist on defining an abstract space whose coordinates are defined by the metric functions and the fields entering in the system. In order to introduce an ansatz resembling the harmonic map ansatz into the functional geodesic formulation (see[8] and[9] for an explanation), we shortly explain the general idea of the harmonic map ansatz method. The field equations of the theory can be written as

$$D(\rho\, D\, \Psi^A) + \rho \{{}^{\,A}_{B\ C}\} D\, \Psi^B\, D\, \Psi^C = 0 \tag{7}$$

where Ψ^A are the potentials of the geodesic formulation and $\{{}^{\,A}_{B\ C}\}$ are the Christoffel symbols of the Riemannian space dS^2 defining the potential space of the theory. We can transform the field equations obtained in terms of the functions $(f, \epsilon, \chi, \psi, \kappa)$ into a set of first order differential equations[6]

$$\begin{aligned} \frac{1}{\rho} D(\rho\, A) &= A\,(A - \bar{A}) + B\, \bar{B}, \\ \frac{1}{\rho} D(\rho\, B) &= \frac{1}{2}\, B\,(A - 3\, \bar{A}) - C\, \bar{B}, \\ \frac{1}{\rho} D(\rho\, C) &= \frac{1}{2}\, \alpha^2\, (B^2 + \bar{B}^2), \end{aligned} \tag{8}$$

where a bar over the functions denotes complex conjugate. Notice that in this way, one can reduce the system of field equations to a set of three first order differential equations for the three functions

$$\begin{aligned} A &= \frac{1}{2\, f}\, [D\, f - i\,(D\, \epsilon - \psi\, D\, \chi)], \\ B &= -\frac{1}{2\, \sqrt{f}}\, (\kappa\, D\, \psi - \frac{i}{\kappa}\, D\, \chi), \\ C &= -\frac{D\, \kappa}{\kappa} \end{aligned} \tag{9}$$

3. The Generalised Harmonic Maps Ansatz

Now we look for invariant transformations of the equations (7), *i.e.*, transformations of the form $\Psi^A = \Psi^A(\lambda^i)$ that leave the field equations (7) invariant, where λ^i are potentials fulfilling the same field equations (7). The potentials λ^i define the Riemannian space V_p. In terms of the potentials λ^i, the field equations (7) read

$$\rho[\Psi^A{}_{,i\,j} - \Gamma^k_{i\,j}\, \Psi^A{}_{,k} + \{{}^{\,A}_{B\ C}\} \Psi^B{}_{,i} \Psi^C{}_{,j}] D\, \lambda^i\, D\, \lambda^j + \Psi^A{}_{,k} [D\,(\rho\, D\, \lambda^k) + \rho \Gamma^k_{i\,j}\, D\, \lambda^i\, D\, \lambda^j] = 0, \tag{10}$$

where $,i = \partial/\partial\lambda^i$ and $\Gamma^i_{j\,k}$ are the Christoffel symbols of V_p. In terms of the Christoffel symbols of the abstract Riemannian space V_p, (10) reads

$$\Psi^A{}_{,i;\,j} + \{{}^{\,A}_{B\;C}\}\Psi^B{}_{,i}\Psi^C{}_{,j} = 0 \tag{11}$$

$$D\,(\rho\,D\,\lambda^k) + \rho\Gamma^k_{i\,j}\,D\,\lambda^i\,D\,\lambda^j = 0 \tag{12}$$

where we have used the field equations for the λ^i's and the fact that the λ^i's are linear independent.

In terms of the complex variable $\varsigma = \rho + i\ z$ and $\bar{\varsigma}$ its complex conjugated, equation (12) reads

$$(\rho\lambda^k{}_{,\varsigma})_{,\bar{\varsigma}} + (\rho\lambda^k{}_{,\bar{\varsigma}})_{,\varsigma} + 2\rho\,\Gamma^k_{i\,j}\,\lambda^i{}_{,\varsigma}\,\lambda^j{}_{,\bar{\varsigma}} = 0. \tag{13}$$

The ansatz consist in to choose an apropiated V_p potential space. Here we will study the V_1 and V_2 spaces because they are the most simple ones, but we will obtain solutions only for the V_1 subspaces.

We start with a two dimensional Riemannian spaces V_2 with constant curvature, parametrising this Riemannian spaces with two harmonic parameters λ, and τ, such that $\lambda, \tau \in \mathbb{R}$. The line element is

$$ds^2 = \frac{2\,(d\,\lambda^2\ + d\,\tau^2)}{(1 - \sigma\,(\lambda^2\ + \tau^2))^2} = \frac{d\xi d\bar{\xi}}{(1 - \sigma\xi\bar{\xi})^2}, \tag{14}$$

where σ is a real constant proportional to the potential space curvature, and $\xi = \lambda + i\tau$, for the case of complex parameters. We know that this is a maximally symmetric space, so it has three killing vectors. If the electromagnetic field vanishes any value for α is similar, because there is not interaction between scalar and electromagnetic fields. But if there is an electromagnetic interaction the situation is different. As we showed in [6], the subalgebras for the potential space with arbitrary α, with three Killing vectors, are such that one of them has to be set to zero, so if the electromagnetic field does not vanish the only case of maximally symmetric V_2 that can be taken is the one with $\sigma = 0$. In this case, the parameters satisfy the usual Laplace equation: $D(\rho\,D\,\lambda) = 0$, $D(\rho\,D\,\tau) = 0$. As in the harmonic maps ansatz case, let us express the functions A, B, C in terms of these parameters as follows

$$\begin{aligned} A &= a_1(\lambda,\tau)\,D\,\lambda + a_2(\lambda,\tau)\,D\,\tau,\\ B &= b_1(\lambda,\tau)\,D\,\lambda + b_2(\lambda,\tau)\,D\,\tau,\\ C &= c_1(\lambda,\tau)\,D\,\lambda + c_2(\lambda,\tau)\,D\,\tau. \end{aligned} \tag{15}$$

Using the harmonic equations, *i.e.* the Laplace equation, for these parameters in the field equations (8), and recalling the fact that $(D\,\lambda)^2, (D\,\tau)^2$, and $D\,\lambda\,D\,\tau$ are independent functions, from the system of equations for A, B, C, (8), is possible to obtain a set of equations equivalent to the field equations in terms of a_i, b_i and c_i for $i = 1, 2$ (10).

Taking the original potential also as functions of the harmonic parameters, is possible to express the functions A, B, and C from (15) in terms of λ, τ and the functions $(f, \epsilon, \chi, \psi, \kappa)$. From which is possible to make the following identification

$$a_1 = \frac{1}{2\,f}\,[f_{,\lambda} - i\,(\epsilon_{,\lambda} - \psi\,\chi_{,\lambda})]; \quad a_2 = \frac{1}{2\,f}\,[f_{,\tau} - i\,(\epsilon_{,\tau} - \psi\,\chi_{,\tau})],$$
$$b_1 = -\frac{1}{2\sqrt{f}}\,(\kappa\,\psi_{,\lambda} - \frac{i}{\kappa}\,\chi_{,\lambda}); \quad b_2 = -\frac{1}{2\sqrt{f}}\,(\kappa\,\psi_{,\tau} - \frac{i}{\kappa}\,\chi_{,\tau}),$$
$$c_1 = -\frac{\kappa_{,\lambda}}{\kappa}; \qquad c_2 = -\frac{\kappa_{,\tau}}{\kappa}. \tag{16}$$

In what follows we proceed to present some solutions to the Einstein-Maxwell-Phantom system in terms of the harmonic functions λ, τ.

4. Monopole Rotating Wormhole

The idea in a nutshell was to use the solution generation techniques developed in the late $80's$ for the Einstein's equations, see[10], where it was possible, within the quiral formulation[11], to derive the Kerr solution starting from the Schwarzschild one, so we thought on using the same techniques applied to the WH proposed by Ellis and Thorne. The final result is that we ended up with the following ansatz for the line element:

$$ds^2 = -f\,(dt + a\,\cos\theta\,d\varphi)^2 + \frac{1}{f}\,\left[dl^2 + \left(l^2 - 2\,l\,l_1 + {l_0}^2\right)\,\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right)\right], \tag{17}$$

where $f = f(l)$ is an unknown function to be determined by the field equations, l_0, l_1 are constant parameters with units of distance, such that $l_0^2 > l_1^2$, and $l_0 \neq 0$, and a is the rotational parameter (angular momentum per unit of mass). In these coordinates the distance l covers the complete manifold, going from minus to plus infinity.

We suppose that some scalar field fluctuation collapses in such a way that it forms a rotating scalar field configuration with three regions; the interior one where the rotation is non-zero and two exterior regions, one on each side in of the throat, where the rotation stops. The inner boundaries of this configuration are defined where the rotation vanishes. The interior field is the source of the WH. We look for a solution to the Einstein equations with an stress-energy tensor describing an opposite sign massless scalar field, ϕ (see[12]), so that the field equations take the form $R_{\mu\nu} = -8\,\pi\,G\,\phi_{,\mu}\,\phi_{,\nu}$, with $R_{\mu\nu}$ the Ricci tensor. For $a \neq 0$ we find the analytic solution:

$$f = 2\,\frac{\phi_0\,\sqrt{D\,\left({l_0}^2 - {l_1}^2\right)}\,e^{\phi_0\,\left(\lambda - \frac{\pi}{2}\right)}}{a^2 + D\,e^{2\,\phi 0\,\left(\lambda - \frac{\pi}{2}\right)}}, \qquad \sqrt{8\pi G}\phi = \sqrt{2 + \frac{{\phi_0}^2}{2}}\,\left(\lambda - \frac{\pi}{2}\right), \tag{18}$$

where the function λ is given by $\lambda = \arctan\left(\frac{l - l_1}{\sqrt{{l_0}^2 - {l_1}^2}}\right)$ and a is the angular momentum parameter. Here $\phi_0 > 0$ in order to preserve the signature of the metric. Then, the whole rotating WH solution reads

$$f = \begin{cases} \exp(\lambda - \frac{\pi}{2}) & \text{if } l > l_b \\ -\frac{4M\sqrt{D}\,e^{\phi_0\left(\lambda-\frac{\pi}{2}\right)}}{a^2+D\,e^{2\,\phi_0\left(\lambda-\frac{\pi}{2}\right)}} & \text{if } l_a \le l \le l_b \\ \phi_1 \exp(\lambda + \frac{\pi}{2}) & \text{if } l < l_a \end{cases} . \tag{19}$$

It can be seen that the interior solution matches with the rhs exterior one provided that the parameter D becomes

$$D = -2M\sqrt{4\,M^2 - \frac{a^2}{E^2}} + 8\,M^2 - \frac{a^2}{E^2} \tag{20}$$

where the constant E is determined by the radio l_b where the two solutions match, it is given by $E = \exp\left[\phi_0(\lambda_b - \frac{\pi}{2})\right]$ where $\lambda_b = \lambda(l_b)$. In order to have a real solution everywhere we impose the constraint that $4\,M^2 = \left({l_0}^2 - {l_1}^2\right){\phi_0}^2 > \frac{a^2}{E^2}$. The matching on the rhs could be smooth if l_b is sufficiently large and the rotation's parameter a is sufficiently small. On the contrary, for small l_b or/and big rotation's parameter the matching on the rhs is continue but not necessarily smooth. On the other side the matching of the interior solution is smooth with the lhs exterior one, if the constant ϕ_1 is chosen such that

$$\phi_1 = -\frac{4M\sqrt{D}\,e^{-\phi_0\,\pi}}{a^2 + De^{-2\,\phi_0\,\pi}}. \tag{21}$$

5. Geodesics

Due to the symmetry of the metric, it is sufficient to study radial geodesics. For the case $\theta = const.$ and $\varphi = 0$ the geodesics reduce to

$$\begin{aligned} &\ddot{l} - \frac{1}{2}\frac{f}{f_{,l}}\dot{l}^2 + \frac{1}{2}ff_{,l}\dot{t}^2 = 0 \\ &\ddot{t} + \frac{f}{f_{,l}}\dot{l}\dot{t} = 0. \end{aligned} \tag{22}$$

These equations can be solved numerically, the result is given in Fig. 1–3. Fig. 1 plots the evolution of the coordinate l with respect to the afin parameter τ. In the same way, Fig. 2 plots the evolution of the coordinate t with respect to the afin parameter τ. Fig. 3 plots the evolution of the coordinate l but now with respect to the coordinate t.

6. Dipole Rotating Wormhole

We consider a stationary and axially symmetric space-time obtained from the original one in Ref.[13]

$$ds^2 = -f\,(dt + \omega\,d\varphi)^2 + \frac{1}{f}\left[\Delta\left(\frac{dl^2}{\Delta_1} + d\theta^2\right) + \Delta_1\,\sin^2\theta\,d\varphi^2\right], \tag{23}$$

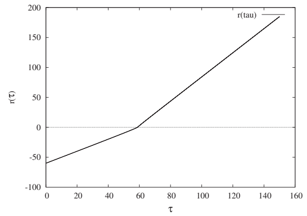

Fig. 1. Plot of the proper distance coordinate l with respect to the afin parameter τ. The values of the parameters are $l_1 = 0.5, l_b = 10, \phi_0 = 2, l_0 = 1, a = 0.6$ and we plot for radius from -20 to 20.

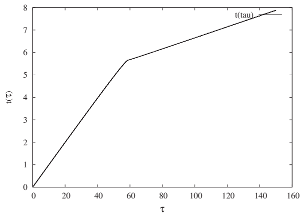

Fig. 2. Plot of the proper distance coordinate t with respect to the afin parameter τ. The values of the parameters are $l_1 = 0.5, l_b = 10, \phi_0 = 2, l_0 = 1, a = 0.6$ and we plot for radius from -20 to 20.

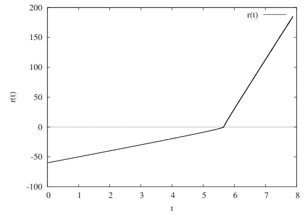

Fig. 3. Plot of the proper distance coordinate l with respect to the coordinate t. The values of the parameters are $l_1 = 0.5, l_b = 10, \phi_0 = 2, l_0 = 1, a = 0.6$ and we plot for radius from -20 to 20.

where $\Delta = (l - l_1)^2 + \left(l_0^2 - l_1^2\right) \cos^2\theta$, $\Delta_1 = (l - l_1)^2 + \left(l_0^2 - l_1^2\right)$. The line element components ω and f, and the parameter λ are given by

$$\omega = a\,\frac{(l - l_1)}{2\,\Delta}\,\sin^2\theta, \qquad f = \frac{\left(a^2 + k_1^2\right)\,e^{\lambda}}{a^2 + k_1^2\,e^{2\,\lambda}} \qquad \lambda = \frac{a^2 + {k_1}^2}{2\,k_1\,\Delta}\,\cos\theta. \tag{24}$$

being l_1, l_0 parameters with units of distance, such that $g_{ll} > 0$, that is $l_0^2 > l_1^2 > 0$. These parameters are related with the size of the throat, while a and k_1 are parameters with units of angular momentum. Line element (23) is a solution of the Einstein's equations $R_{\mu\nu} = -8\,\pi\,G\,\Phi_{,\mu}\Phi_{,\nu}$ for an opposite sign scalar field given by

$$\Phi = \frac{1}{\sqrt{16\,\pi\,G}}\,\lambda. \tag{25}$$

The restriction $l_0^2 > l_1^2 > 0$ avoids that $\Delta_1 = 0$, but $\Delta = 0$ for $\theta = \pi/2$ and $l = l_1$. Thus, metric (23) has a naked singularity at $\Delta = 0$, which represents a ring on the equatorial plane centred in the centre of the wormhole[14].

If we drop out the acceleration $a = 0$, the rotation $\omega = 0$, and metric (23) transforms into

$$ds^2 = -f\,dt^2 + \frac{1}{f}\left[\Delta\left(\frac{dl^2}{\Delta_1} + d\theta^2\right) + \Delta_1\,\sin^2\theta\,d\varphi^2\right], \tag{26}$$

where $f = e^{-\lambda}$, and $\lambda = \frac{k_1}{2\,\Delta}\cos\theta$. Notice that f is discontinuous at the ring singularity[16]. To have a better visualization of the wormhole structure in metric (23), we define a new radial variable[b] $r^2 = \Delta_1$ and set $r_0^2 = l_0^2 - l_1^2$ and then write Eq. (23) as a conformal metric of the form $ds^2 = (K/f)ds_c^2$

$$ds^2 = \frac{K}{f}\left(-\frac{f^2}{K}dt^2 + \frac{dr^2}{1 - r_0^2/r^2} + \frac{r^2}{K}d\Omega_0^2\right), \tag{27}$$

[b]It must be noticed that, in terms of the new variable, the ring singularity is located at $\theta = \pi/2$ and $r = r_0$.

where now we have $K = 1 - \frac{r_0^2}{r^2}\sin^2\theta$, $f = \exp\left(-\frac{k_1}{2}\frac{\cos\theta}{r^2 - r_0^2\sin^2\theta}\right)$ and $d\Omega_0^2 = K d\theta^2 + \sin^2\theta\, d\varphi^2$. Curiously enough, ds_c^2 in resembles the line element of the famous Morris-Thorne (MT) wormhole. The original metric represents a wormhole that is conformally related to the MT solution[c].

7. Geodesics

We are to study now the null geodesics of test particles freely falling into the wormhole. It proves convenient to work with the Hamiltonian of the geodesics:

$$2\mathcal{H} = -\frac{p_t^2}{f} + \frac{f\, p_\varphi^2}{\Delta_1 \sin^2\theta} + \frac{f}{K}\left(p_l^2 + \frac{p_\theta^2}{\Delta_1}\right), \tag{28}$$

which is in itself a constant of motion, i.e. $\mathcal{H} = 0$ along any given null geodesic.

To have a connection with our previous discussion about the throat of the wormhole, we will draw the paths of null geodesics over the embedding profile of the throat of a MT wormhole, setting $p_\varphi = 0$. As we said before, in our case the geodesic paths will lie on the throat's surface too, but will show deviations because of the peculiarities induced upon them by the conformal factor K/f, see Fig. 4. In general terms, we can see that null geodesics are able to avoid the naked singularity of the spacetime. The barrier protecting the singularity is dynamically induced by the same discontinuities of the metric functions[16].

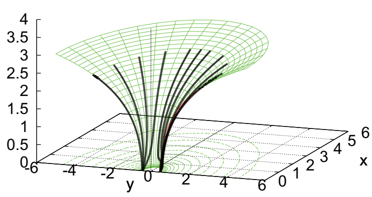

Fig. 4. The surface represents the MT throat's profile, the solid lines are the null geodesics as depicted on the throat, and the ring singularity is marked as a point on the xy-plane at $r = r_0$ and $\theta = \pi/2$.

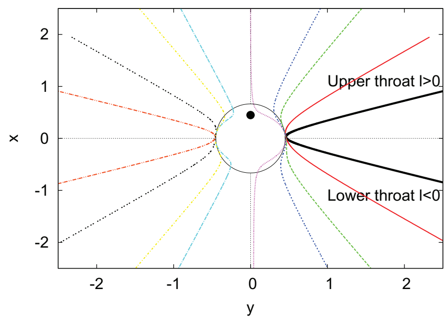

Fig. 5. Null geodesics projected on the xy-plane. In general, geodesics are able to avoid the naked singularity, and its deformations are due to the conformal factor K/f that changes the real throat's surface with respect to MT's at points close to the ring singularity.

[c]However, there is the issue that the conformal transformation used is not well behaved everywhere. Thus, the transformation should work well as long as the problematic points, those of the ring singularity, are left out in our calculations.

In order to see the behavior of the geodesics close to the singularity, we write the geodesic equations explicitly under the approximations $l << l_0$ and $\theta \sim \pi/2$,

$$\dot{p}_l \frac{2\Delta}{f} = F_1\, l\, M - \frac{k_1 l}{l_0^4 \cos\theta} p_\varphi^2 - \frac{k_1 l}{l_0^2 \cos\theta} \exp\left(\frac{k_1}{l_0^2 \cos\theta}\right) p_t^2\,,$$
$$\dot{p}_\theta \frac{2\Delta}{f} = F_2\, M + \frac{k_1}{2 l_0^2 \sin\theta} p_\varphi^2 + \frac{k_1}{2} \sin\theta \exp\left(\frac{k_1}{2 l_0^2 \cos\theta}\right) p_t^2\,, \tag{29}$$

being $M = p_l^2 l_0^2 + p_\theta^2$, and

$$F_1 = -\frac{k_1 - 2\, l_0^2 \cos\theta}{l_0^4 \cos^3\theta}\,, \quad F_2 = \frac{k_1 - 4 l_0^2 \cos\theta}{2 l_0^2 \cos^2\theta}\,.$$

Neglecting again the motion on the φ direction, so that $p_\varphi = 0$, then Eqs. (29) can be integrated exactly on the southern hemisphere as follows.

$$p_l = A_0 \sin(M_1 l l_0) \exp\left(\frac{1}{2} F_1 l^2\right) H\,,$$
$$p_\theta = A_0 \cos(M_1 l l_0) \exp\left(\frac{1}{2} F_1 l^2\right) H\,, \tag{30}$$

where $M_{1,\theta} = F_1$. Thus, we get

$$l^2 = -\frac{k_1 l_0^2 \cos^2\theta}{k_1 - 2 l_0^2 \cos\theta} - \frac{l_0^4 \cos^3\theta}{k_1 - 2 l_0^2 \cos\theta} \ln\left(\frac{p_t^2}{A_0^2}\right) \sim -l_0^2 \cos^2\theta\,, \tag{31}$$

for $\theta \sim \pi/2^+$. It is clear that Eq. (31) does not have solutions for points on the southern hemisphere that are also close to the ring singularity. It is not possible to find a geodesic trajectory that can be in contact with the singularity, and then the latter is not visible to exterior observers[17]. This leads us to the famous Penrose's cosmic censorship conjecture[18]. According to our solution, it is also possible to protect a singularity if we surround it with a wormhole's throat. This may indicate a generalization of Penrose's cosmic censorship conjecture.

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