Hydrodynamic representation and energy balance for the Dirac and Weyl fermions in curved space-times

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Using a generalized Madelung transformation, we derive the hydrodynamic representation of the Dirac equation in arbitrary curved space-times coupled to an electromagnetic field. We obtain Dirac-Euler equations for fermions involving a continuity equation and a first integral of the Bernoulli equation. Using the comparison of the Dirac and Klein-Gordon equations we obtain the balance equation for fermion particles. We also use the correspondence between fermions and bosons to derive the hydrodynamic representation of the Weyl equation which is a chiral form of the Dirac equation.

Keywords: Dirac equation; Weyl equation; Fermions; Hydrodynamics

1. Introduction

The Standard Model of elementary particles establishes that there exist two kinds of particles, fermions and bosons. In previous works,[1,2] the energy balance for bosons was derived starting from the general relativistic Klein-Gordon (KG) equation. In the present work, we study a system of fermions described by the Dirac equation in arbitrary curved space-times taking into account electromagnetic effects. We also use the Weyl equation which is a chiral form of the Dirac equation due to the relationship between the Lie algebras of the symmetry groups for both systems of particles. We give the hydrodynamic representation of the Dirac and Weyl equations for fermions using previous results obtained for boson particles.

Many examples of fermion particles in strong gravitational fields can be found in nature. Indeed, the curvature of space-time plays an important role in a neutron star, in the early Universe, or in a fermion cloud (e.g. a dark matter halo) in the vicinity of a black hole. We need to develop a general framework to identify what are the different energy contributions in such systems. In this work we use the geometrical decomposition of the metric in 3+1 slices and the tetrad formalism to study the particle spin in an arbitrary space-time. We define the gamma matrices in curved space-times and derive the generalized Dirac and Weyl equations. Then, using the Madelung transformation, we introduce a hydrodynamic representation of the Dirac and Weyl spinors. This hydrodynamic representation can help us to describe the fermionic system in a general framework.

2. Field Equations

We use the tetrad formalism for the space-time geometry, and the canonical expansion of the space-time in a 3+1 ADM decomposition,[3–8] such that the coordinate t is the parameter of evolution. The 3+1 metric reads

$$\mathrm{d}s^2 = N^2c^2\mathrm{d}t^2 - h_{ij}\left(\mathrm{d}x^i + N^i c\,\mathrm{d}t\right)\left(\mathrm{d}x^j + N^j c\,\mathrm{d}t\right), \tag{1}$$

where N represents the lapse function which measures the proper time of the observers traveling along the world line, N^i is the shift vector that measures the displacement of the observers between the spatial slices and h_{ij} is the 3-dimensional slice-metric. In what follows $i,j,k,l = 1,2,3$; $a,b,c = 0,1,2,3$ are the internal indices and $\mu,\nu,\alpha = 0,1,2,3$ the space-time indices. We write eq. (1) in the tetrad formalism as $\mathrm{d}s^2 = \eta_{ab}e^a{}_\mu e^b{}_\nu dx^\mu dx^\nu$, where $\eta_{ab} = \mathrm{diag}(1,-1,-1,-1)$. Here $e^a = e^a{}_\mu dx^\mu$ is the set of one-forms base of the cotangent space at the space-time manifold given by

$$\begin{aligned} e^0 &= N\mathrm{d}t, \\ e^k &= \hat{e}^k{}_i\left(\mathrm{d}x^i + N^i c\,\mathrm{d}t\right), \end{aligned} \tag{2}$$

with inverse

$$\begin{aligned} e_0 &= \frac{1}{N}\left(\frac{\partial}{c\,\partial t} - N^j\frac{\partial}{\partial x^j}\right), \\ e_k &= \hat{e}_k{}^j\frac{\partial}{\partial x^j}, \end{aligned} \tag{3}$$

where $\hat{e}^k = \hat{e}^k_i\mathrm{d}x^i$ are the one-form base to the three-dimensional slice of the cotangent manifold, such that $h_{ij} = \delta_{kl}\hat{e}^k_i\hat{e}^l{}_j$. We can also define the set of vectors base of the tangent-space to the space-time as $e_a = e_a^{\ \mu}\partial_\mu$, such that $e^a e_b = \delta^a_b$. We will use the tetrad formalism[3,6–10] to describe the space-time geometry where the fermion particles are located.

The action of a fermion system in curved space-time coupled to an electromagnetic field A_μ is given by $S[\psi,\partial_\mu\psi] = \int\mathcal{L}(\psi,\partial_\mu\psi(x^\mu))\,d^4x$, where $\mathcal{L} = \mathcal{L}(\psi,\partial_\mu\psi(x^\mu))$ is the Lagrangian density[11–13]

$$\mathcal{L} = \sqrt{-g}\frac{i\hbar c}{2}\left[\psi^\dagger B\gamma^\mu\left(D_\mu\psi\right) - \left(D_\mu\psi\right)^\dagger B\gamma^\mu\psi + \frac{2imc}{\hbar}\psi^\dagger B\psi\right]. \tag{4}$$

Here $D_\mu = \nabla_\mu + \dfrac{iq}{\hbar c}A_\mu$ is the total covariant derivative accounting for electromagnetic effects. The covariant derivative of a spinor $\psi = (\psi_{\dot\nu})$ is given by $\nabla_\mu(\psi_{\dot\nu}) = \partial_\mu(\psi_{\dot\nu}) + \Gamma^{\dot\alpha}_{\mu\dot\nu}(\psi_{\dot\alpha})$, where $\Gamma^{\dot\alpha}_{\mu\dot\nu}$ is the spin connection.[3,14] Using the least action principle it is possible to obtain from eq. (4) the corresponding Dirac equation. This equation is given by

$$\left[i\hbar\gamma^\mu(\,\nabla_\mu + iqA_\mu\,) - mc\right]\psi = 0, \tag{5}$$

where $\hbar$, c are the Planck constant and the speed of light respectively, while q, m are the charge and mass of the fermion particle and ψ is its spinor. Besides, the gamma matrices γ^μ are related to the spin and space-time geometry. They can be written as $\gamma^\mu = e^\mu{}_a \tilde{\gamma}^a$, where $\tilde{\gamma}^a$ are the gamma matrices in flat space-time, which are well-know from Quantum Field Theory (QFT).[15–17] Therefore,

$$\begin{aligned} \gamma^0 &= N\tilde{\gamma}^0, \\ \gamma^k &= \hat{e}^k{}_j(\tilde{\gamma}^j + N^j\tilde{\gamma}^0). \end{aligned} \tag{6}$$

In general, these matrices fulfill the following anti-commutation relation[3,18]

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \tag{7}$$

where $g_{\mu\nu}$ represents the metric that describes the space-time geometry. Furthermore, as we know, the gamma matrices in flat space-time are related to the Pauli matrices, which describe the spin of the fermion particles. In general, the gamma matrices obey the following relation[11–13,19]

$$(\gamma^\mu)^\dagger = B\gamma^\mu B^{-1}, \tag{8}$$

where B is a hermitian matrix, i.e. $B^\dagger = B$, that is uniquely determined by the gamma matrices γ^μ. As usual, we denote by $B^\dagger$ the conjugate (or Hermitian) transpose of B. We note that in QFT the relation (8) is fulfilled when $B = \tilde{\gamma}^0$ and the gamma matrices are in flat space-time. From the action (4) of the fermion system we can find the equation for the transpose conjugated spinor by making an infinitesimal variation of this action with respect to ψ. Another way of getting this equation of motion is to take the transpose conjugate of the Dirac equation (5) and using (8). In this manner we find that the transpose conjugated Dirac equation in curved space-time is given by

$$i\left(\nabla_\mu\bar{\psi}\right)\gamma^\mu - i\psi^\dagger\nabla_\mu\left(B\gamma^\mu\right) + i\bar{\psi}\nabla_\mu\gamma^\mu + \bar{\psi}A_\mu\gamma^\mu + m\bar{\psi} = 0. \tag{9}$$

To simplify the notations, here and in the following we use $mc/\hbar \to m$ in natural units ($c = \hbar = 1$). We consider that $(\nabla_\mu\psi)^\dagger = \nabla_\mu\psi^\dagger$, and we denote the adjoint spinor as $\bar{\psi} = \psi^\dagger B$. Using the gamma matrices in flat space-time and the fact that $B = \tilde{\gamma}^0$ we recover the definition of $\bar{\psi}$ in QFT and the transpose conjugated Dirac equation. However, in an arbitrary space-time $\nabla_\mu\gamma^\mu$ is distinct from zero, since $\gamma^\mu = e^\mu{}_a\tilde{\gamma}^a$, so in general $\nabla_\mu e^\mu{}_a$ is non-zero.

We can get the conserved charge from the Noether theorem.[20] The Dirac current is

$$J^\mu = \bar{\psi}\gamma^\mu\psi = \psi^\dagger B\gamma^\mu\psi. \tag{10}$$

To obtain the continuity equation

$$\nabla_\mu J^\mu = 0 \tag{11}$$

for the Dirac current, we take the covariant derivative of eq. (10). This gives

$$\nabla_\mu J^\mu = (\nabla_\mu\bar{\psi})\gamma^\mu\psi + \bar{\psi}\left(\nabla_\mu\gamma^\mu\right)\psi + \bar{\psi}\gamma^\mu\nabla_\mu\psi. \tag{12}$$

If we multiply the Dirac equation (5) by $\bar{\psi}$ and its transpose conjugate (9) by ψ and sum both equations, it follows that

$$\nabla_\mu J^\mu = \psi^\dagger \nabla_\mu \left(B\gamma^\mu\right)\psi. \tag{13}$$

If we require that the continuity equation (11) is fulfilled, i.e., that the number of particles is conserved, then we need $\nabla_\mu \left(B\gamma^\mu\right) = 0$, or equivalently

$$(\nabla_\mu B)\gamma^\mu = -B\nabla_\mu \gamma^\mu. \tag{14}$$

At this point, we want to emphasize the consistency conditions for the continuity equation (11). Some authors[21] impose $\nabla_\mu \gamma^\nu = 0$ while others[22] impose $\nabla_\mu B = 0$. These conditions are independent of each other, i.e., in general the condition of Ref.[21] is not fulfilled in Ref.[22] and *vice versa*. In Refs.[12,13] the authors conclude that the condition $\nabla_\mu(B\gamma^\nu) = 0$ is the most convenient because it is implied by $\nabla_\mu \gamma^\nu = 0$ and $\nabla_\mu B = 0$.

In addition, we can note that the matrix B can be obtained for a general metric (1) by solving the differential equation

$$\left(\nabla_0(BN) + \nabla_j(B\hat{e}_i^j N^i)\right)\tilde{\gamma}^0 - \nabla_j(B\hat{e}_i^j)\tilde{\gamma}^i = 0, \tag{15}$$

which follows from eq. (14). Using the condition (14), it is possible to rewrite the transpose conjugated Dirac equation (9) as

$$i\left(\nabla_\mu \bar{\psi}\right)\gamma^\mu + i\bar{\psi}\nabla_\mu \gamma^\mu + \bar{\psi}A_\mu \gamma^\mu + m\bar{\psi} = 0. \tag{16}$$

In order to find the conserved quantity resulting from the continuity equation, we take an arbitrary surface $\mathcal{S}$ enclosing the volume $\mathcal{V}$ which contains the whole system. Let k^j be an orthonormal vector to $\mathcal{S}$ such that

$$\int_\mathcal{V} \nabla_\mu J^\mu dV = \int_\mathcal{V} \nabla_0 J^0 dV + \int_\mathcal{S} k_j J^j dS = 0. \tag{17}$$

We assume that far away from the source, J^μ is negligible. Then, the surface integral in eq. (17) vanishes, and we obtain

$$\frac{dQ}{dt} = \int_\mathcal{V} \nabla_0 J^0 dV = 0, \tag{18}$$

where $Q = \int_\mathcal{V} J^0 dV$ is the conserved charge and dV is the curved volume element $dV = \sqrt{-g}d^4x$. In QFT this charge is identified with the number of fermions or with the electric charge of the system. In flat space-time we have $B = \tilde{\gamma}^0$, so that $J^0 = \psi^\dagger\psi = n$ represents the number density of fermion particles. In curved space-time J^0 (which is determined by γ^0 and by the generalized gamma matrices) has a different interpretation. The form of B given by eqs. (8) and (14) for each metric is related to the gamma matrices and to the tetrad formalism.

With the Maxwell four-potential we can define the Faraday tensor

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu. \tag{19}$$

In the electromagnetic theory, the Faraday tensor $F_{\mu\nu}$ satisfies the Maxwell field equations

$$\nabla_\nu F^{\nu\mu} = J^{E\mu}, \tag{20}$$

where $J^{E\mu}$ is the four-electromagnetic current.

The problem of the Energy Balance for boson particles in a curved space-time is studied in Ref.[1], where the conserved 4-current associated with the KG equation describing the evolution of a complex scalar field $\Phi(x^\mu)$ is defined. We can generalize this idea by defining a new 4-current J_μ^{KG}, changing the scalar field by a spinor and the complex conjugate scalar field by the conjugate transpose of the spinor. Namely, the KG current is redefined as

$$J_\mu^{KG} = i\frac{q}{2m^2}\left[\psi\left(\nabla_\mu - iqA_\mu\right)\psi^\dagger - \psi^\dagger\left(\nabla_\mu + iqA_\mu\right)\psi\right]. \tag{21}$$

3. Dirac Hydrodynamic Representation

Analogously to the hydrodynamic representation of the Schrödinger equation, which was introduced by Madelung,[23] we derive the hydrodynamic representation of the Dirac equation. We carry out the following generalized Madelung transformation for each component of the spinor $\psi = \psi(x^\mu)$ as follows

$$\psi = \exp(i\theta)R, \tag{22}$$

where R is a spinor and θ is a function. For the case where we consider a Dirac electron-like fermion, $\theta = \theta(x^\mu)$, the spinor ψ reads

$$\psi = \begin{pmatrix} R_{\dot{1}} \\ R_{\dot{2}} \\ R_{\dot{3}} \\ R_{\dot{4}} \end{pmatrix} \exp(i\theta) = R\exp(i\theta), \tag{23}$$

where we use the notation $\dot{\mu}$, $\dot{\nu}$, ...$= \dot{1}, \cdots, \dot{4}$ for the spinor indices such that

$$R = \begin{pmatrix} R_{\dot{1}} \\ R_{\dot{2}} \\ R_{\dot{3}} \\ R_{\dot{4}} \end{pmatrix} = \begin{pmatrix} \sqrt{n_{\dot{1}}} \\ \sqrt{n_{\dot{2}}} \\ \sqrt{n_{\dot{3}}} \\ \sqrt{n_{\dot{4}}} \end{pmatrix}. \tag{24}$$

In the appendix we show some exact solutions of the Dirac equation with this ansatz in flat space-time. Here the dot indices represent elements of each component and we do not use the sum convention when the indices are up and down. On the other hand, we assume this notation $R_{\dot{\mu}} = \sqrt{n_{\dot{\mu}}}$ to (24), where $n_{\dot{\mu}}$ is the number density, represents the modulus of $\psi_{\dot{\mu}}$ and θ is its phase (both are real variables). In general, $n_{\dot{\mu}}$ is different for each component of the spinor. Note that the covariant derivative of the spinor ψ in terms of its decomposition (23) is $\nabla_\mu(\psi_{\dot{\nu}}) = \partial_\mu(R_{\dot{\nu}}e^{i\theta}) + \Gamma^{\dot{\alpha}}_{\mu\dot{\nu}}(R_{\dot{\alpha}}e^{i\theta}) = (\partial_\mu R_{\dot{\nu}})e^{i\theta} + i(\partial_\mu\theta)R_{\dot{\nu}}e^{i\theta} + \Gamma^{\dot{\alpha}}_{\mu\dot{\nu}}(R_{\dot{\alpha}}e^{i\theta})$, implying that $\nabla_\mu\theta = \partial_\mu\theta$.

Using the transformation (23) in eq. (5), the Dirac equation in terms of the variables R and θ reads

$$\exp(i\theta)\gamma^{\mu}\left(i\nabla_{\mu}R-(\nabla_{\mu}\theta)R-qA_{\mu}R-\frac{m}{4}\gamma_{\mu}R\right)=0. \tag{25}$$

To get the last term, we used the property of the gamma matrices that $\gamma_{\mu}\gamma^{\mu}=4\mathbb{I}$, where $\mathbb{I}$ is the 4×4 identity matrix. This property results from the anti-commutation relation of the gamma matrices. Similarly, the continuity equation (11) with (10) can be written with these new variables as

$$\left(\nabla_{\mu}R^{T}\right)K^{\mu}R+R^{T}K^{\mu}\left(\nabla_{\mu}R\right)=0, \tag{26}$$

where R^T denotes the transpose of R and $K^{\mu}=B\gamma^{\mu}$. Observe that K^{μ} is hermitian ($K^{\mu\dagger}=K^{\mu}$).

In conclusion, we have introduced the hydrodynamic representation of the Dirac equation (25) and its conjugate transpose equation by making the change of variables from eq. (22).

4. Dirac-Euler Equation

As for the Klein-Gordon equation,[1,2] we define the 4-velocity v_{μ} by

$$mv_{\mu\dot{\nu}}=\nabla_{\mu}S_{\dot{\nu}}+qA_{\mu}. \tag{27}$$

Here, $S(x^{\mu})$ is a phase with components $S_{\dot{\alpha}}=\theta_{\dot{\nu}}\delta^{\dot{\nu}}{}_{\dot{\alpha}}-\omega_{\dot{\nu}}\delta^{\dot{\nu}}{}_{\dot{\alpha}}t$ where $\omega_{\dot{\nu}}$ are constants that can be related to the mass of the fermion particle by $\omega_{\dot{\nu}}=mc^2/\hbar$ and $\theta_{\dot{\nu}}=\theta$. In this manner we can write

$$\nabla_{\mu}\theta_{\dot{\alpha}}\delta^{\dot{\alpha}}{}_{\dot{\nu}}=(mv_{\mu\dot{\alpha}}-\omega_{\dot{\alpha}}\delta^{0}{}_{\mu})\delta^{\dot{\alpha}}{}_{\dot{\nu}}-qA_{\mu}. \tag{28}$$

We interpret $n_{\dot{\nu}}$ as the density number of fermions and $v_{\mu\dot{\nu}}$ as its velocity. Actually, eq. (25) can be interpreted as the first integral of the Bernoulli equation for fermions in an arbitrary space-time. To see this, we apply the operator $i\gamma^{\mu}D_{\mu}=i\gamma^{\mu}\nabla_{\mu}-q\gamma^{\mu}A_{\mu}$ to the Dirac equation (5) written under the form $i\gamma^{\mu}\nabla_{\mu}\psi=q\gamma^{\mu}A_{\mu}\psi-m\psi$. This yields

$$\begin{aligned}-\gamma^{\mu}\gamma^{\nu}\left(\nabla_{\mu}\nabla_{\nu}\psi+iq(\nabla_{\mu}A_{\nu})\psi+iqA_{\nu}(\nabla_{\mu}\psi)+iqA_{\mu}(\nabla_{\nu}\psi)-q^{2}A_{\mu}A_{\nu}\psi\right)-\\ m^{2}\psi-\gamma^{\mu}(\nabla_{\mu}\gamma^{\nu})(\nabla_{\nu}\psi+iqA_{\nu}\psi)=0.\end{aligned} \tag{29}$$

Using the relation (7) in eq. (29) we obtain

$$\Box_{E}\psi+m^{2}\psi+\frac{i}{2}q\gamma^{\mu}\gamma^{\nu}F_{\mu\nu}\psi+\gamma^{\mu}(\nabla_{\mu}\gamma^{\nu})(D_{\nu}\psi)=0, \tag{30}$$

where we have defined the D'Alambertian operator in the presence of an electromagnetic field by $\Box_{E}=(\nabla_{\mu}+iqA_{\mu})(\nabla^{\mu}+iqA^{\mu})$ and the anti-symmetric Faraday tensor by $F_{\mu\nu}=\nabla_{\mu}A_{\nu}-\nabla_{\nu}A_{\mu}$. Eq. (30) is similar to the Klein-Gordon equation with an electromagnetic source except that here ψ is a spinor instead of a complex SF. The last term of eq. (30) contains the covariant derivative of γ^{μ} which vanishes in a flat space-time. According to Refs.[1,2] if we apply the transformation (22) to eq. (30), we could expect to obtain the continuity equation for the imaginary part and the Bernoulli equation for the real part. However, in the case of the Dirac equation the four components are mixed by the presence of the four dimensional spinor ψ. Hence, we obtain the following expression

$$i\left[2(mv^\mu - \omega\delta_0^\mu)\nabla_\mu R - qA_\mu + q\nabla_\mu(A^\mu R) + \nabla_\mu(mv^\mu - \omega\delta_0^\mu - qA^\mu)R\right] + \left(m^2 v_\mu v^\mu + 2m\omega v^0 + \frac{\omega^2}{N^2} + m^2\right)R - \Box R + \frac{i}{2}q\gamma^\mu\gamma^\nu F_{\mu\nu}R + \gamma^\mu(\nabla_\mu\gamma^\nu)(i(mv_\nu + \omega\nabla_\nu t)R + D_\nu R) = 0. \quad (31)$$

Here, we have defined $\Box = \nabla^\nu\nabla_\nu$ and we have introduced the diagonal matrices $v_\mu = v_{\mu\dot{\nu}}\delta^{\dot{\nu}}_{\dot{\alpha}}$ and $\omega = \omega_{\dot{\nu}}\delta^{\dot{\nu}}_{\dot{\alpha}}$. For bosons, the real and imaginary parts separate into two independent equations, namely, the continuity equation and the Bernoulli equation.[1,2] But in the spinor case, the last line of equation (31) mixes both the imaginary and real parts and there is no natural separation into real and imaginary parts. The system remains coupled.

5. Weyl Representation

The Dirac equation for 1/2-spin particles is associated with the $SO(1,3)$ symmetry group. Nevertheless, we can introduce a new representation as in standard QFT, since there exists a surjective homomorphism between the $SO(1,3)$ and $SU(2)\otimes SU(2)$ Lie groups.

In terms of the Pauli matrices the 4×4 gamma matrices γ^μ can be written as two 2×2 block matrices

$$\gamma^0 = N\tilde{\gamma}^0 = N\begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \quad (32)$$

$$\gamma^j = \hat{e}^j{}_i(\tilde{\gamma}^i + N^i\tilde{\gamma}^0) = \begin{pmatrix} 0 & -\hat{e}^j{}_i(\tilde{\sigma}^i - N^i\mathbb{I}) \\ \hat{e}^j{}_i(\tilde{\sigma}^i + N^i\mathbb{I}) & 0 \end{pmatrix}, \quad (33)$$

where $\tilde{\sigma}^i$ are the 2×2 Pauli matrices in flat space-time and $\mathbb{I}$ is the 2×2 identity matrix. The γ^μ matrices satisfy $\left(\gamma^0\right)^\dagger = \gamma^0$ and $\left(\gamma^j\right)^\dagger = -\gamma^j + 2N^j\gamma^0/N$.

As we know, the special unitary group $SU(2)$ is formed by the set of 2×2 complex matrices A, which satisfy $\det(A) = 1$. Explicitly, we have

$$A = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}, \quad (34)$$

with $\det(A) = |a|^2 + |b|^2 = 1$, where a and b are complex parameters. Equivalently, we have the identity $A^\dagger = A^{-1}$.

The Lie algebra $\mathfrak{su}(2)$ associated to the $SU(2)$ Lie group is given by the exponential map

$$\exp(\mathfrak{su}(2)) \to SU(2). \quad (35)$$

For any element X of the Lie algebra, we have $\exp(X)\exp(X)^\dagger = \mathbb{I}$, implying that $X + X^\dagger = 0$. In what follows, we will indistinctly use $\exp(X)$ and e^X as the exponential map.

In the Weyl representation we can write a Dirac fermion as a four-spinor ψ made of two spinors each of which having two components, for instance

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \tag{36}$$

where ψ_R and ψ_L are the right- and the left- handed Weyl spinors, respectively. If we write the adjoint spinor $\bar{\psi}$ and use the Weyl representation, it follows that

$$\bar{\psi} = \psi^\dagger B = \left(\psi_R^\dagger, \psi_L^\dagger\right) B, \tag{37}$$

where B is the matrix from eqs. (8) and (14). If we use the relation (8) it is straightforward to see that the matrix B must have the following form

$$B = \begin{pmatrix} 0 & B_\zeta \\ B_\zeta & 0 \end{pmatrix}, \tag{38}$$

where the 2×2 matrix B_ζ is a diagonal matrix, $B_\zeta = b\mathbb{I}$, with $b = b(x^\mu)$. Therefore, we get $B = b\tilde{\gamma}^0$ and eq. (15) transforms into

$$\nabla_0(Nb) + \nabla_j(\hat{e}^j_i N^i b) = 0, \tag{39}$$

$$\nabla_j(\hat{e}^j_i b)\tilde{\sigma}^i = 0. \tag{40}$$

Using the definition of the spinor and its adjoint we can write the Dirac quadricurrent J^μ from eq. (10) as

$$J^\mu = \left(\psi_R^\dagger, \psi_L^\dagger\right) B\gamma^\mu \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \tag{41}$$

where the gamma matrices are defined by eqs. (32) and (33) and, in general, B is given by the previously mentioned conditions. This yields

$$J^0 = Nb(\psi_R^\dagger\psi_R + \psi_L^\dagger\psi_L) = Nbn, \tag{42}$$

$$J^j = b\hat{e}^j{}_i(\psi_R^\dagger(\tilde{\sigma}^i + N^i\mathbb{I})\psi_R - \psi_L^\dagger(\tilde{\sigma}^i - N^i\mathbb{I})\psi_L). \tag{43}$$

In order to simplify the notation, we now define the vectors of 2×2 matrices $\mathbb{S}^a = (\mathbb{I}, \tilde{\sigma}^j + N^j\mathbb{I})$ and $\bar{\mathbb{S}}^a = (-\mathbb{I}, \tilde{\sigma}^j - N^j\mathbb{I})$ in terms of the Pauli matrices. $\mathbb{S}^a$ and $\bar{\mathbb{S}}^a$ are the (generalized) Pauli matrices in flat space-time. In terms of these new definitions, the density currents read

$$\begin{aligned} J^j &= b\hat{e}^j{}_i(\psi_R^\dagger\mathbb{S}^i\psi_R - \psi_L^\dagger\bar{\mathbb{S}}^i\psi_L) \\ &= b(\psi_R^\dagger\sigma^i\psi_R - \psi_L^\dagger\bar{\sigma}^i\psi_L), \end{aligned} \tag{44}$$

where we have defined the 2×2 Pauli matrices in a curved space-time by $\sigma^\mu = e^\mu{}_a\mathbb{S}^a$ and $\bar{\sigma}^\mu = e^\mu{}_a\bar{\mathbb{S}}^a$. With this definition, the matrices γ^j read

$$\gamma^j = \begin{pmatrix} 0 & -\bar{\sigma}^j \\ \sigma^j & 0 \end{pmatrix}. \tag{45}$$

Furthermore, observe that the σ^j matrices follow the same commutation relations as the flat space-time Pauli matrices. This means that $[\sigma^i, \bar{\sigma}^j] = -\hat{e}^i_k \hat{e}^j_l [\tilde{\sigma}^k, \tilde{\sigma}^l]$. For the Weyl representation we have to obtain two equations for each Dirac fermion. Thus, we need to redefine the covariant derivative ∇_μ and the spinor affine connection Γ_μ,[14, 24] which can be written as $\nabla_\mu = \partial_\mu + \Gamma_\mu$ and $\Gamma_\mu = \dfrac{1}{4}\bar{\sigma}_\nu \sigma^\nu_{;\mu}$ where $\sigma^\mu_{;\nu} = \partial_\nu \sigma^\mu + \Gamma^\mu_{\alpha\nu}\sigma^\alpha$. Nevertheless, in this representation we need to introduce two other notations due to the presence of $\bar{\sigma}^\mu$. Let $\bar{\nabla}_\mu$ and $\tilde{\Gamma}_\mu$ be the bar covariant derivative and the bar spinor affine connection, respectively, defined by $\bar{\nabla}_\mu = \partial_\mu + \tilde{\Gamma}_\mu$ where $\tilde{\Gamma}_\mu = \dfrac{1}{4}\sigma_\nu \bar{\sigma}^\nu_{;\mu}$ (we stress that we use the greek indices for denoting the objects in curved space-time as the gamma and Pauli matrices).

We can now apply the Weyl representation to rewrite the Dirac equation (5) for a spinor with four components as

$$\begin{pmatrix} i\sigma^\mu \left(\bar{\nabla}_\mu + iqA_\mu\right)\psi_R - m\psi_L \\ i\bar{\sigma}^\mu \left(\nabla_\mu + iqA_\mu\right)\psi_L - m\psi_R \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{46}$$

These are the Weyl equations for a spinor in a curved space-time coupled to an electromagnetic field. If we apply the Weyl representation to the transpose conjugated Dirac equation (16), it is straightforward to obtain the Weyl equation for the adjoint spinor (37). However, we shall not write the adjoint spinor equation explicitly because the results are analogous to the spinor equation as we have seen in the previous sections.

If we set $B = b\tilde{\gamma}^0$, the current density now reads

$$J^\mu = b\left(\psi_R^\dagger \sigma^\mu \psi_R - \psi_L^\dagger \bar{\sigma}^\mu \psi_L\right). \tag{47}$$

Explicitly, we have

$$J^0 = Nbn, \tag{48}$$

$$J^j = b\hat{e}^j_i \left(\psi_R^\dagger \tilde{\sigma}^i \psi_R - \psi_L^\dagger \tilde{\sigma}^i \psi_L + \frac{N^i}{Nb^2} J^0\right). \tag{49}$$

On the other hand, the last line of eq. (31) can be obtained from the identities

$$\gamma^\mu \gamma^\nu F_{\mu\nu}\psi = \begin{cases} (2NN^k F_{0k} + i\hat{F}_{ij}\epsilon^{ij}{}_k \tilde{\sigma}^k)\psi_R \\ -(2NN^k F_{0k} - i\hat{F}_{ij}\epsilon^{ij}{}_k \tilde{\sigma}^k)\psi_L \end{cases}, \tag{50}$$

and using definition (45), we find that

$$\begin{aligned} \gamma^\mu(\nabla_\mu \gamma^\nu)(D_\nu \psi) &= \begin{cases} -\bar{\mathbb{S}}^a \mathbb{S}^b (\hat{\nabla}_a \hat{e}^\nu_b)(D_\nu \psi_R) \\ -\mathbb{S}^a \bar{\mathbb{S}}^b (\hat{\nabla}_a \hat{e}^\nu_b)(D_\nu \psi_L) \end{cases} \\ &= \begin{cases} (N(\nabla_0 N) - \bar{\sigma}^j(\nabla_j N))(D_0\psi_R) + (N(\nabla_0 \sigma^i) - \bar{\sigma}^j(\nabla_j \sigma^i))(D_i \psi_R) \\ (N(\nabla_0 N) + \sigma^j(\nabla_j N))(D_0\psi_L) - (N(\nabla_0 \bar{\sigma}^i) - \sigma^j(\nabla_j \bar{\sigma}^i))(D_i \psi_L) \end{cases} \\ &= \begin{cases} (\hat{\nabla}_0 N - \bar{\mathbb{S}}^k(\hat{\nabla}_k N))(D_0\psi_R) + (\mathbb{S}^k \hat{\nabla}_0 \hat{e}^i_k - \bar{\mathbb{S}}^k \mathbb{S}^l \hat{\nabla}_k \hat{e}^i_l))(D_i \psi_R) \\ (\hat{\nabla}_0 N + \mathbb{S}^k(\hat{\nabla}_k N))(D_0\psi_L) - (\bar{\mathbb{S}}^k \hat{\nabla}_0 \hat{e}^i_k - \mathbb{S}^k \bar{\mathbb{S}}^l (\hat{\nabla}_k \hat{e}^i_l))(D_i \psi_L), \end{cases} \end{aligned} \tag{51}$$

where $\epsilon^{ij}{}_k$ is the usual Levi-Civita tensor, $\hat{F}_{ij} = \hat{e}^l_i \hat{e}^m_j F_{lm}$ is the directional Maxwell tensor $\hat{F}_{ij} = (\hat{e}^l_i \hat{\nabla}_j - \hat{e}^l_j \hat{\nabla}_i)A_l$, and $\hat{\nabla}_a = \hat{e}^\alpha_a \nabla_\alpha$ is the directional covariant derivative which defines the Cartan connection $\hat{\nabla}_c \hat{e}^\nu_b = \Gamma^a_{bc} \hat{e}^\nu_a$. The Cartan connection $\Gamma^a_{bc} = \hat{e}^a_\nu \hat{\nabla}_c \hat{e}^\nu_b$ determines the Cartan first fundamental form $d\hat{e}^a + \Gamma^a_b \wedge \hat{e}^b$ for the connections $\Gamma^a_b = \Gamma^a_{bd} \hat{e}^d$ with the property that $\Gamma_{ab} + \Gamma_{ba} = 0$, where $\Gamma_{ab} = \eta_{ad} \Gamma^d_b$.

6. Weyl Hydrodynamic Representation

We now have all the ingredients to propose a hydrodynamic representation for the Weyl fermions, following the same procedure as the one developed for the Schrödinger and KG equations in Refs.[1,2].

We start to propose our Madelung transformation in the Weyl spinor, using the exponential map, that is

$$\Psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = \begin{pmatrix} R_R \\ R_L \end{pmatrix} e^{i\theta}. \tag{52}$$

Since ψ_R and ψ_L are two spinors, we observe that R_R and R_L are two two-dimensional vectors. The Weyl representation of the adjoint spinor $\bar{\Psi}$ when $B = b\tilde{\gamma}^0$ is

$$\bar{\Psi} = b \left(\psi^\dagger_R, \psi^\dagger_L \right) \tilde{\gamma}^0 = \left(R^\dagger_R, R^\dagger_L \right) e^{-i\theta}. \tag{53}$$

Since R is a real vector, the transposed conjugate is equal to the transposed, that is $R^\dagger = R^T$.

Using the Madelung transformation (52) in the Weyl equations (46) and applying the Lie algebra and the Lie group, we can get the following expression

$$\begin{pmatrix} -\sigma^\mu \left(\bar{\nabla}_\mu \theta_R \right) R_R + i\sigma^\mu \left(\bar{\nabla}_\mu R_R \right) - q\sigma^\mu A_\mu R_R \\ -\bar{\sigma}^\mu \left(\nabla_\mu \theta_L \right) R_L + i\bar{\sigma}^\mu \left(\nabla_\mu R_L \right) - q\bar{\sigma}^\mu A_\mu R_L \end{pmatrix} = \begin{pmatrix} mR_L \\ mR_R \end{pmatrix}. \tag{54}$$

These are the Weyl equations in curved space-time with the Madelung transformation. We can also apply the Madelung transformation (52) and (53) to the current density (47), thereby obtaining

$$J^\mu = b \left(R^T_R \bar{\sigma}^\mu R_R - R^T_L \sigma^\mu R_L \right). \tag{55}$$

Its components are

$$J^0 = Nb(R^T_R R_R + R^T_L R_L) = Nbn, \tag{56}$$

$$J^j = b \left(\hat{e}^j_3 (n_{\dot{1}} - n_{\dot{2}} - n_{\dot{3}} + n_{\dot{4}}) + 2\hat{e}^j_1 (\sqrt{n_{\dot{1}} n_{\dot{2}}} - \sqrt{n_{\dot{3}} n_{\dot{4}}}) + \hat{e}^j_i N^i n \right). \tag{57}$$

We note that the zero component, where $n = \sum_{\dot{\nu}=\dot{1}}^{\dot{4}} n_{\dot{\nu}}$ is the density number of fermions in the system, gives the number of both right- and left-handed particles. We can write the following expressions $|\psi_R|^2 = \psi^\dagger_R \psi_R = R^T_R R_R = n_R$ and $|\psi_L|^2 = \psi^\dagger_L \psi_L = R^T_L R_L = n_L$ for the right- and left-handed spinors, as in the Dirac case. Thus, n_R, n_L are the right- and left- handed particle number and $n = n_R + n_L$ is the total density number.

Furthermore, eq. (31) becomes

$$i\,[2(mv_R^\mu - \omega_R\delta_0^\mu)\nabla_\mu R_R - qA_\mu + q\nabla_\mu(A^\mu R_R) + \nabla_\mu(mv_R^\mu - \omega_R\delta_0^\mu - qA^\mu)R_R] + \left(m^2 v_{R\mu}v_R^\mu + 2m\omega_R v_R^0 + \frac{\omega^2}{N^2} + m^2\right)R_R - \Box R_R + (2NN^k F_{0k} + i\epsilon^{ij}{}_k\hat{F}_{ij}\tilde{\sigma}^k)R_R + (N(\nabla_0 N) - \bar{\sigma}^j(\nabla_j N))((mv_{R0} - \omega_R)R_R + D_0 R_R) + (N(\nabla_0\sigma^i) - \bar{\sigma}^j(\nabla_j\sigma^i))(imv_{Ri}R_R + D_i R_R) = 0. \tag{58}$$

A similar equation is obtained for the left-handed spinor R_L with the substitution $R \longrightarrow L$ and $\mathbb{S} \longleftrightarrow \bar{\mathbb{S}}$ in eq. (58). Simplifying the first line in this equation for $\dot{\nu} = 1, 2$ corresponding to right-handed components, we get

$$i\frac{m}{\sqrt{n_{\dot{\nu}}}}\left[-\frac{\omega_{\dot{\nu}}}{m}\nabla_0 n_{\dot{\nu}} + \nabla_\mu(n_{\dot{\nu}}v_{\dot{\nu}}^\mu) + \frac{\omega_{\dot{\nu}}}{m}\Box t\right] + \sqrt{n_{\dot{\nu}}}\left[m^2 v_{\mu R}v_{\dot{\nu}}^\mu + 2m\omega_{\dot{\nu}}v_{\dot{\nu}}^0 + \frac{\omega_R^2}{N^2} + m^2 - \frac{\Box\sqrt{n_{\dot{\nu}}}}{\sqrt{n_{\dot{\nu}}}}\right] + (2NN^k F_{0k} + i\epsilon^{lj}{}_k\hat{F}_{lj}\tilde{\sigma}^k)R_R - (\hat{\nabla}_a\hat{e}_b^\alpha)\bar{\mathbb{S}}^a\mathbb{S}^b((mv_{R\alpha} - \omega_{\dot{\nu}}\delta_\alpha^0)R_R + D_\alpha R_R) = 0. \tag{59}$$

The equation for the left-handed components $\dot{\nu} = 3, 4$ is obtained by changing $R_R \longrightarrow R_L$ and $\mathbb{S} \longleftrightarrow \bar{\mathbb{S}}$. Here, we have introduced the two-dimensional vectors $v_\mu = (v_{R\mu}, v_{L\mu})$ and $\omega = (\omega_R, \omega_L)$. Explicitly, they are given by

$$R = \begin{pmatrix} R_R \\ R_L \end{pmatrix} = \begin{pmatrix} R_{\dot{1}} \\ R_{\dot{2}} \\ R_{\dot{3}} \\ R_{\dot{4}} \end{pmatrix} = \begin{pmatrix} \sqrt{n_{\dot{1}}} \\ \sqrt{n_{\dot{2}}} \\ \sqrt{n_{\dot{3}}} \\ \sqrt{n_{\dot{4}}} \end{pmatrix}, \tag{60}$$

$$v_\mu = \begin{pmatrix} v_{R\mu} & 0 \\ 0 & v_{L\mu} \end{pmatrix} = \begin{pmatrix} v_{\mu\dot{1}} & 0 & 0 & 0 \\ 0 & v_{\mu\dot{2}} & 0 & 0 \\ 0 & 0 & v_{\mu\dot{3}} & 0 \\ 0 & 0 & 0 & v_{\mu\dot{4}} \end{pmatrix}, \tag{61}$$

$$\omega = \begin{pmatrix} \omega_R & 0 \\ 0 & \omega_L \end{pmatrix} = \begin{pmatrix} \omega_{\dot{1}} & 0 & 0 & 0 \\ 0 & \omega_{\dot{2}} & 0 & 0 \\ 0 & 0 & \omega_{\dot{3}} & 0 \\ 0 & 0 & 0 & \omega_{\dot{4}} \end{pmatrix}. \tag{62}$$

We now write the last two lines of eq. (59) explicitly and separate them into imaginary and real parts, respectively. Using the Pauli representation of the $\tilde{\sigma}^j$ matrices

$$\tilde{\sigma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{\sigma}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tilde{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{63}$$

we obtain for $\dot{\nu} = \dot{1}$:

$$\frac{m}{\sqrt{n_{\dot{1}}}}\left[-\frac{\omega_{\dot{1}}}{m}\nabla_0 n_{\dot{1}} + \nabla_\mu(n_{\dot{1}} v^\mu_{\dot{1}}) + \frac{\omega_{\dot{1}}}{m}\Box t\right] =$$
$$F_{12}\sqrt{n_{\dot{1}}} + F_{23}\sqrt{n_{\dot{2}}} - 2\Gamma^a_{21}((m\hat{v}_{a\dot{1}} - \omega_{\dot{1}}\hat{\delta}^0_a)\sqrt{n_{\dot{1}}} + \hat{D}_a\sqrt{n_{\dot{1}}}) -$$
$$2(\Gamma^a_{21}N^1 - \Gamma^a_{32}N^3 + \Gamma^a_{20} + \Gamma^a_{32})((m\hat{v}_{a\dot{2}} - \omega_{\dot{2}}\hat{\delta}^0_a)\sqrt{n_{\dot{2}}} + \hat{D}_a\sqrt{n_{\dot{2}}}),$$
$$\sqrt{n_{\dot{1}}}\left[m^2 v_{\mu\dot{1}} v^\mu_{\dot{1}} + 2m\omega_{\dot{1}} v^0_{\dot{1}} + \frac{\omega^2_{\dot{1}}}{N^2} + m^2 - \frac{\Box\sqrt{n_{\dot{1}}}}{\sqrt{n_{\dot{1}}}}\right] =$$
$$2N(F_{01}N^1 + F_{02}N^2 + F_{03}N^3)\sqrt{n_{\dot{1}}} - F_{13}\sqrt{n_{\dot{2}}} +$$
$$\left[\Gamma^a_{11}(1-(N^1)^2) + \Gamma^a_{22}(1-(N^2)^2) + \Gamma^a_{33}(1-(N^3)^2)\ +\right.$$
$$\left.2\Gamma^a_{31}N^1 + 2\Gamma^a_{32}N^2 - \Gamma^a_{00} + 2\Gamma^a_{30}\right]((m\hat{v}_{a\dot{1}} - \omega_{\dot{1}}\hat{\delta}^0_a)\sqrt{n_{\dot{1}}} + \hat{D}_a\sqrt{n_{\dot{1}}}) +$$
$$(-2\Gamma^a_{21}N^2 - 2\Gamma^a_{31}N^3 + 2\Gamma^a_{10} + 2\Gamma^a_{31})((m\hat{v}_{a\dot{2}} - \omega_{\dot{2}}\hat{\delta}^0_a)\sqrt{n_{\dot{2}}} + \hat{D}_a\sqrt{n_{\dot{2}}}), \tag{64}$$

for $\dot{\nu} = \dot{2}$:

$$\frac{m}{\sqrt{n_{\dot{2}}}}\left[-\frac{\omega_{\dot{2}}}{m}\nabla_0 n_{\dot{2}} + \nabla_\mu(n_{\dot{2}} v^\mu_{\dot{2}}) + \frac{\omega_{\dot{2}}}{m}\Box t\right] =$$
$$-F_{12}\sqrt{n_{\dot{2}}} + F_{23}\sqrt{n_{\dot{1}}} + 2\Gamma^a_{21}((m\hat{v}_{a\dot{2}} - \omega_{\dot{2}}\hat{\delta}^0_a)\sqrt{n_{\dot{2}}} + \hat{D}_a\sqrt{n_{\dot{2}}}) +$$
$$2(\Gamma^a_{21}N^1 - \Gamma^a_{32}N^3 + \Gamma^a_{20} - \Gamma^a_{32})(m\hat{v}_{a\dot{1}} - \omega_{\dot{1}}\hat{\delta}^0_a)\sqrt{n_{\dot{1}}} + \hat{D}_a\sqrt{n_{\dot{1}}}),$$
$$\sqrt{n_{\dot{2}}}\left[m^2 v_{\mu\dot{2}} v^\mu_{\dot{2}} + 2m\omega_{\dot{2}} v^0_{\dot{2}} + \frac{\omega^2_{\dot{2}}}{N^2} + m^2 - \frac{\Box\sqrt{n_{\dot{2}}}}{\sqrt{n_{\dot{2}}}}\right] =$$
$$2N(F_{01}N^1 + F_{02}N^2 + F_{03}N^3)\sqrt{n_{\dot{2}}} + F_{13}\sqrt{n_{\dot{1}}} +$$
$$\left[\Gamma^a_{11}(1-(N^1)^2) + \Gamma^a_{22}(1-(N^2)^2) + \Gamma^a_{33}(1-(N^3)^2\ ) +\right.$$
$$\left.-2\Gamma^a_{31}N^1 - 2\Gamma^a_{32}N^2 - \Gamma^a_{00} - 2\Gamma^a_{30}\right]((m\hat{v}_{a\dot{2}} - \omega_{\dot{2}}\hat{\delta}^0_a)\sqrt{n_{\dot{2}}} + \hat{D}_a\sqrt{n_{\dot{2}}}) +$$
$$(-2\Gamma^a_{21}N^2 - 2\Gamma^a_{31}N^3 + 2\Gamma^a_{10} - 2\Gamma^a_{31})((m\hat{v}_{a\dot{1}} - \omega_{\dot{1}}\hat{\delta}^0_a)\sqrt{n_{\dot{1}}} + \hat{D}_a\sqrt{n_{\dot{1}}}), \tag{65}$$

for $\dot{\nu} = \dot{3}$:

$$\frac{m}{\sqrt{n_{\dot{3}}}}\left[-\frac{\omega_{\dot{3}}}{m}\nabla_0 n_{\dot{3}} + \nabla_\mu(n_{\dot{3}} v^\mu_{\dot{3}}) + \frac{\omega_{\dot{3}}}{m}\Box t\right] =$$
$$F_{12}\sqrt{n_{\dot{3}}} + F_{23}\sqrt{n_{\dot{4}}} - 2\Gamma^a_{21}((m\hat{v}_{a\dot{3}} - \omega_{\dot{3}}\hat{\delta}^0_a)\sqrt{n_{\dot{3}}} + \hat{D}_a\sqrt{n_{\dot{3}}}) +$$
$$2(\Gamma^a_{21}N^1 - \Gamma^a_{32}N^3 + \Gamma^a_{20} - \Gamma^a_{32})((m\hat{v}_{a\dot{4}} - \omega_{\dot{4}}\hat{\delta}^0_a)\sqrt{n_{\dot{4}}} + \hat{D}_a\sqrt{n_{\dot{4}}}),$$
$$\sqrt{n_{\dot{3}}}\left[m^2 v_{\mu\dot{3}} v^\mu_{\dot{3}} + 2m\omega_{\dot{3}} v^0_{\dot{3}} + \frac{\omega^2_{\dot{3}}}{N^2} + m^2 - \frac{\Box\sqrt{n_{\dot{3}}}}{\sqrt{n_{\dot{3}}}}\right] =$$
$$2N(F_{01}N^1 + F_{02}N^2 + F_{03}N^3)\sqrt{n_{\dot{3}}} - F_{13}\sqrt{n_{\dot{4}}} +$$
$$\left[\Gamma^a_{11}(1-(N^1)^2) + \Gamma^a_{22}(1-(N^2)^2) + \Gamma^a_{33}(1-(N^3)^2)\ +\right.$$
$$\left.-2\Gamma^a_{31}N^1 - 2\Gamma^a_{32}N^2 - \Gamma^a_{00} - 2\Gamma^a_{30}\right]((m\hat{v}_{a\dot{3}} - \omega_{\dot{3}}\hat{\delta}^0_a)\sqrt{n_{\dot{3}}} + \hat{D}_a\sqrt{n_{\dot{3}}}) +$$
$$(2\Gamma^a_{21}N^2 + 2\Gamma^a_{31}N^3 - 2\Gamma^a_{10} + 2\Gamma^a_{31})((m\hat{v}_{a\dot{4}} - \omega_{\dot{4}}\hat{\delta}^0_a)\sqrt{n_{\dot{4}}} + \hat{D}_a\sqrt{n_{\dot{4}}}), \tag{66}$$

and for $\dot{\nu} = \dot{4}$:

$$\frac{m}{\sqrt{n_{\dot{4}}}}\left[-\frac{\omega_{\dot{4}}}{m}\nabla_0 n_{\dot{4}} + \nabla_\mu(n_{\dot{4}}v^\mu_{\dot{4}}) + \frac{\omega_{\dot{4}}}{m}\Box t\right] =$$
$$-F_{12}\sqrt{n_{\dot{4}}} + F_{23}\sqrt{n_{\dot{3}}} + 2\Gamma^a_{21}((m\hat{v}_{a\dot{4}} - \omega_{\dot{4}}\hat{\delta}^0_a)\sqrt{n_{\dot{4}}} + \hat{D}_a\sqrt{n_{\dot{4}}}) -$$
$$2(\Gamma^a_{21}N^1 - \Gamma^a_{32}N^3 + \Gamma^a_{20} + \Gamma^a_{32})((m\hat{v}_{a\dot{3}} - \omega_{\dot{3}}\hat{\delta}^0_a)\sqrt{n_{\dot{3}}} + \hat{D}_a\sqrt{n_{\dot{3}}}),$$
$$\sqrt{n_{\dot{4}}}\left[m^2 v_{\mu\dot{4}}v^\mu_{\dot{4}} + 2m\omega_{\dot{4}}v^0_{\dot{4}} + \frac{\omega^2_{\dot{4}}}{N^2} + m^2 - \frac{\Box\sqrt{n_{\dot{4}}}}{\sqrt{n_{\dot{4}}}}\right] =$$
$$2N(F_{01}N^1 + F_{02}N^2 + F_{03}N^3)\sqrt{n_{\dot{4}}} + F_{13}\sqrt{n_{\dot{3}}} +$$
$$\left[\Gamma^a_{11}(1-(N^1)^2) + \Gamma^a_{22}(1-(N^2)^2) + \Gamma^a_{33}(1-(N^3)^2)\right. +$$
$$\left.2\Gamma^a_{31}N^1 + 2\Gamma^a_{32}N^2 - \Gamma^a_{00} + 2\Gamma^a_{30}\right]((m\hat{v}_{a\dot{4}} - \omega_{\dot{4}}\hat{\delta}^0_a)\sqrt{n_{\dot{4}}} + \hat{D}_a\sqrt{n_{\dot{4}}}) +$$
$$(2\Gamma^a_{21}N^2 + 2\Gamma^a_{31}N^3 - 2\Gamma^a_{10} - 2\Gamma^a_{31})((m\hat{v}_{a\dot{3}} - \omega_{\dot{3}}\hat{\delta}^0_a)\sqrt{n_{\dot{3}}} + \hat{D}_a\sqrt{n_{\dot{3}}}), \tag{67}$$

where we have used that $\Gamma_{ab} + \Gamma_{ba} = 0$ and defined the directional quantities $v_{a\dot{\nu}} = v_{R\alpha\dot{\nu}}\hat{e}^\alpha_a$, $\hat{\delta}^0_a = \delta^0_\alpha\hat{e}^\alpha_a = N\delta^0_a$ and $\hat{D}_a = \hat{e}^\alpha_a D_\alpha$.

The first line of eq. (59) represents the continuity equation of the fermionic fluid. The second line is the Bernoulli equation. In this respect, we note that eq. (25) is the first integral of this equation. Finally, the last three lines of eq. (59) are the source of the fermionic fluid, something that is not present in the case of bosons. This is because the Dirac equation was introduced in Ref.[25] in order to eliminate the negative probability problem of the KG equation. As a result, the Dirac equation involves only first derivatives while the KG equation is a second order equation.

Observe that the structure of equations (64)-(67) is

$$\frac{m}{\sqrt{n_{\dot{\nu}}}}\left[-\frac{\omega_{\dot{\nu}}}{m}\nabla_0 n_{\dot{\nu}} + \nabla_\mu(n_{\dot{\nu}}v^\mu_{\dot{\nu}}) + \frac{\omega_{\dot{\nu}}}{m}\Box t\right] =$$
$$e_{1\dot{\nu}}F_{12}\sqrt{n_{\dot{\nu}}} + F_{23}\sqrt{n_{\ddot{\nu}}} - 2e_{1\dot{\nu}}\Gamma^a_{21}((m\hat{v}_{a\dot{\nu}} - \omega_{\dot{\nu}}\hat{\delta}^0_a)\sqrt{n_{\dot{\nu}}} + \hat{D}_a\sqrt{n_{\dot{\nu}}}) -$$
$$2(\Gamma^a_{21}N^1 - \Gamma^a_{32}N^3 + \Gamma^a_{20} + e_{2\dot{\nu}}\Gamma^a_{32})((m\hat{v}_{a\ddot{\nu}} - \omega_{\ddot{\nu}}\hat{\delta}^0_a)\sqrt{n_{\ddot{\nu}}} + \hat{D}_a\sqrt{n_{\ddot{\nu}}}),$$
$$\sqrt{n_{\dot{\nu}}}\left[m^2 v_{\mu\dot{\nu}}v^\mu_{\dot{\nu}} + 2m\omega_{\dot{\nu}}v^0_{\dot{\nu}} + \frac{\omega^2_{\dot{\nu}}}{N^2} + m^2 - \frac{\Box\sqrt{n_{\dot{\nu}}}}{\sqrt{n_{\dot{\nu}}}}\right] =$$
$$2N(F_{01}N^1 + F_{02}N^2 + F_{03}N^3)\sqrt{n_{\dot{\nu}}} - e_{1\dot{\nu}}F_{13}\sqrt{n_{\ddot{\nu}}} +$$
$$\left[\Gamma^a_{11}(1-(N^1)^2) + \Gamma^a_{22}(1-(N^2)^2) + \Gamma^a_{33}(1-(N^3)^2)\right. +$$
$$\left.2e_{2\dot{\nu}}(\Gamma^a_{31}N^1 + \Gamma^a_{32}N^2 + \Gamma^a_{30}) - \Gamma^a_{00}\right]((m\hat{v}_{a\dot{\nu}} - \omega_{\dot{\nu}}\hat{\delta}^0_a)\sqrt{n_{\dot{\nu}}} + \hat{D}_a\sqrt{n_{\dot{\nu}}}) +$$
$$(-2e_{3\dot{\nu}}(\Gamma^a_{21}N^2 + \Gamma^a_{31}N^3 - \Gamma^a_{10}) + 2e_{1\dot{\nu}}\Gamma^a_{31})((m\hat{v}_{a\ddot{\nu}} - \omega_{\ddot{\nu}}\hat{\delta}^0_a)\sqrt{n_{\ddot{\nu}}} + \hat{D}_a\sqrt{n_{\ddot{\nu}}}), \tag{68}$$

where the coefficients $e_{i\dot{\nu}}$ are ± 1 with $e_{1\dot{\nu}} = (+,-,+,-)$, $e_{2\dot{\nu}} = (+,-,+,-)$ and $e_{3\dot{\nu}} = (+,+,-,-)$, and the sub-index $\ddot{\nu}$ is the conjugate of the sub-index $\dot{\nu}$ such that $\ddot{1} = \dot{2}$, $\ddot{2} = \dot{1}$, $\ddot{3} = \dot{4}$ and $\ddot{4} = \dot{3}$.

7. Energy Balance

We can also write equation (30) as a Schrödinger-like equation. If we perform the transformation $\psi = \Psi e^{i\omega_0 t}$, where Ψ is a four spinor that depends on all the variables x^μ, equation (30) becomes

$$
\begin{aligned}
& i\nabla^0\Psi - \frac{1}{2\omega_0}\Box_E\Psi + \frac{m^2}{2\omega_0}\Psi + \left(-\frac{\omega_0}{N^2} - 2qA^0 + i\Box t\right)\Psi + \\
& \frac{1}{2\omega_0}\begin{pmatrix} 2NN^kF_{0k} + i\hat{F}_{ij}\epsilon^{ij}{}_k\tilde{\sigma}^k & 0 \\ 0 & -2NN^kF_{0k} + i\hat{F}_{ij}\epsilon^{ij}{}_k\tilde{\sigma}^k \end{pmatrix}\Psi - \\
& \frac{1}{2\omega_0}\begin{pmatrix} \bar{\mathbb{S}}^a\mathbb{S}^b & 0 \\ 0 & \mathbb{S}^a\bar{\mathbb{S}}^b \end{pmatrix}\Gamma^d_{ba}(\hat{D}_d\Psi + i\omega_0 N\delta^0_d\Psi) = 0.
\end{aligned} \tag{69}
$$

Equation (69) is the generalization of the Schrödinger equation for fermions with electromagnetic field interaction in an arbitrary space-time.

Finally, from equation (59) we can identify the different energy contributions to the Fermi gas, and obtain an energy balance equation for fermions analogous to the one obtained for bosons in Refs.[1,2]. In order to simplify the notations, we can re-write equation (59) in terms of the $\dot{\nu}$ coefficients with the understanding that the subindex R refers to each component $R = \dot{1}, \dot{2}$ individually. We get

$$
\begin{aligned}
& i\left[-\omega_{\dot{\nu}}\nabla_0\ln(n_{\dot{\nu}}) + \frac{m\nabla_\mu(n_{\dot{\nu}}v^\mu_{\dot{\nu}})}{n_{\dot{\nu}}} + \frac{\omega_{\dot{\nu}}}{n_{\dot{\nu}}}\Box t\right] + \\
& 2m^2\left(K_{\dot{\nu}} + \frac{1}{m}\omega_{\dot{\nu}}v^0_{\dot{\nu}} + \frac{1}{2}U^N_{\dot{\nu}} + U^Q_{\dot{\nu}}\right) + E_{\dot{\nu}} + U^S_{\dot{\nu}} = 0.
\end{aligned} \tag{70}
$$

This equation is valid for right handed fermions. The result is the same for left handed fermions changing $R_R \longrightarrow R_L$ in the first line, and $\mathbb{S} \longleftrightarrow \bar{\mathbb{S}}$ in the second line.

The first line in eq. (70) describes the free density evolution of the fermions, while the contribution of the different energy terms appears in the second line. The first one is the kinetic energy $K_{\dot{\nu}}$ defined as

$$
K_{\dot{\nu}} = \frac{1}{2}v_{\dot{\nu}\mu}v^\mu_{\dot{\nu}}. \tag{71}
$$

The lapse potential $U^N_{\dot{\nu}}$ is given by

$$
U^N_{\dot{\nu}} = \frac{\omega^2_{\dot{\nu}}}{m^2}\frac{1}{N^2} + 1. \tag{72}
$$

It represents the energy contribution due to the chosen lapse function N. The quantum potential $U^Q_{\dot{\nu}}$ is defined as

$$
U^Q_{\dot{\nu}} = -\frac{1}{2m^2}\frac{\Box\sqrt{n_{\dot{\nu}}}}{\sqrt{n_{\dot{\nu}}}}. \tag{73}
$$

The contribution of the electromagnetic interaction $E_{\dot\nu}$ is given by

$$\begin{aligned} E_{\dot\nu} &= (2NN^kF_{0k} + i\epsilon^{lj}{}_k\hat F_{lj}\tilde\sigma^k)|_{\dot\nu}, \\ &= 2N(F_{01}N^1 + F_{02}N^2 + F_{03}N^3) - e_{1\dot\nu}F_{13}\sqrt{\frac{n_{\ddot\nu}}{n_{\dot\nu}}} + i\left(e_{1\dot\nu}F_{12} + F_{23}\sqrt{\frac{n_{\ddot\nu}}{n_{\dot\nu}}}\right). \end{aligned} \tag{74}$$

It depends on the Faraday tensor, shift vector and lapse function that are related to the Pauli matrices. This relationship is due to the interaction between the electromagnetic field and the fermionic spin. Finally, the potential $U^S_{\dot\nu}$ describes the interaction between the spin and the geometry of space-time. It is given by

$$\begin{aligned} U^S_{\dot\nu} &= -\left((m\hat v_{Rd} - \omega_{\dot\nu}\hat\delta^0_d) + \frac{\hat D_\alpha\sqrt{n_{\dot\nu}}}{\sqrt{n_{\dot\nu}}}\right)\Gamma^d_{ba}\bar{\mathbb{S}}^a\mathbb{S}^b|_{\dot\nu}, \\ &= \big[\Gamma^a_{11}(1-(N^1)^2) + \Gamma^a_{22}(1-(N^2)^2) + \Gamma^a_{33}(1-(N^3)^2) \\ &\quad + 2e_{2\dot\nu}(\Gamma^a_{31}N^1 + \Gamma^a_{32}N^2 + \Gamma^a_{30}) - \Gamma^a_{00}\big]\left((m\hat v_{a\dot\nu} - \omega_{\dot\nu}\hat\delta^0_a) + \frac{\hat D_a\sqrt{n_{\dot\nu}}}{\sqrt{n_{\dot\nu}}}\right) \\ &\quad + (-2e_{3\dot\nu}(\Gamma^a_{21}N^2 + \Gamma^a_{31}N^3 - \Gamma^a_{10}) + 2e_{1\dot\nu}\Gamma^a_{31})\left((m\hat v_{a\ddot\nu} - \omega_{\ddot\nu}\hat\delta^0_a)\sqrt{\frac{n_{\ddot\nu}}{n_{\dot\nu}}} + \frac{\hat D_a\sqrt{n_{\ddot\nu}}}{\sqrt{n_{\dot\nu}}}\right) \\ &\quad + i\Bigg[-2e_{1\dot\nu}\Gamma^a_{21}\left((m\hat v_{a\dot\nu} - \omega_{\dot\nu}\hat\delta^0_a) + \frac{\hat D_a\sqrt{n_{\dot\nu}}}{\sqrt{n_{\dot\nu}}}\right) \\ &\quad - 2(\Gamma^a_{21}N^1 - \Gamma^a_{32}N^3 + \Gamma^a_{20} + e_{2\dot\nu}\Gamma^a_{32})\left((m\hat v_{a\ddot\nu} - \omega_{\ddot\nu}\hat\delta^0_a)\sqrt{\frac{n_{\ddot\nu}}{n_{\dot\nu}}} + \frac{\hat D_a\sqrt{n_{\ddot\nu}}}{\sqrt{n_{\dot\nu}}}\right)\Bigg] \end{aligned} \tag{75}$$

for $\dot\nu = \dot 1, \dot 2$, and by making the substitution $\mathbb{S} \longleftrightarrow \bar{\mathbb{S}}$ for $\dot\nu = \dot 3, \dot 4$. In the foregoing equations, the notation $|_{\dot\nu}$ means that we have to evaluate the quantity at the corresponding $\dot\nu$. Note that $U^S_{\dot\nu}$ disappears if we assume a flat space-time or if we consider particles without spin. Furthermore, $U^S_{\dot\nu}$ is constructed with the generalized gamma matrices (45), which are related to the spin (the Pauli matrices) and to the space-time geometry (tetrads).

8. Conclusions

The main difference between the hydrodynamic representation of bosons[1,2] and fermions concerns the form of the Bernoulli equation. For bosons, after making the Madelung transformation, we can separate the KG equation into real and imaginary parts. By contrast, for fermion particles we have to work with the complete equations of motion because the real and imaginary parts cannot be separated easily. This is related to the fact that the gamma matrices are a representation of the SO(1, 3) group.

The spin is a fundamental outcome of the Dirac equation[25] which combines both elements of special relativity and quantum mechanics and was introduced to solve the problem of negative probability present in the KG equation – first proposed as a relativistic generalization of the Schrödinger equation. Here, we observe that the general relativistic Dirac equation involves an additional contribution due to geometry and spin through the generalized gamma and Pauli matrices. These terms arise from endowing a quantum field with a curvature (geometry) given by a metric in General Relativity. Such a contribution is absent in a flat space-time and in a system without spin as for a scalar field.

With this work we open the possibility of studying in detail the behavior of fermions in different situations (such as massive stars or dark matter halos harboring a central black hole) where general relativity effects may be important. We solved the problem of energy balance for both bosons and fermions. In this manner, we can compare the result of the hydrodynamic representation for classical and quantum fluids in the various geometries mentioned above.

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Appendix A. Solutions to the Dirac equation in flat space-time

Equation (5) in flat space-time, using the Pauli matrices (63), reads

$$\begin{bmatrix} \frac{\partial}{\partial t}\psi_y - \frac{\partial}{\partial x}\psi_z + i\frac{\partial}{\partial y}\psi_z - \frac{\partial}{\partial z}\psi_y - m\psi_t \\ \frac{\partial}{\partial t}\psi_z - \frac{\partial}{\partial x}\psi_y - i\frac{\partial}{\partial y}\psi_y + \frac{\partial}{\partial z}\psi_z - m\psi_x \\ \frac{\partial}{\partial t}\psi_t + \frac{\partial}{\partial x}\psi_x - i\frac{\partial}{\partial y}\psi_x + \frac{\partial}{\partial z}\psi_t - m\psi_y \\ \frac{\partial}{\partial t}\psi_x + \frac{\partial}{\partial x}\psi_t + i\frac{\partial}{\partial y}\psi_t - \frac{\partial}{\partial z}\psi_x - m\psi_z \end{bmatrix} = 0, \tag{A.1}$$

where we have defined the spinor as $\psi = (\psi_{\dot{\mu}}) = (\psi_x, \psi_y, \psi_z, \psi_t)^T$. In order to find an exact solution of the previous equation, we use the ansatz $\psi_{\dot{\mu}} = R_{0\dot{\mu}} \exp(i(x_0 x + y_0 y + z_0 z + t_0 t))$, where $x_0 \cdots t_0$ and $R_{0\dot{\mu}}$ are constants. Here we have the simplest solutions of the Dirac equation where the exponential is the same for all components.

We obtain four linear equations

$$
\begin{aligned}
iR_{0z}\zeta_0^* + iR_{0y}\eta_0 + mR_{0t} &= 0, \\
iR_{0y}\zeta_0 - iR_{0z}\xi_0 + mR_{0x} &= 0, \\
R_{0x}\zeta_0^* + R_{0t}\xi_0 + imR_{0y} &= 0, \\
R_{0t}\zeta_0 - R_{0x}\eta_0 + imR_{0z} &= 0,
\end{aligned} \tag{A.2}
$$

where $\zeta_0 = x_0 + iy_0$, $\eta_0 = z_0 - t_0$, and $\xi_0 = z_0 + t_0$. The solution of these equations is

$$
\begin{aligned}
R_{0t} &= -\frac{1}{m}(iR_{0y}\eta_0 + iR_{0z}\zeta_0), \\
R_{0x} &= \frac{1}{m}(iR_{0z}\xi_0 - iR_{0y}\zeta_0^*),
\end{aligned} \tag{A.3}
$$

where $x_0^2 + y_0^2 + z_0^2 - t_0^2 = m^2$.

Now we use the ansatz $\psi_\mu = R_{0\mu}\exp(i\theta)$, where θ is an arbitrary function of the coordinates. Substituting this ansatz into (A.1) we obtain

$$
\begin{aligned}
iR_{0z}Z_0^* + iR_{0y}E_0 + mR_{0t} &= 0, \\
iR_{0y}Z_0 - iR_{0z}F_0 + mR_{0x} &= 0, \\
R_{0x}Z_0^* + R_{0t}F_0 + imR_{0y} &= 0, \\
R_{0t}Z_0 - R_{0x}E_0 + imR_{0z} &= 0,
\end{aligned} \tag{A.4}
$$

where $Z_0 = \theta_{,x} + i\theta_{,y}$, $E_0 = \theta_{,z} - \theta_{,t}$, and $F_0 = \theta_{,z} + \theta_{,t}$. The solution of the previous system of differential equations is

$$
\begin{aligned}
\theta = {} & F(X) - it \\
& + \frac{m}{2R_{0t}R_{0z} + 2R_{0x}R_{0y}}\left(i\zeta_0^*(R_{0x}^2 - R_{0z}^2) - i\zeta_0(R_{0y}^2 - R_{0t}^2)\right),
\end{aligned} \tag{A.5}
$$

where $F(X)$ is an arbitrary function of

$$
X = \frac{R_{0t}(-\zeta R_{0y} - \zeta^* R_{0x} + \xi R_{0y} - \eta R_{0z})}{2R_{0t}R_{0z} + 2R_{0x}R_{0y}}. \tag{A.6}
$$

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