Exact Solutions of G-invariant Chiral Equations
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Abstract

We give a methodology for solving the chiral equations $(\alpha g_{,z} g^{-1})_{,\overline{z}} + (\alpha g_{,\overline{z}} g^{-1})_{,z} = 0$ where g belongs to some Lie group G. The solutions are writing in terms of Harmonic maps. The method could be used even for some infinite lie groups.

One of the most important set of differential equations in mathematical physics is perhaps the chiral equations. The most popular of them are the SU(N)-invariant one. In general relativity chiral equations appear very often for the groups SU(1,1), SU(2,1), SL(2, R) etc. [1]. One of the first aproaches for solving them is by means of the Inverse Scatering Method (ISM), given by Belinsky ans Zacharov [2]. They gave a Lax Pair respresentation of the chiral equatons and solve them using the ISM. An equivalent aproach using the polinomial ansatz is given in ref [3]. In this last work we used the generalized Bäcklund transformations for finding exact solutions of the chiral equations.

Recently, Hussain [4] showed that self or antiself dual gravity reduces to chiral equation-like systems but the group remain here infinite. Algebra brackets are replaced by Poisson brackets and all of these methods mentioned above can not be aplied in this case. In this paper we generalize the method of Harmonic maps for solving the chiral equations for all Lie groups. All the solutions obtained by the ISM or by Bäcklund transformations can be derived with this method choosing the Harmonic map conveniently. For example, all the solutions found by ISM in general relativity can also be found by this mathod [10]. With the method of Harmonic maps the solutions apear clasified in classes naturaly. The solutions can be given in terms of arbitrary harmonic maps and the method can be used even for infinite groups.

Let g be a mapping $g : \mathcal{C} \otimes \overline{\mathcal{C}} \to G$, $g \to g(z, \overline{z}) \epsilon G$, where G is a paracompact Lie group.

The chiral equations for g

$$(\alpha g_{,z} g^{-1})_{,\overline{z}} + (\alpha g_{,\overline{z}} g^{-1})_{,z} = 0 \qquad (1)$$

$(\alpha^2 = det\ g)$ are a set of non-linear, coupled, second order partial diffential equations that appear in many topics in physics. Normally $g(z, \overline{z})$ is given in a representation of

G. One of the most important features of the chiral equations is that they can be derived from the Lagrangian.

$$\mathcal{L} = \alpha tr\ (g_{,z}g^{-1}g_{,\overline{z}}g^{-1}). \tag{2}$$

This Lagrangian represents a topological quantum field theory with gauge group G, then it is of great interest to know explicit expression of the elements $g\epsilon G$ in terms of the local coordinates z and $\overline{z}$.

Let G_c be a subgroup $G_c \subset G$ such that $c\epsilon G_c$ implies $c_{,z} = 0,\ c_{,\overline{z}} = 0$. Then equation (1) is invariant under the left action L_c of G_c over G. We say that the chiral equations (1) are invariant under this group.

Proposition 1. Let β be a complex function defined by

$$\beta_{,z} = \frac{1}{4(ln\alpha)_{,z}}\ tr\ (g_{,z}g^{-1})^2, \quad g\epsilon G \tag{3}$$

and $\beta_{,\overline{z}}$ with $\overline{z}$ in place of z. If g fulfills the chiral equations, then β is integrable.

Proof. $\beta_{,z\overline{z}} = \beta_{,\overline{z}z}$.

Let $\mathcal{G}$ be the corresponding Lie algebra of G. The Maurer-Cartan form ω_g of G defined by $\omega_g = L_{g-1*}(g)$ is a one-form on G with value on $\mathcal{G}, \omega_g\epsilon T_g^*G \otimes \mathcal{G}(T_xM$ represents the tangent space of the Manifold M at the point x). Let us define the mappings

$$A_z : G \rightharpoonup \mathcal{G}$$
$$g \rightharpoonup A_z(g) = g_{,z}g^{-1}$$

$$A_{\overline{z}} : G \rightharpoonup \mathcal{G}$$
$$g \rightharpoonup A_{\overline{z}}(g) = g_{,\overline{z}}g^{-1} \tag{4}$$

If g is given in a representation of G, then we can write the one-form $\omega(g) = \omega_g$ as

$$\omega = A_z dz + A_{\overline{z}} d\overline{z} \tag{5}$$

We can now define a metric on $\mathcal{G}$ in a standard manner. Since ω_g can be written as in (50), the tensor

$$l = tr(dgg^{-1} \otimes dgg^{-1}) \tag{6}$$

on G defines a metric on the tangent bundle of G.

Theorem 1. The submanifold of solutions of the chiral equations $S \subset G$, is a symmetric manifold with metric (6).

This theorem was proved by Neugebauer and Kramer in reference [5] and we will only outline here the proof.

Proof. We take a parametrization $\lambda^a \quad a = 1, \cdots n$, of G. The set $\{\lambda^a\}$ is a local coordinate system of the n-dimensional differential manifold G. In terms of this parametrization the Maurer-Cartan one form ω can be written as

$$\omega = A_a d\lambda^a, \tag{7}$$

where $A_a(g) = (\frac{\partial}{\partial \lambda^a} g) g^{-1}$. The chiral equations then read

$$\nabla_b A_a(g) + \nabla_a A_b(g) = 0, \tag{8}$$

with ∇_a the convariant derivative defined by (6).

One infers the relation

$$\nabla_b A_a(g) = \frac{1}{2} [A_a, A_b](g). \tag{9}$$

With this, one can calculate the Riemannian curvature $\mathcal{R}$ of (6). Their components read

$$R_{abcd} = \frac{1}{4} tr(A_{[a} A_{b]} A_{[c} A_{d]}) \tag{10}$$

($[a, b]$ means index commutation) this can be done, because G is a paracompact manifold. It follows that $\nabla \mathcal{R} = 0$

Proposition 2. The function $\alpha = \det g$ is harmonic.

Proof. The trace of the chiral equations implies $\alpha_{,z\bar{z}} = 0$.

In the following we will explain the method for calculating explicit chiral field.

Let V_p be a complete totally geodesic submanifold of G and let $\{\lambda^i\} \quad i = 1, \cdots, p$ be a set of local coordiantes on V_p. Because we know the Manifold G, it is possible to know V_p. In fact V_p is a subgroup of G and since V_p is a complete totally geodesic submanifold, V_p is also symmetric. The symmetries of G and V_p are in fact isometries, since both of them are paracompact manifolds, with Riemannian metrics (6) and $i_* l$ respectivelly. (i is the restriction of V_p into G). Let us suppose that V_p possesses d isometies. Thus

$$(\alpha \lambda^i_{,z})_{,\bar{z}} + (\alpha \lambda^i_{,\bar{z}})_{,z} + 2\alpha \sum_{ijk} \Gamma^i_{jk} \lambda^j_{,z} \lambda^k_{,\bar{z}} = 0 \tag{11}$$

$$i, j, k = 1, \cdots, p$$

where Γ^i_{jk} are the Christoffel symboles of $i_* l$ and λ^i are the totally geodesic parameters on V_p. In terms of the parameters λ^i the chiral equations read

$$\nabla_i A_j(g) + \nabla_j A_i(g) = 0 \tag{12}$$

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where ∇_i is the covariant derivative of V_p. Equation (12) is the Killing equation on V_p for the components of A_i. Since we know the manifold V_p, we know its isometries and therefore its Killing vector space. Let $\xi_s, s = 1, \cdots, d$, be a base of the Killing vector space of V_p and Γ^s be a base of the subalgebra corresponding to V_p. Then we can write

$$A_i(g) = \sum_s \xi_s^i \, \Gamma^s \tag{13}$$

where $\xi_s = \sum_j \xi_s^j \frac{\partial}{\partial \lambda^j}$. The covariant derivative on V_p is given by

$$\nabla_j A_i(g) = -\frac{1}{2}[A_i, A_j](g) \tag{14}$$

where A_i fulfills the integrability conditions

$$F_{ij} = \nabla_j A_i(g) - \nabla_i A_j(g) - [A_j, A_i](g) = 0 \tag{15}$$

i.e., A_i has a pure gauge form.

The left action of G_c over $G, L : G_c \otimes G \rightharpoonup G$ must be defined in a convenient manner in order to preserve the properties of the elements of S.

Knowing $\{\xi_s\}$ and $\{\Gamma^s\}$ one could integrate the elements of S, since $A_i(g)\epsilon\mathcal{G}$ can be mapped into the group by means of the exponential map. Nevertheless it is not possible to map all the elements one by one. Fortunately we have the following proposition.

Proposition 3. The relation $A_i^c r A_i$ iff there exist $c\epsilon G_c$ such that $A^c = A \circ L_c$, is an equivalence relation.

Proof. Trivial.

This equivalence relation separates the set $\{A_i\}$ into equivalence classes $[A_i]$. Let TB be a set of representatives of each class, $TB = \{[A_i]\}$. Now we map the elements of $TB \subset \mathcal{G}$ into the group S by means of the exponential map or by integration. Let us define B as the set of elements of the group, mapped from each representative $B = \{g\epsilon S | g = exp(A_i), A_i\epsilon TB\} \subset G$. The elements of B are also elements of S because A_i fulfills the chiral equations, $i.e.B \subset S$. For constructing all the set S we have the following Theorem.

Theorem 2. (S, B, π, G_c, L) is a principal fibre bundle with projection $\pi(L_c(g)) = g$; $\quad L(c, g) = L_c(g)$.

Proof. The fibres of G are the orbits of the group G_c on $G, F_g = \{g'\epsilon G | g' = L_c(g)\}$ for some $g\epsilon B$. The topology of B is its relative topology with respect to G. Let α_F be the bundle $\alpha_F = (G_c \times U_\alpha, U_\alpha, \pi)$, where $\{U_\alpha\}$ is an open covering of B. We have the following lemma.

Lemma 1. The bundle α_F and $\alpha = (\pi^{-1}(U_\alpha), U_\alpha, \pi|_{\pi^{-1}(U_a)})$ are isomorphic.

Proof. The mapping

$$\psi_\alpha : \phi^{-1}(U_\alpha) = \{g\epsilon S | g' = L_c(g), g\epsilon U_\alpha\}_{c\epsilon G_c} \rightharpoonup G_c \times U_\alpha$$

$$g' \rightharpoonup \psi_\alpha(g') = (c, g)$$

is an homeomorphism and $\pi|_{\pi^{-1}(U_\alpha)}(g') = g = \pi_2 \circ \psi_\alpha(g')$.

By lemma 1 the bundle α is locally trivial. To end the proof of the Theorem it is sufficient to prove that the G_c spaces (S, G_c, L) and $(G_c \times U_\alpha, G_c, \delta)$, are isomorphic,but that is so because

$$\delta \circ id|_{G_c} \times \psi_\alpha = \psi_\alpha \circ L|_{G_c \times \pi^{-1}(U_\alpha)}$$

With this theorem it is now possible to explain the method.

a) Given the chiral equations (1), invariant under the group G, chose a symmetric Riemannian space V_p with d killing vectors, $p \leq n = \dim G$.

b) Look for a representation for the corresponding Lie Algebra $\mathcal{G}$ compatible with the commutating relations of the killing vectors, via equation (14).

c) Write the matrices $A_i(g)$ explicitly in terms of the geodesic parameters of the symmetric space V_p.

d) Use proposition 2 for finding the equivalence classes in $\{A_i\}$ and choose a set of representatives.

e) Map the lie algebra representatives into the group.

The solutions can be constructed by means of the left action of the G_c group into G.

Let us give an example. Suppose $G = SL(2, \mathbf{R})$.

a) We choose the one dimensional space V_1 with $i_* l = d\lambda^2$. It is Riemannian and symmetric with one killing vector

b) The algebra $sl(2, \mathbf{R})$ is the set of traceless and real 2×2 matrices. The Killing equation reduces to the Laplace equation $(\alpha\lambda_{,z})_{,\bar{z}} + (\alpha\lambda_{,\bar{z}})_{,z} = 0$.

c) Using (14) we obtain

$$g_{,\lambda}g^{-1} = A = constant.$$

d) The representative of the set of trasless and real constant 2×2 matrices $\{A_i\}$ is

$$\{[A_i]\} = \left\{ \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix} \right\}$$

(see also ref. [6]).

e) The mapping of the Lie algebra representatives can be done by interaction of the matrix differential equation

$$g_{,\lambda} = [A_i]g.$$

The solutions depend on the characteristical polynomial of $[A_i]$. We obtain three cases: $a > 0$, $a < 0$, $a = 0$. For each case the matrix g reads

$$a > 0 \quad g = b \begin{pmatrix} 1 & a \\ a & a^2 \end{pmatrix} e^{a\lambda} + c \begin{pmatrix} 1 & -a \\ -a & a^2 \end{pmatrix} e^{-a\lambda}, \ 4bca^2 = 1$$

$$a < 0 \quad g = b \begin{pmatrix} \cos(a\lambda + \psi_0) & -a\sin(a\lambda + \psi_0) \\ -a\sin(a\lambda + \psi_0) & -a^2\cos(a\lambda + \psi_0) \end{pmatrix}, a^2 b^2 = -1$$

$$a = 0 \quad g = \begin{pmatrix} b\lambda + c & b \\ b & 0 \end{pmatrix} \qquad b^2 = -1$$

where a, b, c and ψ_0 are constants. So, for each solution λ of the Harmonic equation $(\alpha\lambda_{,\bar{z}})_{,z} + (\alpha\lambda_{,z})_{,\bar{z}} = 0$ we will have a new solution of the chiral one. The left action of $SL_c(2, R)$ over $SL(2, R)$ is represented as

$$g' = CgD$$

where $C, D \epsilon SL_c(2, R)$. g' will be also a solution of the chiral equations.

If we choose a V_2 manifold we can fiend an other class of solutions. All V_2 manifold is conformally flat, therefore the metric on V_2 reads

$$dl^2 = \frac{d\tau d\lambda}{(l + k\lambda z)^2}$$

But V_2 symmetric implies $k =$ constan. A V_2 manifold with constant curvature has three Killing vectors. Let

$$\xi_1 = \frac{1}{2V^2} \left[(k\tau^2 + 1)d\lambda + (k\lambda + 1)d\tau \right]$$

$$\xi_2 = \frac{1}{V^2} [-\tau d\lambda + \lambda d\tau] \qquad V = 1 + k\lambda\tau$$

$$\xi_3 = \frac{1}{2V^2} [(k\tau^2 - 1)d\lambda + (1 - k\lambda^2)d\tau]$$

be a base of the Killing vector spaces of V_2. The correspondings commutation relations of the dual vectors are

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$$[\Gamma^1, \Gamma^2] = -4k\Gamma^3 \tag{16}$$
$$[\Gamma^2, \Gamma^3] = 4k\Gamma^2$$
$$[\Gamma^3, \Gamma^1] = -4\Gamma^2$$

We have to put $k = -1$ in order to have the commutation relations of $sl(2, R)$. A representation of $sl(2, R)$, compatible with (16) is

$$\Gamma^1 = 2 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Gamma^2 = \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix}, \quad \Gamma^3 = \begin{pmatrix} 0 & -b \\ a & 0 \end{pmatrix} \quad ab = 1$$

Then equation (13) reads

$$g_{,\lambda} g^{-1} = \frac{1}{V^2} \begin{pmatrix} \tau^2 - 1 & b(1 - \tau)^2 \\ -a(1 + \tau)^2 & 1 - \tau^2 \end{pmatrix}$$
$$g_{,\tau} g^{-1} = A_\tau = \frac{1}{V^2} \begin{pmatrix} \lambda^2 - 1 & b(1 - \lambda)^2 \\ -a(1 + \lambda)^2 & 1 - \lambda^2 \end{pmatrix}$$

wich integration is

$$g = \frac{1}{1 - \lambda\tau} \begin{pmatrix} c(1 - \lambda)(1 - \tau) & e(\tau - \lambda) \\ e(\tau - \lambda) & d(1 + \lambda)(1 + \tau) \end{pmatrix}$$
$$cd = -e^2, \quad a = \frac{e}{c}, \quad b = -\frac{e}{d} \tag{17}$$

The harmonic equation in V_2 reads

$$(\alpha\lambda_{,z})_{,\overline{z}} + (\alpha\lambda_{,\overline{z}})_{,z} + \frac{4\tau}{1 + \lambda\tau}\alpha\lambda_{,z}\lambda_{,\overline{z}} = 0$$
$$(\alpha\tau_{,z})_{,\overline{z}} + (\alpha\tau_{,\overline{z}})_{,z} + \frac{4\lambda}{1 + \lambda\tau}\alpha\tau_{,z}\tau_{,\overline{z}} = 0 \tag{18}$$

So, for each solution of (18) into (17) we fiend a new solution of the chiral equations for $SL(2, R)$.

A general algorithm of integration even for the Lie Algebra $sl(N, \mathbf{R})$, is given in [7]. The explicity example for $SL(4, R)$ is presented in ref. [8]. The application of the method for infinite groups will be published elsewhere [9].

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