A Numerical Study of Boson Star Binaries

by

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ABSTRACT

In this thesis we present a numerical study of general relativistic boson stars in both spherical symmetry... Choptuik [1, 2].

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ACKNOWLEDGEMENTS

I would like to my family whose all its efforts and sacrificies made my journey possible so far.

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2.1.1 In any flat coordinate system

Under Maximal slicing condition and conformal flat condition for the spatial metric, the equations of motion reduce to 5 elliptical equations corresponding to the 5 geometric quantities to be calculated, ψ, α and $\vec{\beta}$.

The maximal slicing condition along with the evolution equation for $\dot{K}_i^i$ and hamiltonian constraint yields a elliptical equation for the lapse α:

$$\nabla^2\alpha = -\frac{2}{\psi}\vec{\nabla}\psi \cdot \vec{\nabla}\alpha + \alpha\psi^4\left(K_{ij}K^{ij} + 4\pi\left(\rho + S\right)\right) \tag{2.1}$$

The elliptical equation for the conformal factor ψ is obtained from the hamiltonian constraint and has the following functional form:

$$\nabla^2\psi = -\frac{\psi^5}{8}\left(K_{ij}K^{ij} + 16\pi\rho\right) \tag{2.2}$$

When the momentum constraint equations are used to fix the shift vector, $\vec{\beta}$, a set of 2nd order elliptical equations are obtained:

$$\nabla^2\beta^j = -\frac{1}{3}\hat{\gamma}^{ij}\partial_i\left(\vec{\nabla}\cdot\vec{\beta}\right) + \alpha\psi^4 16\pi J^j - \partial_i\left[ln\left(\frac{\psi^6}{\alpha}\right)\right]\left[\hat{\gamma}^{ik}\partial_k\beta^j + \hat{\gamma}^{jk}\partial_k\beta^i - \frac{2}{3}\hat{\gamma}^{ij}\left(\vec{\nabla}\cdot\vec{\beta}\right)\right] \tag{2.3}$$

where $\hat{\gamma}$ is the spatial flat metric related to the physical metric through a conformal transformation:

$$\gamma^{ij} = \psi\hat{\gamma}^{ij} \tag{2.4}$$

and $\vec{\nabla}$ is the usual flat operator notation. In terms of the flat spatial covariant derivative and the flat metric, it can be identified as:

$$\vec{\nabla}\cdot\vec{\beta} \equiv \hat{D}_k\beta^k \tag{2.5}$$

$$\nabla^2\beta^k \equiv \hat{\gamma}^{ij}\hat{D}_i\hat{D}_j\beta^k \tag{2.6}$$

Note that $K_{ij}K^{ij}$ can also be expressed in terms of the flat operators. It ends up being expressed as flat derivatives of the shift vector:

$$K_{ij}K^{ij} = \frac{1}{2\alpha^2}\left(\hat{\gamma}_{kn}\hat{\gamma}^{ml}\hat{D}_m\beta^k\hat{D}_l\beta^n + \hat{D}_m\beta^l\hat{D}_l\beta^m - \frac{2}{3}\hat{D}_l\beta^l\hat{D}_k\beta^k\right) \tag{2.7}$$

As we are interested in studying the strong-field dynamics of gravitationally compact objects with no imposed symmetries, the geometric quantities are functions of all 3 spatial coordinates (we say then that it is a 3d problem). The most convenient, or maybe easier, coordinate system to work with in this case is the cartesian one, since it doesn't feature any kind of coordinate singularity or similar pathologies.

In cartesian coordinates:

The maximal slicing condition equation becomes:

$$\frac{\partial^2\alpha}{\partial x^2} + \frac{\partial^2\alpha}{\partial y^2} + \frac{\partial^2\alpha}{\partial z^2} = -\frac{2}{\psi}\left[\frac{\partial\psi}{\partial x}\frac{\partial\alpha}{\partial x} + \frac{\partial\psi}{\partial y}\frac{\partial\alpha}{\partial y} + \frac{\partial\psi}{\partial z}\frac{\partial\alpha}{\partial z}\right] + \alpha\psi^4\left(K_{ij}K^{ij} + 4\pi\left(\rho + S\right)\right) \tag{2.8}$$

on the other hand, the hamiltonian equation yields:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = -\frac{\psi^5}{8}\left(K_{ij}K^{ij} + 16\pi\rho\right) \tag{2.9}$$

At last the momentum constraint equations gives the 3 shift equations:

$$\begin{aligned}\frac{\partial^2\beta^x}{\partial x^2} + \frac{\partial^2\beta^x}{\partial y^2} + \frac{\partial^2\beta^x}{\partial z^2} \quad = \quad & -\frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial\beta^x}{\partial x} + \frac{\partial\beta^y}{\partial y} + \frac{\partial\beta^z}{\partial z}\right) + \alpha\psi^4\, 16\pi J^x \\ & -\frac{\partial}{\partial x}\left[ln\left(\frac{\psi^6}{\alpha}\right)\right]\left[\frac{4}{3}\frac{\partial\beta^x}{\partial x} - \frac{2}{3}\left(\frac{\partial\beta^y}{\partial y} + \frac{\partial\beta^z}{\partial z}\right)\right] \\ & -\frac{\partial}{\partial y}\left[ln\left(\frac{\psi^6}{\alpha}\right)\right]\left[\frac{\partial\beta^x}{\partial y} + \frac{\partial\beta^y}{\partial x}\right] - \frac{\partial}{\partial z}\left[ln\left(\frac{\psi^6}{\alpha}\right)\right]\left[\frac{\partial\beta^x}{\partial z} + \frac{\partial\beta^z}{\partial x}\right]\end{aligned} \tag{2.10}$$

$$\begin{aligned}\frac{\partial^2\beta^y}{\partial x^2} + \frac{\partial^2\beta^y}{\partial y^2} + \frac{\partial^2\beta^y}{\partial z^2} \quad = \quad & -\frac{1}{3}\frac{\partial}{\partial y}\left(\frac{\partial\beta^x}{\partial x} + \frac{\partial\beta^y}{\partial y} + \frac{\partial\beta^z}{\partial z}\right) + \alpha\psi^4\, 16\pi J^y \\ & -\frac{\partial}{\partial y}\left[ln\left(\frac{\psi^6}{\alpha}\right)\right]\left[\frac{4}{3}\frac{\partial\beta^y}{\partial y} - \frac{2}{3}\left(\frac{\partial\beta^x}{\partial x} + \frac{\partial\beta^z}{\partial z}\right)\right] \\ & -\frac{\partial}{\partial x}\left[ln\left(\frac{\psi^6}{\alpha}\right)\right]\left[\frac{\partial\beta^x}{\partial y} + \frac{\partial\beta^y}{\partial x}\right] - \frac{\partial}{\partial z}\left[ln\left(\frac{\psi^6}{\alpha}\right)\right]\left[\frac{\partial\beta^y}{\partial z} + \frac{\partial\beta^z}{\partial y}\right]\end{aligned} \tag{2.11}$$

$$\begin{aligned}
\frac{\partial^2\beta^z}{\partial x^2}+\frac{\partial^2\beta^z}{\partial y^2}+\frac{\partial^2\beta^z}{\partial z^2} \quad = \quad & -\frac{1}{3}\frac{\partial}{\partial z}\left(\frac{\partial\beta^x}{\partial x}+\frac{\partial\beta^y}{\partial y}+\frac{\partial\beta^z}{\partial z}\right)+\alpha\psi^4\,16\pi J^z \\
& -\frac{\partial}{\partial z}\left[ln\left(\frac{\psi^6}{\alpha}\right)\right]\left[\frac{4}{3}\frac{\partial\beta^z}{\partial z}-\frac{2}{3}\left(\frac{\partial\beta^x}{\partial x}+\frac{\partial\beta^y}{\partial y}\right)\right] \\
& -\frac{\partial}{\partial y}\left[ln\left(\frac{\psi^6}{\alpha}\right)\right]\left[\frac{\partial\beta^z}{\partial y}+\frac{\partial\beta^y}{\partial z}\right]-\frac{\partial}{\partial x}\left[ln\left(\frac{\psi^6}{\alpha}\right)\right]\left[\frac{\partial\beta^x}{\partial z}+\frac{\partial\beta^z}{\partial x}\right]
\end{aligned} \tag{2.12}$$

and $K_{ij}K^{ij}$ is given by:

$$\begin{aligned}
K_{ij}K^{ij} = & \frac{1}{2\alpha^2}\Bigg[\left(\frac{\partial\beta^x}{\partial x}\right)^2+\left(\frac{\partial\beta^x}{\partial y}\right)^2+\left(\frac{\partial\beta^x}{\partial z}\right)^2+\left(\frac{\partial\beta^y}{\partial x}\right)^2+\left(\frac{\partial\beta^y}{\partial y}\right)^2+\left(\frac{\partial\beta^y}{\partial z}\right)^2 \\
& +\left(\frac{\partial\beta^z}{\partial x}\right)^2+\left(\frac{\partial\beta^z}{\partial y}\right)^2+\left(\frac{\partial\beta^z}{\partial z}\right)^2+\frac{\partial}{\partial x}\left(\beta^x\frac{\partial\beta^x}{\partial x}+\beta^y\frac{\partial\beta^x}{\partial y}+\beta^z\frac{\partial\beta^x}{\partial z}\right) \\
& +\frac{\partial}{\partial y}\left(\beta^x\frac{\partial\beta^y}{\partial x}+\beta^y\frac{\partial\beta^y}{\partial y}+\beta^z\frac{\partial\beta^y}{\partial z}\right)+\frac{\partial}{\partial z}\left(\beta^x\frac{\partial\beta^z}{\partial x}+\beta^y\frac{\partial\beta^z}{\partial y}+\beta^z\frac{\partial\beta^z}{\partial z}\right) \\
& -\frac{2}{3}\left(\frac{\partial\beta^x}{\partial x}+\frac{\partial\beta^x}{\partial x}+\frac{\partial\beta^x}{\partial x}\right)^2\Bigg]
\end{aligned} \tag{2.13}$$

2.1.2 Scalar field evolution equations and related field densities in ADM formalism

The matter content is described by the scalar field:

$$\Phi = \phi_1 + i\phi_2 \tag{2.14}$$

where ϕ_1 and ϕ_2 are real-valued. The Lagrangian density associated with this field is given by:

$$L_\Phi = -\frac{1}{8\pi}(g^{ab}\nabla_a\Phi\nabla_b\Phi^* + U(|\Phi|^2)) = -\frac{1}{8\pi}(g^{ab}\nabla_a\Phi\nabla_b\Phi^* + m^2\Phi\Phi^*) \tag{2.15}$$

Extremizing this action with respect to each component of the scalar field, we get the Klein-Gordon equation

$$\Box\phi_A - m^2\phi_A = 0 \qquad A = 1, 2 \tag{2.16}$$

From the point of view of ADM formalism, the Hamiltonian formulation of the dynamics of scalar field is more useful. The conjugate momentum field is defined as

$$\hat{\Pi}_A \equiv \frac{\delta(\sqrt{-g}L_{\phi_A})}{\delta\dot{\phi}_A} \tag{2.17}$$

or explicitly, for the Klein-Gordon field:

$$\hat{\Pi}_A = \frac{\sqrt{-g}}{4\pi\alpha^2}\left[\dot{\phi}_A - \beta^i\partial_i\phi_A\right] \tag{2.18}$$

In terms of these fields, the dynamical equations are given by

$$\partial_t\phi_A = \frac{\alpha^2}{\sqrt{-g}}\Pi_A + \beta^i\partial_i\phi_A \tag{2.19}$$

$$\partial_t\Pi_A = \partial_i(\beta^i\Pi_A) + \partial_i(\sqrt{-g}\gamma^{ij}\partial_j\phi_A) - \sqrt{-g}m^2\phi_A \tag{2.20}$$

where

$$\Pi_A = 4\pi\hat{\Pi}_A \tag{2.21}$$

Since

$$\sqrt{-g} \equiv \alpha\sqrt{\gamma} \tag{2.22}$$

the evolution equations for the fields and their conjugate momenta can be rewritten as

$$\partial_t\phi_A = \frac{\alpha}{\sqrt{\gamma}}\Pi_A + \beta^i\partial_i\phi_A \tag{2.23}$$

$$\partial_t\Pi_A = \partial_i(\beta^i\Pi_A) + \partial_i(\alpha\sqrt{\gamma}\gamma^{ij}\partial_j\phi_A) - \alpha\sqrt{\gamma}m^2\phi_A \tag{2.24}$$

Extremizing the action with respect to the contravariant metric tensor g^{ab} gives the Einstein equations besides defining the stress-energy tensor in terms of the field Lagrangian:

$$T_{ab} = -2\frac{\delta L_\Phi}{\delta g^{ab}} + g^{ab} L_\Phi \tag{2.25}$$

This gives the following stress-energy tensor for the complex scalar field:

$$T_{ab} = \frac{1}{8\pi}\left[\nabla_a\Phi\nabla_b\Phi^* + \nabla_b\Phi\nabla_a\Phi^* - g_{ab}(g^{cd}\nabla_c\Phi\nabla_d\Phi^* + m^2\Phi\Phi^*)\right] \tag{2.26}$$

or in its coordinate basis components explicitly in terms of the field components:

$$T_{\alpha\beta} = \sum_{A=1}^{2}\frac{1}{8\pi}\left[\partial_\alpha\phi_A\partial_\beta\phi_A + \partial_\beta\phi_A\partial_\alpha\phi_A - g_{\alpha\beta}(g^{\mu\nu}\partial_\mu\phi_A\partial_\nu\phi_A + m^2\phi_A^2)\right] \tag{2.27}$$

The stress-energy tensor T_{ab} is a type $(0,2)$ symmetric tensor. As can be checked, a generic tensor of this type has the following decomposition in the 3+1 formalism:

$$T_{ab} = \perp T_{ab} - 2n_{(a} \perp T_{b)\hat{n}} + n_a n_b T_{\hat{n}\hat{n}} \tag{2.28}$$

or

$$T_{ab} = S_{ab} - 2J_{(a}n_{b)} + n_a n_b \rho \tag{2.29}$$

where

$$\rho \equiv T_{\hat{n}\hat{n}} = T_{ab}n^a n^b \tag{2.30}$$

$$J_a \equiv \perp T_{a\hat{n}} = \perp (T_{ab}n^b) \tag{2.31}$$

$$J^a \equiv \perp T^{a\hat{n}} = - \perp (T^{ab}n_b) \tag{2.32}$$

$$S_{ab} \equiv \perp T_{ab} \tag{2.33}$$

It's worth noticing that from the above $J's$ definitions the momentum constraint equations are slightly different from one case to the other:

$$\perp G^{a\hat{n}} = D_b K^{ab} - D^a K = 8\pi J^a \tag{2.34}$$

$$\perp G_{a\hat{n}} = -D^b K_{ab} + D_a K = 8\pi J_a \tag{2.35}$$

In the ADM coordinate system n^a and n_a have components $n^\mu = (\frac{1}{\alpha}; -\frac{\beta^i}{\alpha})$ and $n_\mu = (-\alpha; 0)$ respectively. Then the field densities can be expressed in the ADM coordinate system as:

$$\rho = \frac{T_{00}}{\alpha^2} - 2\frac{\beta^i T_{0i}}{\alpha^2} + \frac{\beta^i \beta^j T_{ij}}{\alpha^2} \tag{2.36}$$

$$J_i = \frac{T_{i0}}{\alpha} - \frac{T_{ij}\beta^j}{\alpha} \tag{2.37}$$

$$S_{ij} = T_{ij} \tag{2.38}$$

For the complex scalar field, the field densities can be then expressed as:

$$\rho = \frac{1}{8\pi}\sum_{A=1}^{2}\left[\frac{\Pi_A^2}{\gamma} + \gamma^{ij}\partial_i\phi_A\partial_j\phi_A + m^2\phi_A^2\right] \tag{2.39}$$

$$J_i = \frac{1}{8\pi}\sum_{A=1}^{2}\left[2\frac{\Pi_A}{\sqrt{\gamma}}\partial_i\phi_A\right] \tag{2.40}$$

$$S_{ij} = \frac{1}{8\pi}\sum_{A=1}^{2}\left\{2\partial_i\phi_A\partial_j\phi_A + \gamma_{ij}\left[\frac{\Pi_A^2}{\gamma} - \gamma^{pq}\partial_p\phi_A\partial_q\phi_A - m^2\phi_A^2\right]\right\} \tag{2.41}$$

as we are also looking for the combined quantity $\rho + S$, let's then trace the stress tensor:

$$S_i^i = \frac{1}{8\pi}\sum_{A=1}^{2}\left\{3\frac{\Pi_A^2}{\gamma} - \gamma^{ij}\partial_i\phi_A\partial_j\phi_A + 3m^2\phi_A^2\right] \tag{2.42}$$

$$\rho + S = \frac{1}{2\pi}\sum_{A=1}^{2}\left[\frac{\Pi_A^2}{\gamma} + m^2\phi_A^2\right] \tag{2.43}$$

2.1.3 Scalar field equations and related field densities in ADM formalism for a conformally flat spatial metric in cartesian coordinates

The conformally flat spatial metric can be written in cartesian coordinates as:

$$^{(3)}ds^2 = \psi^4(dx^2 + dy^2 + dz^2) \tag{2.44}$$

For this metric, the field equations can then be written as:

$$\partial_t\phi_A = \frac{\alpha}{\psi^6}\Pi_A + \beta^i\partial_i\phi_A \tag{2.45}$$

$$\begin{aligned}\partial_t\Pi_A &= \partial_x\left(\beta^x\Pi_A + \alpha\psi^2\partial_x\phi_A\right) + \partial_y\left(\beta^y\Pi_A + \alpha\psi^2\partial_y\phi_A\right) \\ &+ \partial_z\left(\beta^z\Pi_A + \alpha\psi^2\partial_z\phi_A\right) - \alpha\psi^6 m^2\phi_A\end{aligned} \tag{2.46}$$

and the field densities as:

$$\rho = \frac{1}{8\pi}\sum_{A=1}^{2}\left[\frac{\Pi_A^2}{\psi^{12}} + \frac{1}{\psi^4}\left[(\partial_x\phi_A)^2 + (\partial_y\phi_A)^2 + (\partial_z\phi_A)^2\right] + m^2\phi_A^2\right] \quad (2.47)$$

$$J_i = \frac{1}{8\pi}\sum_{A=1}^{2}\left[2\frac{\Pi_A}{\psi^6}\partial_i\phi_A\right] \quad (2.48)$$

$$S_{ij} = \frac{1}{8\pi}\sum_{A=1}^{2}\left\{2\partial_i\phi_A\partial_j\phi_A + \psi^4\delta_{ij}\left[\frac{\Pi_A^2}{\psi^{12}} + \frac{1}{\psi^4}\left[(\partial_x\phi_A)^2 + (\partial_y\phi_A)^2 + (\partial_z\phi_A)^2\right] - m^2\phi_A^2\right]\right\} \quad (2.49)$$

for the combined quantity $\rho + S$, we have:

$$S_i^i = \frac{1}{8\pi}\sum_{A=1}^{2}\left\{3\frac{\Pi_A^2}{\psi^{12}} - \frac{1}{\psi^4}\left[(\partial_x\phi_A)^2 + (\partial_y\phi_A)^2 + (\partial_z\phi_A)^2\right] + 3m^2\phi_A^2\right] \quad (2.50)$$

$$\rho + S = \frac{1}{2\pi}\sum_{A=1}^{2}\left[\frac{\Pi_A^2}{\psi^{12}} + m^2\phi_A^2\right] \quad (2.51)$$

2.2 Compactification of the spatial domain

Compactification of the spatial domain means to map $\mathbb{R}$ into a finite subinterval $M \in \mathbb{R}$. As this subinterval can be remapped in any other finite one, there is no loss of generality if $[-1, 1]$ interval is chosen. Then all that is left is to find a particular function $\xi = \xi(x) \in C^2$ to do this map:

$$\xi : \mathbb{R} \longrightarrow [-1, 1] \quad (2.52)$$

One possible choice - and the one to be used from now on - is tangent hyperbolic,

$$\xi = \tanh(x) \quad (2.53)$$

It can be easily shown that its inverse and first derivative are:

$$x(\xi) = \ln\sqrt{\frac{1+\xi}{1-\xi}} \quad (2.54)$$

$$\frac{d\xi}{dx} = 1 - \tanh^2(x) \equiv 1 - \xi^2 \quad (2.55)$$

Then the first derivative in $u(x)$ in the compactified coordinates becomes:

$$\frac{du(x)}{dx} = (1 - \xi^2)\frac{d\bar{u}(\xi)}{d\xi} \quad (2.56)$$

as does the second derivative:

$$\frac{d^2u(x)}{dx^2} = (1-\xi^2)\frac{d}{d\xi}\left[(1-\xi^2)\frac{d\bar{u}(\xi)}{d\xi}\right] \tag{2.57}$$

2.2.1 Discretization

The discrete versions for the above operators are given by:

$$\frac{u_{i+1}-u_{i-1}}{2h} = (1-\xi_i^2)\frac{(\bar{u}_{i+1}-\bar{u}_{i-1})}{2h_c} \tag{2.58}$$

where

$$h_c = \frac{\xi_{max}-\xi_{min}}{N-1} = \frac{2}{N-1} \tag{2.59}$$

and

$$\begin{aligned}\frac{u_{i+1}-2u_i+u_{i-1}}{h^2} &= \frac{(1-\xi_i^2)}{h_c}\left\{(1-\xi^2)\frac{d\bar{u}(\xi)}{d\xi}\bigg|_{i+1/2} - (1-\xi^2)\frac{d\bar{u}(\xi)}{d\xi}\bigg|_{i-1/2}\right\} \\ &= \frac{(1-\xi_i^2)}{h_c}\left\{\left[1-\left(\frac{\xi_{i+1}+\xi_i}{2}\right)^2\right]\frac{(\bar{u}_{i+1}-\bar{u}_i)}{h_c}\right. \\ &\qquad \left. - \left[1-\left(\frac{\xi_i+\xi_{i-1}}{2}\right)^2\right]\frac{(\bar{u}_i-\bar{u}_{i-1})}{h_c}\right\}\end{aligned} \tag{2.60}$$

2.3 3d EOM in 3+1 formalism under CFC approximation and maximal slicing for massive, complex scalar field in a spatial compactified domain.

2.3.1 3d equations in compactified cartesian coordinates

To compactify the 3d equations we use the compactification function explained in the last section for each spatial dimension. Therefore:

$$x = \tanh(\chi) \tag{2.61}$$

$$y = \tanh(\eta) \tag{2.62}$$

$$z = \tanh(\zeta) \tag{2.63}$$

$$\tag{2.64}$$

The maximal slicing condition equation becomes:

$$
\begin{aligned}
(1-\chi^2)\frac{\partial}{\partial\chi}\left[(1-\chi^2)\frac{\partial\alpha}{\partial\chi}\right]+(1-\eta^2)\frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial\alpha}{\partial\eta}\right]+(1-\zeta^2)\frac{\partial}{\partial\zeta}\left[(1-\zeta^2)\frac{\partial\alpha}{\partial\zeta}\right]=\\
-\frac{2}{\psi}\left[(1-\chi^2)^2\frac{\partial\psi}{\partial\chi}\frac{\partial\alpha}{\partial\chi}+(1-\eta^2)^2\frac{\partial\psi}{\partial\eta}\frac{\partial\alpha}{\partial\eta}+(1-\zeta^2)^2\frac{\partial\psi}{\partial\zeta}\frac{\partial\alpha}{\partial\zeta}\right]\\
+\alpha\psi^4\left(K_{ij}K^{ij}+4\pi\left(\rho+S\right)\right)
\end{aligned}
\tag{2.65}
$$

on the other hand, the hamiltonian equation yields:

$$
(1-\chi^2)\frac{\partial}{\partial\chi}\left[(1-\chi^2)\frac{\partial\psi}{\partial\chi}\right]+(1-\eta^2)\frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial\psi}{\partial\eta}\right]+(1-\zeta^2)\frac{\partial}{\partial\zeta}\left[(1-\zeta^2)\frac{\partial\psi}{\partial\zeta}\right]=-\frac{\psi^5}{8}\left(K_{ij}K^{ij}+16\pi\rho\right)
\tag{2.66}
$$

At last the momentum constraint equations gives the 3 shift equations:

$$
\begin{aligned}
(1-\chi^2)\frac{\partial}{\partial\chi}\left[(1-\chi^2)\frac{\partial\beta^\chi}{\partial\chi}\right]+(1-\eta^2)\frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial\beta^\chi}{\partial\eta}\right]+(1-\zeta^2)\frac{\partial}{\partial\zeta}\left[(1-\zeta^2)\frac{\partial\beta^\chi}{\partial\zeta}\right]=\\
-\frac{1}{3}(1-\chi^2)\frac{\partial}{\partial\chi}\left((1-\chi^2)\frac{\partial\beta^\chi}{\partial\chi}+(1-\eta^2)\frac{\partial\beta^\eta}{\partial\eta}+(1-\zeta^2)\frac{\partial\beta^\zeta}{\partial\zeta}\right)+\alpha\psi^4\,16\pi J^\chi\\
-(1-\chi^2)\frac{\partial}{\partial\chi}\left[ln\left(\frac{\psi^6}{\alpha}\right)\right]\left[\frac{4}{3}(1-\chi^2)\frac{\partial\beta^\chi}{\partial\chi}-\frac{2}{3}\left((1-\eta^2)\frac{\partial\beta^\eta}{\partial\eta}+(1-\zeta^2)\frac{\partial\beta^\zeta}{\partial\zeta}\right)\right]\\
-(1-\eta^2)\frac{\partial}{\partial\eta}\left[ln\left(\frac{\psi^6}{\alpha}\right)\right]\left[(1-\eta^2)\frac{\partial\beta^\chi}{\partial\eta}+(1-\chi^2)\frac{\partial\beta^\eta}{\partial\chi}\right]\\
-(1-\zeta^2)\frac{\partial}{\partial\zeta}\left[ln\left(\frac{\psi^6}{\alpha}\right)\right]\left[(1-\zeta^2)\frac{\partial\beta^\chi}{\partial\zeta}+(1-\chi^2)\frac{\partial\beta^\zeta}{\partial\chi}\right]
\end{aligned}
\tag{2.67}
$$

$$
\begin{aligned}
(1-\chi^2)\frac{\partial}{\partial\chi}\left[(1-\chi^2)\frac{\partial\beta^\eta}{\partial\chi}\right]+(1-\eta^2)\frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial\beta^\eta}{\partial\eta}\right]+(1-\zeta^2)\frac{\partial}{\partial\zeta}\left[(1-\zeta^2)\frac{\partial\beta^\eta}{\partial\zeta}\right]=\\
-\frac{1}{3}(1-\eta^2)\frac{\partial}{\partial\eta}\left((1-\chi^2)\frac{\partial\beta^\chi}{\partial\chi}+(1-\eta^2)\frac{\partial\beta^\eta}{\partial\eta}+(1-\zeta^2)\frac{\partial\beta^\zeta}{\partial\zeta}\right)+\alpha\psi^4\,16\pi J^\eta\\
-(1-\eta^2)\frac{\partial}{\partial\eta}\left[ln\left(\frac{\psi^6}{\alpha}\right)\right]\left[\frac{4}{3}(1-\eta^2)\frac{\partial\beta^\eta}{\partial\eta}-\frac{2}{3}\left((1-\chi^2)\frac{\partial\beta^\chi}{\partial\chi}+(1-\zeta^2)\frac{\partial\beta^\zeta}{\partial\zeta}\right)\right]\\
-(1-\chi^2)\frac{\partial}{\partial\chi}\left[ln\left(\frac{\psi^6}{\alpha}\right)\right]\left[(1-\eta^2)\frac{\partial\beta^\chi}{\partial\eta}+(1-\chi^2)\frac{\partial\beta^\eta}{\partial\chi}\right]\\
-(1-\zeta^2)\frac{\partial}{\partial\zeta}\left[ln\left(\frac{\psi^6}{\alpha}\right)\right]\left[(1-\zeta^2)\frac{\partial\beta^\eta}{\partial\zeta}+(1-\eta^2)\frac{\partial\beta^\zeta}{\partial\eta}\right]
\end{aligned}
\tag{2.68}
$$

$$(1-\chi^2)\frac{\partial}{\partial\chi}\left[(1-\chi^2)\frac{\partial\beta^\zeta}{\partial\chi}\right]+(1-\eta^2)\frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial\beta^\zeta}{\partial\eta}\right]+(1-\zeta^2)\frac{\partial}{\partial\zeta}\left[(1-\zeta^2)\frac{\partial\beta^\zeta}{\partial\zeta}\right]=$$
$$-\frac{1}{3}(1-\zeta^2)\frac{\partial}{\partial\zeta}\left((1-\chi^2)\frac{\partial\beta^\chi}{\partial\chi}+(1-\eta^2)\frac{\partial\beta^\eta}{\partial\eta}+(1-\zeta^2)\frac{\partial\beta^\zeta}{\partial\zeta}\right)+\alpha\psi^4\,16\pi J^\zeta$$
$$-(1-\zeta^2)\frac{\partial}{\partial\zeta}\left[ln\left(\frac{\psi^6}{\alpha}\right)\right]\left[\frac{4}{3}(1-\zeta^2)\frac{\partial\beta^\zeta}{\partial\zeta}-\frac{2}{3}\left((1-\eta^2)\frac{\partial\beta^\eta}{\partial\eta}+(1-\chi^2)\frac{\partial\beta^\chi}{\partial\chi}\right)\right]$$
$$-(1-\eta^2)\frac{\partial}{\partial\eta}\left[ln\left(\frac{\psi^6}{\alpha}\right)\right]\left[(1-\eta^2)\frac{\partial\beta^\zeta}{\partial\eta}+(1-\zeta^2)\frac{\partial\beta^\eta}{\partial\zeta}\right]$$
$$-(1-\chi^2)\frac{\partial}{\partial\chi}\left[ln\left(\frac{\psi^6}{\alpha}\right)\right]\left[(1-\zeta^2)\frac{\partial\beta^\chi}{\partial\zeta}+(1-\chi^2)\frac{\partial\beta^\zeta}{\partial\chi}\right] \quad (2.69)$$

and $K_{ij}K^{ij}$ is given by:

$$\begin{aligned}K_{ij}K^{ij}=&\frac{1}{2\alpha^2}\Bigg[\left((1-\chi^2)\frac{\partial\beta^\chi}{\partial\chi}\right)^2+\left((1-\eta^2)\frac{\partial\beta^\chi}{\partial\eta}\right)^2+\left((1-\zeta^2)\frac{\partial\beta^\chi}{\partial\zeta}\right)^2\\
&+\left((1-\chi^2)\frac{\partial\beta^\eta}{\partial\chi}\right)^2+\left((1-\eta^2)\frac{\partial\beta^\eta}{\partial\eta}\right)^2+\left((1-\zeta^2)\frac{\partial\beta^\eta}{\partial\zeta}\right)^2\\
&+\left((1-\chi^2)\frac{\partial\beta^\zeta}{\partial\chi}\right)^2+\left((1-\eta^2)\frac{\partial\beta^\zeta}{\partial\eta}\right)^2+\left((1-\zeta^2)\frac{\partial\beta^\zeta}{\partial\zeta}\right)^2\\
&+(1-\chi^2)\frac{\partial}{\partial\chi}\left(\beta^\chi(1-\chi^2)\frac{\partial\beta^\chi}{\partial\chi}+\beta^\eta(1-\eta^2)\frac{\partial\beta^\chi}{\partial\eta}+\beta^\zeta(1-\zeta^2)\frac{\partial\beta^\chi}{\partial\zeta}\right)\\
&+(1-\eta^2)\frac{\partial}{\partial\eta}\left(\beta^\chi(1-\chi^2)\frac{\partial\beta^\eta}{\partial\chi}+\beta^\eta(1-\eta^2)\frac{\partial\beta^\eta}{\partial\eta}+\beta^\zeta(1-\zeta^2)\frac{\partial\beta^\eta}{\partial\zeta}\right)\\
&+(1-\zeta^2)\frac{\partial}{\partial\zeta}\left(\beta^\chi(1-\chi^2)\frac{\partial\beta^\zeta}{\partial\chi}+\beta^\eta(1-\eta^2)\frac{\partial\beta^\zeta}{\partial\eta}+\beta^\zeta(1-\zeta^2)\frac{\partial\beta^\zeta}{\partial\zeta}\right)\\
&-\frac{2}{3}(1-\chi^2)^2\left(\frac{\partial\beta^\chi}{\partial\chi}+\frac{\partial\beta^\chi}{\partial\chi}+\frac{\partial\beta^\chi}{\partial\chi}\right)^2\Bigg]\end{aligned}$$

On the other hand the scalar field equations as well as the related field densities become:

$$\frac{\partial\phi_A}{\partial t} = \frac{\alpha}{\psi^6}\Pi_A+\beta^\chi(1-\chi^2)\frac{\partial\phi_A}{\partial\chi}+\beta^\eta(1-\eta^2)\frac{\partial\phi_A}{\partial\eta}+\beta^\zeta(1-\zeta^2)\frac{\partial\phi_A}{\partial\zeta} \quad (2.70)$$

$$\begin{aligned}\frac{\partial\Pi_A}{\partial t} =& (1-\chi^2)\frac{\partial}{\partial\chi}\left(\beta^\chi\Pi_A+\alpha\psi^2(1-\chi^2)\frac{\partial\phi_A}{\partial\chi}\right) \quad (2.71)\\
&+ (1-\eta^2)\frac{\partial}{\partial\eta}\left(\beta^\eta\Pi_A+\alpha\psi^2(1-\eta^2)\frac{\partial\phi_A}{\partial\eta}\right)\\
&+ (1-\zeta^2)\frac{\partial}{\partial\zeta}\left(\beta^\zeta\Pi_A+\alpha\psi^2(1-\zeta^2)\frac{\partial\phi_A}{\partial\zeta}\right)-\alpha\psi^6m^2\phi_A\end{aligned}$$

and the field densities as:

$$\rho = \frac{1}{8\pi}\sum_{A=1}^{2}\left\{\frac{\Pi_A^2}{\psi^{12}} + \frac{1}{\psi^4}\left[\left((1-\chi^2)\frac{\partial\phi_A}{\partial\chi}\right)^2 + \left((1-\eta^2)\frac{\partial\phi_A}{\partial\eta}\right)^2 + \left((1-\zeta^2)\frac{\partial\phi_A}{\partial\zeta}\right)^2\right] + m^2\phi_A^2\right\} \tag{2.72}$$

$$J_i = \frac{1}{8\pi}\sum_{A=1}^{2}\left[2\frac{\Pi_A}{\psi^6}(1-\chi_i^2)\frac{\partial\phi_A}{\partial\chi_i}\right] \tag{2.73}$$

$$S_{ij} = \frac{1}{8\pi}\sum_{A=1}^{2}\left\{2(1-\chi_i^2)\frac{\partial\phi_A}{\partial\chi_i}(1-\chi_j^2)\frac{\partial\phi_A}{\partial\chi_j} + \psi^4\delta_{ij}\left[\frac{\Pi_A^2}{\psi^{12}} + \frac{1}{\psi^4}\left[\left((1-\chi^2)\frac{\partial\phi_A}{\partial\chi}\right)^2 + \left((1-\eta^2)\frac{\partial\phi_A}{\partial\eta}\right)^2 + \left((1-\zeta^2)\frac{\partial\phi_A}{\partial\zeta}\right)^2\right] - m^2\phi_A^2\right]\right\} \tag{2.74}$$

for the combined quantity $\rho + S$, we have:

$$S_i^i = \frac{1}{8\pi}\sum_{A=1}^{2}\left\{3\frac{\Pi_A^2}{\psi^{12}} - \frac{1}{\psi^4}\left[\left((1-\chi^2)\frac{\partial\phi_A}{\partial\chi}\right)^2 + \left((1-\eta^2)\frac{\partial\phi_A}{\partial\eta}\right)^2 + \left((1-\zeta^2)\frac{\partial\phi_A}{\partial\zeta}\right)^2\right] + 3m^2\phi_A^2\right\} \tag{2.75}$$

$$\rho + S = \frac{1}{2\pi}\sum_{A=1}^{2}\left[\frac{\Pi_A^2}{\psi^{12}} + m^2\phi_A^2\right] \tag{2.76}$$

Explicitly and according to the projection convention introduced by York, the current densities can be written as:

$$J^\chi \equiv -J_\chi = -\frac{1}{8\pi}\sum_{A=1}^{2}\left[2\frac{\Pi_A}{\psi^6}(1-\chi^2)\frac{\partial\phi_A}{\partial\chi}\right] \tag{2.77}$$

$$J^\eta \equiv -J_\eta = -\frac{1}{8\pi}\sum_{A=1}^{2}\left[2\frac{\Pi_A}{\psi^6}(1-\eta^2)\frac{\partial\phi_A}{\partial\eta}\right] \tag{2.78}$$

$$J^\zeta \equiv -J_\zeta = -\frac{1}{8\pi}\sum_{A=1}^{2}\left[2\frac{\Pi_A}{\psi^6}(1-\zeta^2)\frac{\partial\phi_A}{\partial\zeta}\right] \tag{2.79}$$

2.4 Initial value problem in spherical symmetry

2.4.1 Maximal-isotropic coordinates

The equations of motion are reduced to a set of ODEs:

$$\psi' = \Psi \tag{2.80}$$

$$\Psi' = -\frac{2\Psi}{r} - \frac{1}{4}\left[\left(\frac{\omega^2\phi_0^2}{\alpha^2} + U(\phi_0^2)\right)\psi^5 + \Phi_0^2\psi\right] \tag{2.81}$$

$$\alpha' = A \tag{2.82}$$

$$A' = -2\left(\frac{1}{r} + \frac{\Psi}{\psi}\right)A + \psi^4\alpha\left(\frac{2\omega^2\phi_0^2}{\alpha^2} - U(\phi_0^2)\right) \tag{2.83}$$

$$\phi_0' = \Phi_0 \tag{2.84}$$

$$\Phi_0' = -\left(\frac{A}{\alpha} + \frac{2}{r} + \frac{2\Psi}{\psi}\right)\Phi_0 + \psi^4\left(\frac{dU(\phi_0^2)}{d\phi_0^2} - \frac{\omega^2}{\alpha^2}\right)\phi_0 \tag{2.85}$$

or in second order form:

$$\frac{1}{r^2}\frac{d}{dr}(r^2\Psi) + \frac{1}{4}\left[\left(\frac{\omega^2\phi_0^2}{\alpha^2} + U(\phi_0^2)\right)\psi^5 + \Phi_0^2\psi\right] = 0 \tag{2.86}$$

$$\frac{1}{r^2}\frac{d}{dr}(r^2 A) + 2\frac{\Psi}{\psi}A - \psi^4\alpha\left(\frac{2\omega^2\phi_0^2}{\alpha^2} - U(\phi_0^2)\right) = 0 \tag{2.87}$$

$$\frac{1}{r^2}\frac{d}{dr}(r^2\Phi_0) + \left(\frac{A}{\alpha} + \frac{2\Psi}{\psi}\right)\Phi_0 - \psi^4\left(\frac{dU(\phi_0^2)}{d\phi_0^2} - \frac{\omega^2}{\alpha^2}\right)\phi_0 = 0 \tag{2.88}$$

Compactification of the spatial domain

As we want to generate initial data to be interpolated in the 3d grid, it would be convenient to use the same compactification function as in the 3d case, i.e.:

$$\xi(r) = \tanh(r) \tag{2.89}$$

$$r(\xi) = \frac{1}{2}\ln\left(\frac{1+\xi}{1-\xi}\right) \tag{2.90}$$

$$\frac{d\xi}{dr} = 1 - \tanh^2(r) = 1 - \xi^2 \tag{2.91}$$

$$\frac{du(r)}{dr} = (1-\xi^2)\frac{du(\xi)}{d\xi} \tag{2.92}$$

$$\frac{d^2u(r)}{dr} = (1-\xi^2)\frac{d}{d\xi}\left[(1-\xi^2)\frac{du}{d\xi}\right] \tag{2.93}$$

$$\tag{2.94}$$

Note that with this compactification function the auxiliary functions becomes:

$$\Phi_0(r) = (1-\xi^2)\Phi_0(\xi) \tag{2.95}$$

$$\Psi(r) = (1-\xi^2)\Psi(\xi) \tag{2.96}$$

$$A(r) = (1-\xi^2)A(\xi) \tag{2.97}$$

The equations of motion then becomes:

$$\frac{(1-\xi^2)}{\left[\ln\left(\frac{1+\xi}{1-\xi}\right)\right]^2}\frac{d}{d\xi}\left[(1-\xi^2)\left[\ln\left(\frac{1+\xi}{1-\xi}\right)\right]^2\frac{d\psi}{d\xi}\right]+\frac{1}{4}\left[\left(\frac{\omega^2\phi_0^2}{\alpha^2}+U(\phi_0^2)\right)\psi^5+(1-\xi^2)^2\Phi_0^2\psi\right]=0$$

$$\frac{(1-\xi^2)}{\left[\ln\left(\frac{1+\xi}{1-\xi}\right)\right]^2}\frac{d}{d\xi}\left[(1-\xi^2)\left[\ln\left(\frac{1+\xi}{1-\xi}\right)\right]^2\frac{d\alpha}{d\xi}\right]+2(1-\xi^2)^2\frac{\Psi}{\psi}A-\psi^4\alpha\left(\frac{2\omega^2\phi_0^2}{\alpha^2}-U(\phi_0^2)\right)=0$$

$$\frac{(1-\xi^2)}{\left[\ln\left(\frac{1+\xi}{1-\xi}\right)\right]^2}\frac{d}{d\xi}\left[(1-\xi^2)\left[\ln\left(\frac{1+\xi}{1-\xi}\right)\right]^2\frac{d\phi}{d\xi}\right]+(1-\xi^2)^2\left(\frac{A}{\alpha}+\frac{2\Psi}{\psi}\right)\Phi_0-\psi^4\left(\frac{dU(\phi_0^2)}{d\phi_0^2}-\frac{\omega^2}{\alpha^2}\right)\phi_0=0$$

As we can see, this set of equations are too complicated to be solved analytically in terms of orthogonal polynomials. The use of a different compactification function is a need. The following function provides a much simpler set of equations. The idea is to solve this set of equations and then do a coordinate transformation back to the previous compactification function.

$$\zeta(r) = \frac{r}{1+r} \tag{2.98}$$

$$r(\zeta) = \frac{\zeta}{1-\zeta} \tag{2.99}$$

$$\frac{d\zeta}{dr} = (1-\zeta)^2 \tag{2.100}$$

$$\frac{du(r)}{dr} = (1-\zeta)^2\frac{du(\zeta)}{d\zeta} \tag{2.101}$$

$$\frac{d^2u(r)}{dr} = (1-\zeta)^2\frac{d}{d\zeta}\left[(1-\zeta)^2\frac{du}{d\zeta}\right] \tag{2.102}$$

Finally the compactified equations are:

$$3\frac{d}{d\zeta^3}(\zeta^2\frac{d\psi}{d\zeta}) + \frac{1}{4}\left[\left(\frac{\omega^2\phi_0^2}{\alpha^2} + U(\phi_0^2)\right)\frac{\psi^5}{(1-\zeta)^4} + \Phi_0^2\psi\right] = 0 \quad (2.103)$$

$$3\frac{d}{d\zeta^3}(\zeta^2\frac{d\alpha}{d\zeta}) + 2\frac{\Psi}{\psi}A - \frac{\psi^4\alpha}{(1-\zeta)^4}\left(\frac{2\omega^2\phi_0^2}{\alpha^2} - U(\phi_0^2)\right) = 0 \quad (2.104)$$

$$3\frac{d}{d\zeta^3}(\zeta^2\frac{d\phi_0}{d\zeta}) + \left(\frac{A}{\alpha} + \frac{2\Psi}{\psi}\right)\Phi_0 - \frac{\psi^4}{(1-\zeta)^4}\left(\frac{dU(\phi_0^2)}{d\phi_0^2} - \frac{\omega^2}{\alpha^2}\right)\phi_0 = 0 \quad (2.105)$$

Once the solution for this set of equations is found, a coordinate transformation can be performed:

$$\xi(r) = \tanh(r) \quad (2.106)$$

$$\zeta(r) = \frac{r}{1+r} \quad (2.107)$$

$$r(\zeta) = \frac{\zeta}{1-\zeta} \quad (2.108)$$

$$\xi(\zeta) = \tanh\left(\frac{\zeta}{1-\zeta}\right) \quad (2.109)$$

and

$$\alpha(\xi) = \alpha(\xi(\zeta)) = \bar{\alpha}(\zeta) \quad (2.110)$$

$$\phi_0(\xi) = \phi_0(\xi(\zeta)) = \bar{\phi}_0(\zeta) \quad (2.111)$$

$$\psi(\xi) = \psi(\xi(\zeta)) = \bar{\psi}(\zeta) \quad (2.112)$$

2.4.2 Chebyshev polynomials

$$\frac{d}{dx}\left[\sqrt{1-x^2}\frac{dT_k(x)}{dx}\right] + \frac{k^2}{\sqrt{1-x^2}}T_k(x) = 0 \quad (2.113)$$

$$T_k(1) = 1 \quad (2.114)$$

$$\theta = \arccos(x) \quad (2.115)$$

$$T_k = \cos(k\theta) \quad (2.116)$$

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x) \quad (2.117)$$

$$T_0(x) = 1 \qquad T_1(x) = x \quad (2.118)$$

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