Lectures on Numerical Relativity

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Outline

- Refer to the lecture syllabus for details.
- We will start with Formalisms
  - 3+1 formalism (Today and July 6th)
  - Generalized harmonic formalism (July 20th)

3+1 Formalism

- ADM (Arnott-Deser-Misner, 1962) + trivial modification $\rightarrow$ 3+1 formulation.
- 3+1 line element

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

  where γ_{ij} is the (3-dim) metric of the hypersurface and α and β^i are the lapse and shift functions respectively.
- Define the Extrinsic curvature, along the normal vector, $n^\mu = -\alpha \nabla^\mu t$,

$$K_{ij} = -\frac{1}{2}\mathcal{L}_n \gamma_{ij}$$

- Take (γ_{ij}, K_{ij}) as dynamical variables.
- By considering various projections, 10 Einstein field equations (in 4-dim) becomes 4 elliptic (constraint) equations plus 12 hyperbolic (evolution) equations.

3+1 Formalism in vacuum

- Constraint equations

$$
{}^{(3)}R + K^2 - K_{ij}K^{ij} = 0
$$
$$
D_j(K^{ij} - \gamma^{ij}K) = 0
$$

where ${}^{(3)}R$ is 3-dim Ricci scalar, D_i the covariant derivatives associated with the 3-dim metric, γ_{ij}, and K is trace of K_{ij}.

- Evolution equations

$$
\begin{aligned}
(\partial_t - \mathcal{L}_\beta)\gamma_{ij} &= -2\alpha K_{ij} \\
(\partial_t - \mathcal{L}_\beta)K_{ij} &= -D_i D_j \alpha + \alpha(R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j)
\end{aligned}
$$

- Note that in reality, practitioners use variations of the ADM formalism, e.g. BSSN (Baumgarte-Shapiro-Shibata-Nakamura, 1995, 1998) formulation.

3+1 Formalism

- Look at spacetime from the point of view of the Cauchy Problem.
- (Classical) gravitational field is the time history of the geometry of a spacelike hypersurface.
- To construct solutions (grav. field in 4D), solve the initial-value problem, and integrate the dynamical eqns. along the prescribed coordinate system.
- If matter is present, their initial value and evolution eqns. must also be taken into account.
- I will follow the approach taken by J. W. York, and L. Smarr.

Foliations

- A foliation Σ ia a family of three surfaces that fills spacetime manifold M with metric g_{ab}.
- We assume that each of the surfaces(slices) are spacelike and labeled by a scalar function, τ which we will identify with the coordinate time.
- Define 1-form (dual vector field), $\Omega \equiv d\tau$ (gradient of level surface) or

$$\Omega_a = \nabla_a \tau \quad (1)$$

Ω is closed ($d\Omega = 0$) and has the norm given by

$$g^{ab}\Omega_a\Omega_b = -\alpha^{-2} \quad (2)$$

for strictly positive function, α, called lapse function.

- We can define the unit-norm dual vector field associated with Σ,

$$n_a = -\alpha \Omega_a = -\alpha \nabla_a \tau \tag{3}$$

where the minus sign is chosen so that the associated unit-norm, hypersurface-orthogonal vector field, n^a,

$$n^a = g^{ab} n_b \tag{4}$$

is future-directed.

- Note the vector field, n^a, can be viewed as the 4-velocity field of a congruence of observers moving orthogonally to the slices.

Projection Tensor

- To perform 3+1 split of the Einstein equations, we are interested in decomposing 4D spacetime tensors into hypersurface-tangential and hypersurface-orthogonal pieces.
- To determine the hypersurface-orthogonal ("temporal") part of a tensor is straightforward, we simply contract with n^a.
- Following J. W. York, we define, for a vector field W^a,

$$W^{\hat{n}} = -W^a n_a \tag{5}$$

and for a dual vector field, we define,

$$W_{\hat{n}} = +W_a n^a \tag{6}$$

- Introduce the notion of a projection tensor, which projects tensors onto the hypersurface. Define,

$$\perp^a_b \equiv \delta^a_b + n^a n_b \tag{7}$$

- By construction of the projection tensor,

$$\perp n^a \equiv \perp^a_b n^b = (\delta^a_b + n^a n_b) n^b = n^a - n^a = 0 \tag{8}$$

where we have introduced the notation that a $\perp$ with no incides, operating on an arbitrary tensor expression, means applying the projection tensor to every free tensor index in the expression. Thus, for example,

$$\perp S^a_{bc} \equiv \perp^a_d \perp^e_b \perp^f_c S^d_{ef} \tag{9}$$

- Applying the projection tensor to the spacetime metric itself, g_{ab},

$$\begin{aligned} \perp g_{ab} &\equiv \perp^c_a \perp^d_b g_{cd} = (\delta^c_a + n^c n_a)(\delta^d_b + n^d n_b) g_{cd} \\ &= g_{ab} + n^c n_a g_{cb} + n^d n_b g_{ad} + n_a n_b n^c n_d \\ &= g_{ab} + n_a n_b + n_a n_b - n_a n_b = g_{ab} + n_a n_b \end{aligned} \tag{10}$$

we get the spatial metric, γ_{ab},

$$\gamma_{ab} = g_{ab} + n_a n_b. \tag{11}$$

- The contravariant form of the spatial metric is given by

$$\gamma^{ab} = g^{ac} g^{bd} \gamma_{cd} = g^{ab} + n^a n^b \tag{12}$$

- Note $\gamma^a_b = \perp^a_b$.
- Tr$\perp \equiv \perp^a_a = \delta^a_a + n^a n_a = 4 - 1 = 3$.
- Note that all tensor indices continue to be raised and lowered with the spacetime metric, g_{ab} and only spatial tensors can equally well have their indices raised and lowered with γ_{ab}.

Spatial Derivative Operator

- To define a natural derivative operator, D_a for spatial tensors, we can use the projection tensor.

$$D_a \equiv \perp \nabla_a \tag{13}$$

- For a scalar field, ψ, for example, we have

$$D_a \psi \equiv \perp \nabla_a \psi = \perp_a^b \nabla_b \psi \tag{14}$$

and for a (sparial) vector field W^a

$$D_a W^b \equiv \perp \nabla_a W^b = \perp_a^c \perp_d^b \nabla_c W^d \tag{15}$$

- The action of D_a on an arbitrary spatial tensor then defined in the similar fashion.
- The reason we say D_a is natural is because it is compatible with the spatial metric, γ_{ab}.

$$D_a \gamma_{bc} = \perp \nabla_a \gamma_{bc} = \perp \nabla_a (g_{bc} + n_b n_c) = \perp \nabla_a (n_b n_c) = \perp (n_c \nabla_a n_b + n_b \nabla_a n_c) = 0 \tag{16}$$

- Similarly for $D_a \gamma^{bc} = 0$.
- Define the intrinsic curvature of the three-dimensional hypersurface by the Riemann tensor associated with the spatial metric, $R_{abc}{}^{d}$. Define it by requiring that for an arbitrary spatial dual vector, W_a, we have,

$$(D_a D_b - D_b D_a) W_c = R_{abc}{}^{d} W_d \tag{17}$$

$$R_{abc}{}^{d} n_d = R_{abc}{}^{d} n^a = R_{abc}{}^{d} n^b = R_{abc}{}^{d} n^c = 0 \tag{18}$$

The second condition simply says $R_{abc}{}^{d}$ itself is spatial tensor.

- $R_{abc}{}^{d}$ has the usual symmetries:

$$R_{abcd} = R_{[ab]cd} = R_{ab[cd]} \tag{19}$$

$$R_{[abc]d} = 0, R_{abcd} = R_{cdab} \tag{20}$$

We can construct the spatial Ricci tensor, R_{ab} and spatial Ricci scalar, R, in the usual manner,

$$R_{ab} = R_{acb}{}^{c} \tag{21}$$

$$R = R^{a}{}_{a} \tag{22}$$

Extrinsic Curvature Tensor

- Spatial metric describes internal geometry on the hypersurface. The embedding of the slices in the spacetime is described by the extrinsic curvature tensor. We can define extrinsic curvature, K_{ab},

$$K_{ab} = - \perp \nabla_a n_b \tag{23}$$

- Before we unpack this definition, let us first define a 4-acceleration, $a^b = n^a \nabla_a n^b$, and consider the following result

$$\perp \nabla_a n_b = \nabla_a n_b + n_a a_b \tag{24}$$

- To show, eqn. (24),

$$\begin{aligned}
\perp \nabla_a n_b &= \perp_a^c \perp_b^d \nabla_c n_d = (\delta_a^c + n^c n_a)(\delta_b^d + n^d n_b) \nabla_c n_d && (25)\\
&= \nabla_a n_b + n_a n^c \nabla_c n_b + n_b n^d \nabla_c n_d + n_a n^c n_b n^d \nabla_c n_d && (26)\\
&= \nabla_a n_b + n_a n^c \nabla_c n_b && (27)\\
&= \nabla_a n_b + n_a a_b && (28)
\end{aligned}$$

- where we used, $n^d \nabla_a n_d = n_d \nabla_a n^d = \frac{1}{2} \nabla_a (n^d n_d) = \frac{1}{2} \nabla_a (-1) = 0$.

- Now we have from the definition of the extrinsic curvature and eqn. (24),

$$\nabla_a n_b = -K_{ab} - n_a a_b \tag{29}$$

- Note that this explicitly displays the decomposition of the derivative of the normal field into a hypersurface-tangetial piece (the extrinsic curvature) and a hyperbolic-orthogonal piece (the 4-acceleration).
- Note $n^a K_{ab} = 0$ because K_{ab} is a spatial tensor.
- Then, immediately follows the definition of trace of the extrinsic curvature

$$K \equiv g^{ab} K_{ab} = g^{ab}(-\nabla_a n_b - n_a a_b) \tag{30}$$

$$= -\nabla_a n^a \tag{31}$$

where we used the fact, $a^b n_b = 0$.

- Another way to define the extrinsic curvature is in terms of the Lie derivative along the normal vector field.

$$K_{ab} = -\frac{1}{2}\mathcal{L}_n \gamma_{ab} = -\frac{1}{2} \perp \mathcal{L}_n g_{ab} \tag{32}$$

- To see this second defintion, first note that

$$K_{ab} = K_{(ab)} = -(\nabla_{(a} n_{b)} + n_{(a} a_{b)}) \tag{33}$$

$\mathcal{L}_n \gamma_{ab}$ can be written as

$$\begin{aligned} \mathcal{L}_n \gamma_{ab} &= n^c \nabla_c \gamma_{ab} + \gamma_{cb} \nabla_a n^c + \gamma_{ab} \nabla_b n^c \\ &= n^c \nabla_c (g_{ab} + n_a n_b) + (g_{cb} + n_c n_b) \nabla_a n^c + (g_{ab} + n_a n_c) \nabla_b n^c \\ &= n^c \nabla_c (n_a n_b) + \nabla_a n_b + \nabla_b n_a \\ &= 2 \left(n_{(a} a_{b)} + \nabla_{(a} n_{b)} \right) \end{aligned} \tag{34}$$

where we used the fact that $n^c \nabla_a n_c = 0$.

- Note that

$$\mathcal{L}_n g_{ab} = n^c \nabla_c g_{ab} + g_{cb} \nabla_a n^c + g_{ab} \nabla_b n^c = \nabla_a n_b + \nabla_b n_a = 2 \nabla_{(a} n_{b)} \tag{35}$$

so using the eqn (24),

$$K_{ab} = -\frac{1}{2} \perp \mathcal{L}_n g_{ab} \tag{36}$$

The Gauss-Codazzi Equations

- Having defined spatial derivative operator, D_a, spatial curvature, and the extrinsic curvature, K_{ab}, let us now compute projections of the 4-dimensional Riemann curvature tensor, ${}^{(4)}R_{abcd}$. Note that I am using ${}^{(4)}$ notation to distinguish from spatial curvature tensors.
- First consider the 4-dimensional Ricci identity as applied to a spatial dual-vector, v_a:

$$\begin{aligned} v^a \perp^{(4)} R_{abcd} &= \perp (v^{a\,(4)} R_{abcd}) = \perp (v_a {}^{(4)} R^a_{bcd}) = \perp (R_d c b^a v_a) \\ &= \perp (\nabla_d \nabla_c v_b - \nabla_c \nabla_d v_b) \end{aligned} \tag{37}$$

where we used the fact $n^a v_a = 0$, definition of 4-Riemann.

- Now we have

$$\begin{aligned} \perp \nabla_c v_b &= \perp_c^e \perp_b^f \nabla_e n_f \\ &= \nabla_c v_b + n_b n^e \nabla_c v_e + n_c n^f \nabla_f v_b + n_c n^e n_b n^f \nabla_f v_e \\ &= \nabla_c n_b - n_b v_e \nabla_c n^e + n_c n^f \nabla_f v_b - n_c n_b v_e a^e \end{aligned} \tag{38}$$

where definition of a_b and the fact $n^f \nabla_e v_f = -v_f \nabla_e n^f$ were used.

- Next let's consider $\perp(\nabla_d \perp \nabla_c v_b)$.

$$\begin{aligned}
\perp(\nabla_d \perp \nabla_c v_b) &= D_d D_c v_b \\
&= \perp \nabla_d \nabla_c v_b + \perp \nabla_d (n^c n^f \nabla_f v_b - n_b v_e \nabla_c n^e - n_c n_b v_e a^e) \\
&= \perp \nabla_d \nabla_c v_b - \perp (\nabla_d n_b)(\nabla_c n_e) v^e \\
&= \perp \nabla_d \nabla_c v_b - K_{db} K_{ca} v^a
\end{aligned} \tag{39}$$

which can be written as

$$\perp \nabla_d \nabla_c v_b = D_d D_c v_b + K_{db} K_{ca} v^a \tag{40}$$

- So now we have

$$\begin{aligned}
\perp^{(4)} R^a{}_{bcd} v_a &= \perp (\nabla_d \nabla_c v_b - \nabla_c \nabla_d v_b) \\
&= D_d D_c v_b - D_c D_d v_b + K_{db} K_{ca} v^a - K_{cb} K_{da} v^a \\
&= (R_{dcba} + K_{db} K_{ca} - K_{cb} K_{da}) v^a = (R_{abcd} + K_{db} K_{ca} - K_{cb} K_{da}) v^a \\
&= v^a \perp^{(4)} R_{abcd}
\end{aligned} \tag{41}$$

note here R_{abcd}, etc are all 3-Riemann.

- Now we have,

$$\perp {}^{(4)}R_{abcd} = R_{abcd} + K_{db}K_{ca} - K_{cb}K_{da} \tag{42}$$

- Also let's compute, $\perp {}^{(4)}R_{abc\hat{n}}$; applying the Ricci identity to n^a and projecting,

$$\begin{aligned}
\perp {}^{(4)}R_{\hat{n}bcd} &= \perp ({}^{(4)}R_{abcd}n^a) = \perp ({}^{(4)}R_{dcba}n^a) \\
&= \perp (\nabla_d \nabla_c n_b - \nabla_c \nabla_d n_b) \\
&= -\perp (\nabla_d (K_{cb} + n_c a_b) - \nabla_c (K_{db} + n_d a_b)) \\
&= -\perp (\nabla_d K_{cb} - \nabla_c K_{db} + (\nabla_d n_c - \nabla_c n_d) a_b)) \\
&= -\perp (\nabla_d K_{cb} - \nabla_c K_{db}) \\
&= -D_d K_{cb} + D_c K_{db}
\end{aligned} \tag{43}$$

where we used the fact that $\perp \nabla_{[a} n_{b]} = 0$. Reshuffling indeces,

$$\perp {}^{(4)}R_{abc\hat{n}} = D_b K_{ac} - D_a K_{bc} \tag{44}$$

- Equations (42) and (44) are known as the *Gauss-Codazzi* equations.

The Constraint Equations

- Now let us turn to deriving the constraint equations.
- Consider

$$\begin{aligned} \perp {}^{(4)}R_{ab} &= \perp (g^{cd\,(4)}R_{acbd}) \\ &= \perp (\gamma^{cd\,(4)}R_{acbd}) - \perp (n^c n^{d\,(4)}R_{acbd}) \\ &= \perp (\gamma^{cd\,(4)}R_{acbd}) - \perp ({}^{(4)}R_{a\hat{n}b\hat{n}}) \end{aligned} \quad (45)$$

- Note that

$$\perp (\gamma^{cd\,(4)}R_{acbd}) = \gamma^{cd} \perp {}^{(4)}R_{acbd} = g^{cd} \perp {}^{(4)}R_{acbd}. \quad (46)$$

- Thus we have

$$\perp {}^{(4)}R_{ab} = g^{cd} \perp {}^{(4)}R_{acbd} - \perp {}^{(4)}R_{a\hat{n}b\hat{n}} \quad (47)$$

- We will use the following 3+1 decomposition for a generic (0,2) symmetric tensor, $\sigma_{ab} = \sigma_{(ab)}$,

$$\sigma_{ab} = \perp \sigma_{ab} - 2n_{(a} \perp \sigma_{b)\hat{n}} + n_a n_b \sigma_{\hat{n}\hat{n}} \tag{48}$$

- Applying the eqn (48) to ${}^{(4)}R_{a\hat{n}b\hat{n}}$, we get

$${}^{(4)}R_{a\hat{n}b\hat{n}} = \perp {}^{(4)}R_{a\hat{n}b\hat{n}} - 2n_{(a} \perp {}^{(4)}R_{b)\hat{n}\hat{n}\hat{n}} + n_a n_b {}^{(4)}R_{\hat{n}\hat{n}\hat{n}\hat{n}} \tag{49}$$

- Since ${}^{(4)}R_{abcd}$ is antisymmetric for its first two and the last two indices, the last two terms in eqn (49) vanish. Then we have

$$\perp {}^{(4)}R_{a\hat{n}b\hat{n}} = {}^{(4)}R_{a\hat{n}b\hat{n}} \tag{50}$$

- Contracting eqn (47), we get,

$$g^{ab} \perp {}^{(4)}R_{ab} = -{}^{(4)}R_{\hat{n}\hat{n}} + g^{ab} g^{cd} \perp {}^{(4)}R_{acbd} \tag{51}$$

- Note that by applying the Eqn (48) to ${}^{(4)}R_{ab}$, we get

$$\perp {}^{(4)}R_{ab} = {}^{(4)}R_{ab} + 2n_{(a} \perp {}^{(4)}R_{b)\hat{n}} - n_a n_b {}^{(4)}R_{\hat{n}\hat{n}} \tag{52}$$

- Contracting, we find,

$$g^{ab} \perp {}^{(4)}R_{ab} = {}^{(4)}R + {}^{(4)}R_{\hat{n}\hat{n}} \tag{53}$$

- Equating Eqns (51) and (53), we get

$${}^{(4)}R = -2{}^{(4)}R_{\hat{n}\hat{n}} + g^{ab}g^{cd} \perp {}^{(4)}R_{acbd} \tag{54}$$

- Let's finally look at Einstein field equation in General Relativity.

$$G_{ab} = {}^{(4)}R_{ab} - \frac{1}{2}g_{ab}{}^{(4)}R = 8\pi T_{ab} \tag{55}$$

- We are going to consider tt component and ti components of the equation.

Hamiltonian Constraint Equation

- Contracting both indices with the normal vector field n^a,

$$G_{ab}n^a n^b =^{(4)} R_{ab}n^a n^b - \frac{1}{2} g_{ab} n^a n^{b\,(4)} R = 8\pi T_{ab} n^a n^b \qquad (56)$$

- Then, with definition, $\rho \equiv T_{\hat{n}\hat{n}} = T_{ab}n^a n^b$, tt part of Einstein equation becomes

$$\begin{aligned} {}^{(4)}R_{\hat{n}\hat{n}} + \frac{1}{2}{}^{(4)}R &= {}^{(4)}R_{\hat{n}\hat{n}} + \frac{1}{2}(-2^{(4)}R_{\hat{n}\hat{n}} + g^{ab}g^{cd} \perp {}^{(4)}R_{acbd}) \\ &= \frac{1}{2} g^{ab} g^{cd} \perp {}^{(4)}R_{acbd} = 8\pi\rho \end{aligned} \qquad (57)$$

- Recall the *Gauss-Codazzi* equations, (42),

$$\perp {}^{(4)}R_{abcd} = R_{abcd} + K_{db}K_{ca} - K_{cb}K_{da} \qquad (58)$$

or

$$\perp {}^{(4)}R_{acbd} = R_{acbd} + K_{dc}K_{ba} - K_{bc}K_{da} \qquad (59)$$

- Now we finally obtain the Hamiltonian Constraint Equation.

$$R + K^2 - K^a{}_b K^b{}_a = 16\pi\rho \tag{60}$$

where $K \equiv K^a{}_a$.

Momentum Constraint Equations

- Let's consider now the Einstein equation and contract only one index with $-n_a$. Starting from

$$G^{ab} = {}^{(4)}R^{ab} - \frac{1}{2} g^{ab\,(4)}R = 8\pi T^{ab}, \tag{61}$$

and contracting with $-n_a$, we get (note sign convention)

$$G^{a\hat{n}} = {}^{(4)}R^{a\hat{n}} + \frac{1}{2} n^{a\,(4)}R = 8\pi T^{a\hat{n}} \tag{62}$$

- Then projecting the remaining index onto the 3-hypersurface and using the definition, $j^a \equiv \perp T^{a\hat{n}} \equiv - \perp (T^{ab} n_b)$., we get

$$\perp G^{a\hat{n}} = \perp {}^{(4)}R^{a\hat{n}} = 8\pi \perp T^{a\hat{n}} = 8\pi j^a \tag{63}$$

- Now recall the eqn (47) (with b being replaced by $\hat{n}$)

$$\perp {}^{(4)}R_{a\hat{n}} = g^{cd} \perp {}^{(4)}R_{ac\hat{n}d} - \perp {}^{(4)}R_{a\hat{n}\hat{n}\hat{n}} = g^{cd} \perp {}^{(4)}R_{ac\hat{n}d} \tag{64}$$

- Now using the second *Gauss-Codazzi* eqn. (44), we get

$$\perp {}^{(4)}R_{a\hat{n}} = -g^{cd}(D_c K_{ad} - D_a K_{cd}) = D_a K - D^b K_{ab} \tag{65}$$

- Raising the indices we have (note sign convection)

$$\perp {}^{(4)}R^{a\hat{n}} = -(D^a K - D_b K^{ab}) \tag{66}$$

- Thus we finally find the momentum constraint equation.

$$D_b K^{ab} - D^a K = 8\pi j^a \tag{67}$$

- Note that the Hamiltonian and momentum constraint eqautions involve only spatial tensors and spatial derivatives and do not involve explicit time derivatives of spatial tensors. Therefore the equations are equations of constraint which must be satisfied by the 3+1 variables, γ_{ab}, K_{ab} at all times (i.e. on all slices).