Lectures on Numerical Relativity #2

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Outline

- Review of the Lecture #1
- Derivation of the evolution equations
- Analogy with E&M theory
- Example
  - Spherical symmetry
- Next lectures (tentative plan)
  - Generalized harmonic formulation (?) (July 20th)
  - Initial data (August 3rd, 17th and 31st)

3+1 Formalism

- ADM (Arnott-Deser-Misner, 1962) formulation is the most well known 3+1 formulation. There are variations, most notably BSSN formulation.
- 3+1 line element is written as

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

  where γ_{ij} is the (3-dim) metric of the hypersurface and α and β^i are the lapse and shift functions respectively.
- Define the Extrinsic curvature, along the normal vector, $n^\mu = -\alpha \nabla^\mu t$,

$$K_{ij} = -\frac{1}{2}\mathcal{L}_n \gamma_{ij}$$

- Take $\{\gamma_{ij}, K_{ij}\}$ as dynamical variables.
- By considering various projections, 10 Einstein field equations (in 4-dim) becomes 4 elliptic (constraint) equations plus 12 hyperbolic (evolution) equations.

3+1 Formalism in vacuum

- Constraint equations

$$
\begin{aligned}
{}^{(3)}R + K^2 - K_{ij}K^{ij} &= 0 \\
D_j(K^{ij} - \gamma^{ij}K) &= 0
\end{aligned}
$$

where ${}^{(3)}R$ is 3-dim Ricci scalar, D_i the covariant derivatives associated with the 3-dim metric, γ_{ij}, and K is trace of K_{ij}.

- Evolution equations

$$
\begin{aligned}
(\partial_t - \mathcal{L}_\beta)\gamma_{ij} &= -2\alpha K_{ij} \\
(\partial_t - \mathcal{L}_\beta)K_{ij} &= -D_i D_j \alpha + \alpha(R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j)
\end{aligned}
$$

- Note that in reality, practitioners use variations of the ADM formalism, e.g. BSSN (Baumgarte-Shapiro-Shibata-Nakamura, 1995, 1998) formulation.
- Usually, *free* evolution approach is used: solve constraints to set up initial data and use evolution equations to evolve the initial data forward in time. Just monitor (violations on) constraint equations to assess the quality of simulations.

3+1 Formalism

- Look at spacetime from the point of view of the Cauchy Problem.
- (Classical) gravitational field is the time history of the geometry of a spacelike hypersurface.
- To construct solutions (grav. field in 4D), solve the initial-value problem, and integrate the dynamical eqns. along the prescribed coordinate system.
- If matter is present, their initial value and evolution eqns. must also be taken into account.

Review

- M, g_{ab}.
- Foliation Σ defines *spacelike* hypersurfaces that are labeled by scalar function τ.
- (cf. characteristic approaches based on null surfaces)
- Define unit normal (to-the-hypersurface) vector, n^a associated with Σ.

$$n_a = -\alpha\Omega_a = -\alpha\nabla_a\tau \tag{1}$$
$$n^a = g^{ab}n_b \tag{2}$$

- Note the vector field, n^a, can be viewed as the 4-velocity field of a congruence of observers moving orthogonally to the slices.
- Define projection operator to project tensors onto the hypersurface.

$$\perp^a_b \equiv \delta^a_b + n^a n_b \tag{3}$$

- Define "spatial derivative operator", $D_a \equiv \perp \nabla_a$.
- Define 3-Riemann associated with D_a. $(D_a D_b - D_b D_a)W_c = R_{abc}{}^d W_d$ (W_a is spatial dual vector)

Review (con't)

- Define extrinsic curvature, $K_{ab} = - \perp \nabla_a n_b$ or $K_{ab} = -\frac{1}{2}\mathcal{L}_n \gamma_{ab} = -\frac{1}{2} \perp \mathcal{L}_n g_{ab}$.
- Derived *Gauss-Codazzi equations*

$$\perp {}^{(4)}R_{abcd} = R_{abcd} + K_{db}K_{ca} - K_{cb}K_{da} \quad (4)$$

$$\perp {}^{(4)}R_{abc\hat{n}} = D_b K_{ac} - D_a K_{bc} \quad (5)$$

- Einstein field equations: $G_{ab} = {}^{(4)}R_{ab} - \frac{1}{2} g_{ab} {}^{(4)}R = 8\pi T_{ab}$
- Hamiltonian Constraint equations: contract both indices with n^a, $G_{ab}n^a n^b =^{(4)} R_{ab}n^a n^b - \frac{1}{2} g_{ab} n^a n^{b\,(4)}R = 8\pi T_{ab} n^a n^b$.

$$R + K^2 - K^a{}_b K^b{}_a = 16\pi\rho \quad (6)$$

where $K \equiv K^a{}_a$.

- Momentum constraint equations: contract one index with n^a and project the remaining index onto the 3-hypersurface, $\perp G^{a\hat{n}} = \perp {}^{(4)}R^{a\hat{n}} + \perp \frac{1}{2} n^{a\,(4)}R = 8\pi \perp T^{a\hat{n}} = 8\pi j^a$.

$$D_b K^{ab} - D^a K = 8\pi j^a \quad (7)$$

- Note that the Hamiltonian and momentum constraint eqautions involve only spatial tensors and spatial derivatives and do not involve explicit time derivatives of spatial tensors. Therefore the equations are equations of constraint which must be satisfied by the 3+1 variables, $\{\gamma_{ab}, K_{ab}\}$ at all times (i.e. on all slices).

Evolution equations

- Now let's turn our attention to the evolution equations.
- First we need to compute another projection of the spacetime curvature tensor, $\perp {}^{(4)}R_{a\hat{n}b\hat{n}}$.
- From the definition of Riemann tensor, recalling the definitions of K_{ab} and a_b, (note $\nabla_a n_b = -K_{ab} - n_a a_b$)

$$
\begin{aligned}
{}^{(4)}R_{a\hat{n}b\hat{n}} &= {}^{(4)}R_{acbd} n^c n^d = n^c(\nabla_b \nabla_c n_a - \nabla_c \nabla_b n_a) \\
&= n^c \nabla_c (K_{ba} + n_b a_a) - n^c \nabla_b (K_{ca} + n_c a_a) \\
&= n^c \nabla_c K_{ba} + n^c n_b \nabla_c a_a + (n^c \nabla_c n_b) a_a - n^c \nabla_b K_{ca} + \nabla_b a_a \quad (8) \\
\perp {}^{(4)}R_{a\hat{n}b\hat{n}} &= \perp (n^c \nabla_c K_{ba} + a_b a_a - n^c \nabla_b K_{ca} + \nabla_b a_a) \quad (9)
\end{aligned}
$$

- Note that since $n^c K_{ca} = 0$, we have $-n^c \nabla_b K_{ca} = K_{ca} \nabla_b n^c$.
- Then

$$
\begin{aligned}
\perp {}^{(4)}R_{a\hat{n}b\hat{n}} &= \perp (n^c \nabla_c K_{ba} + K_{ca} \nabla_b n^c + K_{bc} \nabla_a n^c - K_{bc} \nabla_a n^c + a_b a_a + \nabla_b a_a) \\
&= \perp (\mathcal{L}_n K_{ab} - K_{bc} \nabla_a n^c + a_a a_b + \nabla_b a_a) \\
&= \perp (\mathcal{L}_n K_{ab} + K_{bc} K_a{}^c + a_a a_b + \nabla_b a_a) \quad (10)
\end{aligned}
$$

- Let us show $a_b = D_b \ln \alpha$.
- Consider first

$$\begin{aligned} \nabla_a n_b = -\nabla(\alpha\Omega_b) &= -(\nabla_a\alpha)\Omega_b - \alpha(\nabla_a\Omega_b) & (11) \\ n_a n^c \nabla_c n_b &= \alpha^2\Omega_a\Omega^c\left(-(\nabla_c\alpha)\Omega_b - \alpha(\nabla_c\Omega_b)\right) \\ &= -\alpha^2\Omega^c(\nabla_c\alpha)\Omega_a\Omega_b - \alpha^3\Omega_a\Omega^c\nabla_c\Omega_b \\ &= -\alpha^2\Omega^c(\nabla_c\alpha)\Omega_a\Omega_b - \alpha^3\Omega_a\Omega^c\nabla_b\Omega_c \\ &= -\alpha^2\Omega^c(\nabla_c\alpha)\Omega_a\Omega_b - \frac{\alpha^3}{2}\Omega_a\nabla_b(-\alpha^{-2}) \\ &= -\alpha^2\Omega^c(\nabla_c\alpha)\Omega_a\Omega_b - \Omega_a\nabla_b\alpha & (12) \end{aligned}$$

where we used the facts $\nabla_{[a}\Omega_{b]} = \nabla_{[a}\nabla_{b]}\tau = 0$ and $\nabla_b(-\alpha^{-2}) = \nabla_b(g^{cd}\Omega_c\Omega_d) = 2g^{cd}\Omega_c\nabla_b\Omega_d$.

- By contracting with n^a and noting the fact $n^a\Omega_a = \alpha^{-1}$, we get,

$$a_b = n^c\nabla_c n_b = \alpha\Omega^c(\nabla_c\alpha)\Omega_b + \alpha^{-1}\nabla_b\alpha \qquad (13)$$

- On the other hand,

$$D_b \ln\alpha = \perp^c{}_b \nabla_c \ln\alpha = (\delta^c{}_b + n^c n_b)(\alpha^{-1}\nabla_c\alpha) = \alpha\Omega_c(\nabla_c\alpha)\Omega_b + \alpha^{-1}\nabla_b\alpha \quad (14)$$

- Therefor, $\alpha_b = D_b \ln\alpha$.
- Now consider $\perp(a_a a_b + \nabla_b a_a)$:

$$\begin{aligned}\perp(a_a a_b + \nabla_b a_a) &= \perp(D_a \ln\alpha D_b \ln\alpha + \nabla(\alpha^{-1} D_a\alpha)) \\ &= \perp(\alpha^{-2} D_a\alpha D_b\alpha - \alpha^{-2} D_b\alpha D_a\alpha + \alpha^{-1}(\nabla_b D_a\alpha)) \\ &= \alpha^{-1} D_a D_b \alpha. \end{aligned} \quad (15)$$

- Since K_{ab} is a spatial tensor, we have,

$$\perp \mathcal{L}_n K_{ab} = \mathcal{L}_n K_{ab} = \alpha^{-1}\mathcal{L}_N K_{ab} \quad (16)$$

where $N^a = \alpha n^a$. Note that $N^a\Omega_a = 1$.

- Combining Eqns (10,15,16), we obtain,

$$\perp {}^{(4)}R_{a\hat{n}b\hat{n}} = \alpha^{-1}\mathcal{L}_N K_{ab} + K_{ac}K^c{}_b + \alpha^{-1} D_a D_b \alpha \quad (17)$$

Evolution Equations for Spatial Metric

- The evolution equations for the spatial metric essentially come from definition of the extrinsic curvature:

$$K_{ab} = -\frac{1}{2}\mathcal{L}_n g_{ab} \tag{18}$$

- However, note that since n^a is normal vector, Lie-derivative along n^a is only “normal time derivatives”. But we would like to allow more general definition of time derivative along “time directions”, t^a.
- Constant coordinate observer, t^a, therefore, can be defined via

$$t^a = N^a + \beta^a = \alpha n^a + \beta^a \tag{19}$$

$$\beta^a n_a = 0 \tag{20}$$

- Note that the above definition provides the *natural* normalization for the time derivatives, $t^a \Omega_a = 1$.

- Using a fundamental property of the Lie derivative for arbitrary vector fields v^a and w^a, and arbitrary tensor fields $\mathbf{S}$:

$$\mathcal{L}_{v+w}\mathbf{S} = \mathcal{L}_v\mathbf{S} + \mathcal{L}_w\mathbf{S} \tag{21}$$

  we have

$$\mathcal{L}_t\gamma_{ab} = \alpha\mathcal{L}_n\gamma_{ab} + \mathcal{L}_\beta\gamma_{ab} \tag{22}$$

- Or

$$\mathcal{L}_t\gamma_{ab} = -2\alpha K_{ab} + \mathcal{L}_\beta\gamma_{ab} \tag{23}$$

Evol. Equations for Extrinsic Curvature

- First note that Einstein field equations can be written as

$$ {}^{(4)}R_{ab} = 8\pi T_{ab} + \frac{1}{2} g_{ab} {}^{(4)}R = 8\pi (T_{ab} - \frac{1}{2} g_{ab} T) \tag{24} $$

where we used the fact $G = -R = 8\pi T$.

- Projecting onto the hypersurface, we have

$$ \perp {}^{(4)}R_{ab} = 8\pi (\perp T_{ab} - \frac{1}{2} \gamma_{ab} T) \tag{25} $$

- Note from the 3+1 decomposition of a general symmetric type(0,2) tensor

$$ S_{ab} \equiv \perp T_{ab} = T_{ab} + 2 n_{(a} \perp T_{b)\hat{n}} - n_a n_b T_{\hat{n}\hat{n}} \tag{26} $$

- Contracting the above eqn.

$$ S = T + T_{\hat{n}\hat{n}} = T + \rho \tag{27} $$

$$ T = S - \rho \tag{28} $$

- Eqn. (29 becomes

$$\perp^{(4)} R_{ab} \quad = \quad 8\pi(S_{ab} - \frac{1}{2}\gamma_{ab}(S - \rho)) \tag{29}$$

- Recall from the Lecture #1,

$$\perp{}^{(4)}R_{ab} = -\perp{}^{(4)}R_{a\hat{n}b\hat{n}} + g^{cd}\perp{}^{(4)}R_{acbd} \tag{30}$$

- Using eqn. (17) and one of the Gauss-Codazzi eqn. (**??**), we get,

$$\begin{aligned} \perp{}^{(4)}R_{ab} &= -(\alpha^{-1}\mathcal{L}_N K_{ab} + K_{ac}K^c{}_b + \alpha^{-1}D_a D_b\alpha) \\ &+ g^{cd}(R_{abcd} + K_{ab}K_{cd} - K_{ad}K_{cb}) \\ &= -\alpha^{-1}\mathcal{L}_N K_{ab} - 2K_{ac}K^c{}_b - \alpha^{-1}D_a D_b\alpha + R_{ab} + KK_{ab} \end{aligned} \tag{31}$$

- Equating Eqns (29) and (31) and using the fact that $\mathcal{L}_N K_{ab} = \mathcal{L}_{t-\beta}K_{ab} = \mathcal{L}_t K_{ab} - \beta K_{ab}$, we obtain the evolution eqns for K_{ab}:

$$\mathcal{L}_t K_{ab} = \mathcal{L}_\beta K_{ab} - D_a D_b\alpha + \alpha\left(R_{ab} + KK_{ab} - 2K_{ac}K^c{}_b - 8\pi(S_{ab} - \frac{1}{2}\gamma_{ab}(S - \rho))\right)$$

Analogy with E&M

- ADM Evolution equations:

$$\mathcal{L}_t \gamma_{ab} = -2\alpha K_{ab} + \mathcal{L}_\beta \gamma_{ab}$$

$$\mathcal{L}_t K_{ab} = \mathcal{L}_\beta K_{ab} - D_a D_b \alpha + \alpha \Big(R_{ab} + K K_{ab} - 2K_{ac}K^c{}_b - 8\pi(S_{ab} - \frac{1}{2}\gamma_{ab}(S - \rho)$$

- Compare ADM equations to the Maxwell's equations in flat space.
- Constraint equations (Gauss law)

$$D_i E^i = 4\pi\rho_e \quad (33)$$

$$D_i B^i = 0 \quad (34)$$

- Evolution equations (Faraday's law, Ampere's Circuital Law)

$$\partial_t E_i = \epsilon_{ijk} D^j B^k - 4\pi J_i \quad (35)$$

$$\partial_t B_i = -\epsilon_{ijk} D^j E^k \quad (36)$$

where ρ_e and J_i are charge and current densities, respectively.

E&M

- Continuity of charge is expressed by

$$D_i J^i = -\partial_t \rho_e \tag{37}$$

- Note that the evolution equations preserve the constraints so that, if they are satisfied at any time, they are automatically satisfied at all times.
- Often it is useful to introduce the vector potential and write B^i as a curl of A_i

$$B^i = \epsilon^{ijk} D_j A_k \tag{38}$$

so that the second constraint, $D_i B^i = 0$, is automatically satisfied.

- Then, we can rewrite Maxwell's equations in terms or A_i and E_i. Evolution equations are given by

$$\partial_t A_i = -E_i - D_i \Phi \tag{39}$$

$$\partial_t E_i = -D_i D_j A_i + D_i D^j A_j - 4\pi J_i \tag{40}$$

together with the constraint eqn. (33).

- Note that Φ plays a role of gauge just like the lapse and shift in the ADM equations in the sense that it can be chosen independently, but they don't directly affect the physics.
- It is interesting to note that the equations, (39) and (40) are quite similar to the ADM equations.
  - $A_i \leftrightarrow \gamma_{ij}$
  - $E_i \leftrightarrow K_{ij}$
  - Field variable on the RHS for A_i/γ_{ij}
  - Second spatial derivatives of field variable on the RHS for E_i/K_{ij}
- We will use E&M analogy when we discuss continuum issues later.
- E&M theory provides a laboratory to think about issues in NR with a more simple form of equations.

Example: spherical symmetry

- The most general form of a spherically symmetric 4d spacetime metric is given by

$$ds^2 = -a(t,\tilde{r})dt^2 + 2b(t,\tilde{r})dtd\tilde{r} + \tilde{g}_{rr}(t,\tilde{r})d\tilde{r}^2 + \tilde{g}_{\theta\theta}(t,\tilde{r})d\Omega^2 \tag{41}$$

- Using a new coordinate, $\tilde{r} \to r \equiv (\tilde{g}_{\theta\theta}/\tilde{g}_{rr})^{1/2}$, we can write down the metric, (after dropping all the tilde's), in the isotropic form.

$$ds^2 = -a(t,r)dt^2 + 2b(t,r)dtdr + g_{rr}(t,r)(dr^2 + r^2 d\Omega^2) \tag{42}$$

- By comparing with the general 3+1 line element,
$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$, we identify
$b(t,r) = \beta_r, a(t,r) = \alpha^2 - \beta_r\beta^r$ and rewrite, $g_{rr} = \psi$, we have

$$ds^2 = -(\alpha^2 - \beta^r\beta_r)dt^2 + 2\beta_r dtdr + \psi^4(dr^2 + r^2 d\Omega) \tag{43}$$

- Setting $\beta^r \equiv \beta$, we get

$$ds^2 = -(\alpha^2 - \psi^4\beta^2)dt^2 + 2\psi^4\beta dtdr + \psi^4(dr^2 + r^2 d\Omega) \tag{44}$$

where $\alpha \equiv \alpha(t,r), \beta \equiv \beta(t,r), \psi \equiv \psi(t,r)$

- The only non-vanishing, spatial connection coeffients are

$$\Gamma^r{}_{rr} = 2\frac{\psi'}{\psi} \tag{45}$$

$$\Gamma^r{}_{\theta\theta} = -2\frac{r^2\psi'}{\psi} - r \tag{46}$$

$$\Gamma^r{}_{\phi\phi} = \sin^2\theta\Gamma^r{}_{\theta\theta} \tag{47}$$

$$etc. \tag{48}$$

- You can do it by hand or use a symbolic algebra package such as MAPLE or MATHEMATICA. Tensor packages are available for both.
- “By-hand” method becomes extremely painful very quickly as soon as the expressions get complex.

- Maple script using Matt Choptuik Tensor package. (Note there are other freely available packages such as GRTensorII.)

```
######################################################################
#
# (c) Dale Choi 2007 --
#
# Maple script to verify (2.61) and (2.62) in Baumgarte and Shapiro r
#
######################################################################
read `/root/maple/TensorV6/TENSOR`;

alias(alpha=alpha);
alias(beta=beta);
alias(psi=psi);
laliasl([alpha,beta,psi],[t,r]);

Coords[4] := [t,r,theta,phi];

Lmetric[4]:= makeTENSOR(g4,makeIS(4,L,L));

g4tt[1,1] := - (alpha^2 - psi^4 * beta^2)
```

- Maple script (con't)

```
g4_LL[1,1]:= - (alpha^2 - psi^4 * beta^2);
g4_LL[1,2]:= psi^4* beta;
g4_LL[2,1]:= g_LL[1,2];

g4_LL[2,2]:= psi^4;
g4_LL[3,3]:= r^2 * psi^4;
g4_LL[4,4]:= r^2 * psi^4 * sin(theta)^2;

Umetric[4]:=makeINDEXEDinverse(Lmetric[4]);

Coords[3]:= [r,theta,phi];
Lmetric[3]:=projectN(Lmetric[4]);
Umetric[3]:=makeINDEXEDinverse(Lmetric[3]);

Chris1[3]:=makeChris1(3);
Chris2[3]:=raise(Chris1[3]);

Geometry3(on);
```

- Maple output:

```
[TENSOR, Ch3, [IS, 3, U, L, L], ARRAY([1 .. 3, 1 .. 3, 1 .. 3],
 [(1, 1, 1) = 2*(diff(psi, r))/psi, (1, 1, 2) = 0, (1, 1, 3) = 0,
  (1, 2, 1) = 0, (1, 2, 2) = -r*(psi+2*r*(diff(psi, r)))/psi, (1, 2,
  (1, 3, 1) = 0, (1, 3, 2) = 0, (1, 3, 3) = -r*sin(theta)^2*(psi+2*r*
  (2, 1, 1) = 0, (2, 1, 2) = (psi+2*r*(diff(psi, r)))/(psi*r), (2, 1,
  (2, 2, 1) = (psi+2*r*(diff(psi, r)))/(psi*r), (2, 2, 2) = 0, (2, 2,
  (2, 3, 1) = 0, (2, 3, 2) = 0, (2, 3, 3) = -sin(theta)*cos(theta),
  (3, 1, 1) = 0, (3, 1, 2) = 0, (3, 1, 3) = (psi+2*r*(diff(psi, r)))/
  (3, 2, 1) = 0, (3, 2, 2) = 0, (3, 2, 3) = cos(theta)/sin(theta),
  (3, 3, 1) = (psi+2*r*(diff(psi, r)))/(psi*r), (3, 3, 2) = cos(theta
```