Lectures on Numerical Relativity #4

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Outline

- (Lecture #3)
- GR as Cauchy problem
- Initial data (constraint) equations
- York-Lichnerowicz conformal decomposition
  - Example: Brill waves
- Conformal TT(Transverse-Traceless) decomposition
- Physical TT(Transverse-Traceless) decomposition
- Thin Sandwich Decomposition (Original and Extended)
- Next lectures (tentative plan)
  - Initial data: binary black hole initial data (August 17th)
  - Initial data: numrical methods (August 31st)

incomplete eigenvectors

- Consider first order linear system with constant coefficients 1D:

$$\partial_t u = A\partial_x u \tag{1}$$

$$u(x,0) = f(x) \tag{2}$$

where A is $n \times n$ matrix.

- In the case of repeated eigenvalues, there may not be a complete set of n *distinct* eigenvectors associated with those eigenvalues. This is called "generalized eigenvectors".
- If the matrix has an incomplete set of eigenvectors, and therefore a set of generalized eigenvectors, the matrix cannot be diagonalized, but can be converted into Jordan canonical form.

$$T^{-1}AT = J \tag{3}$$

where $T = [v_1, v_2, ..., v_n]$ is the matrix of all the eigenvectors.

- Note that for the *complete* set of eigenvectors, T transforms A to diagonal form

$$T^{-1}AT = \Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & 0 & \lambda_n \end{pmatrix} \tag{4}$$

- For the case where there exists a complete set of eigenvectors, we can show that the growth of the solution has the limit that is independent of fourier mode and the initial data.
- For the case where there is an incomplete set of eigenvectors, because of the presence of Jordan blocks, we have solutions that grow like $|\omega|^{n-1}$. Which implies we cannot find constants, K, α such that solution growth can be limited

$$||u(\cdot, t)|| \leq Ke^{\alpha(t-t_0)}||u(\cdot, t_0)|| \tag{5}$$

BSSN Formalism

- Variables:

$$\gamma_{ij} = e^{4\phi}\tilde{\gamma}_{ij}(\tilde{\gamma}_{ij} = e^{-4\phi}\gamma_{ij}) \tag{6}$$

$$\gamma^{ij} = e^{-4\phi}\tilde{\gamma}^{ij} \tag{7}$$

$$K_{ij} = \tilde{A}_{ij}e^{4\phi} + \frac{1}{3}\gamma_{ij}K \tag{8}$$

$$\tilde{A}^{ij} = e^{4\phi}A^{ij}(\tilde{A}_{ij} = e^{-4\phi}A_{ij}) \tag{9}$$

$$\tilde{\Gamma}^{i} \equiv \tilde{\gamma}^{jk}\tilde{\Gamma}^{i}_{jk} = -\partial_j\tilde{\gamma}^{ij} \tag{10}$$

where $\tilde{\Gamma}^{i}_{jk}$ is the Christoffel symbol w.r.t to $\tilde{\gamma}_{ij}$.

- BSSN constraints

$$\det(\gamma) = 1 \tag{11}$$

$$\tilde{A}^{i}_{i} = 0 \tag{12}$$

$$\tilde{\Gamma}^{i} + \partial_j\tilde{\gamma}^{ij} = 0 \tag{13}$$

- BSSN evolution equations for $\{\phi, \tilde{\gamma}_{ij}, K, \tilde{A}_{ij}, \tilde{\Gamma}^i\}$:

$$
\begin{aligned}
\partial_t \phi - \mathcal{L}_\beta \phi &= -\frac{1}{6}\alpha K && (14) \\
\partial_t K - \mathcal{L}_\beta K &= -\gamma^{ij} D_i D_j \alpha + \alpha(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2) \\
&+ \frac{1}{2}\alpha(\rho + S) && (15) \\
\partial_t \tilde{\gamma}_{ij} - \mathcal{L}_\beta \tilde{\gamma}_{ij} &= -2\alpha \tilde{A}_{ij} && (16) \\
\partial_t \tilde{A}_{ij} - \mathcal{L}_\beta \tilde{A}_{ij} &= e^{-4\phi}[-D_i D_j \alpha + \alpha(R_{ij} - S_{ij})]^{TF} \\
&+ \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{il}\tilde{A}^l_j) && (17) \\
\partial_t \tilde{\Gamma}^i &= \tilde{\gamma}^{jk}\partial_j\partial_k\beta^i + \frac{1}{3}\tilde{\gamma}^{ij}\partial_j\partial_k\beta^k + \beta^j\partial_j\tilde{\Gamma}^i \\
&- \tilde{\Gamma}^j\partial_j\beta^i + \frac{2}{3}\tilde{\Gamma}^i\partial_j\beta^j - 2\tilde{A}^{ij}\partial_j\alpha \\
&+ 2\alpha(\tilde{\Gamma}^i_{jk}\tilde{A}^{jk} + 6\tilde{A}^{ij}\partial_j\phi - \frac{2}{3}\tilde{\gamma}^{ij}\partial_j K) && (18)
\end{aligned}
$$

- where Lie derivative terms are given by

$$\mathcal{L}_\beta \phi = \beta^k \partial_k \phi + \frac{1}{6} \partial_k \beta^k \quad (19)$$

$$\mathcal{L}_\beta K = \beta^k \partial_k K \quad (20)$$

$$\mathcal{L}_\beta \tilde{\gamma}_{ij} = \beta^k \partial_k \tilde{\gamma}_{ij} + \tilde{\gamma}_{ik} \partial_j \beta^k + \tilde{\gamma}_{jk} \partial_i \beta^k - \frac{2}{3} \tilde{\gamma}_{ij} \partial_k \beta^k \quad (21)$$

$$\mathcal{L}_\beta \tilde{A}_{ij} = \beta^k \partial_k \tilde{A}_{ij} + \tilde{A}_{ik} \partial_j \beta^k + \tilde{A}_{jk} \partial_i \beta^k - \frac{2}{3} \tilde{A}_{ij} \partial_k \beta^k \quad (22)$$

- BSSN well-posedness
  - Nagy, Ortiz, and Reula, PRD **70**, 044012 (2004), “Strongly hyperbolic second order Einstein's evolution equations”.
  - Gundlach, and Martin-Garcia, PRD, **70**, 044032 (2004), “Symmetric hyperbolicity and consistent boundary conditions for second-order Einstein equations”.
  - Beyer and Sarbach, PRD, **70**, 104004 (2005), “Well-posedness of the BSSN formulation of Einstein's field equations”.

GR as Cauchy problem

- View-point: pose general relativity as a *dynamical* theory (geometrodynamics).
- View geometry of spacetime (i.e. solutions to the Einstein field equations) as "time history" of the geometry of a *spacelike hypersurface*, $\Sigma(t)$.
- ADM (Arnott-Deser-Misner, 1962) formulation and its several variations (e.g. BSSN) are the most well known and most used 3+1 formulations.
- 3+1 line element is written as

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

  where γ_{ij} is the (3-dim) metric of the hypersurface and α and β^i are the lapse and shift functions respectively.
- Define the Extrinsic curvature, along the normal vector, $n^\mu = -\alpha \nabla^\mu t$,

$$K_{ij} = -\frac{1}{2}\mathcal{L}_n \gamma_{ij}$$

- Two types of variables:
  - Kinematical: lapse function, α, and shift vector, β^i (4 functions).
  - Dynamical: 3-metric, γ_{ij}, and extrinsic curvature, K_{ij} (12 functions).
- Lapse and shifts, which define “coordinates”, are freely specifiable and embody coordinate freedom (general covariance) of general relativity.
- Question of specifying lapse and shift functions is referred as gauge (coordinate) conditions. We will discuss on that later in the Numerical Relativity Lecture series.
- By considering various projections, 10 Einstein field equations (in 4-dim) decompose into two classes.
  - Constraint equations: involves only dynamical variables and their *spatial* derivatives.
  - Evolution equations: involves time derivatives of dynamical variables as well.
- Constraint equations must be satisfied on all hypersurfaces, $\Sigma(t)$, including specifically the *initial* hypersurface, which we usually label by $t = 0$.

3+1 equations in vacuum

- Constraint equations

$$R + K^2 - K_{ij}K^{ij} = 0$$
$$D_j(K^{ij} - \gamma^{ij}K) = 0$$

where R is 3-dim Ricci scalar, D_i the covariant derivatives associated with the 3-dim metric, γ_{ij}, and K is trace of K_{ij}.

- Evolution equations

$$\begin{aligned}(\partial_t - \mathcal{L}_\beta)\gamma_{ij} &= -2\alpha K_{ij} \\ (\partial_t - \mathcal{L}_\beta)K_{ij} &= -D_iD_j\alpha + \alpha(R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j)\end{aligned}$$

- The initial value problem involves a specification of the topology of the initial spacelike hypersurface, $\Sigma(t=0)$, and a specification of dynamical variables $\{g_{ij}, K_{ij}\}$ in such a way the constraint equations are satisfied.
- In other words, out of 12 variables, $\{g_{ij}, K_{ij}\}$, only 8 are freely specifiable and the other 4 variables are fixed (constrained) by the 4 constraint equations.

Initial data problem in 3+1 formalism

- How to decide which 8 are freely specified and which 4 are fixed by constraints?
- Historically, the most used and studied approach is based on the works by Lichnerowicz, OMurchadha, and York.
- (At least a part of) motivation for their approach is to cast the constraint equations into a set of 4 quasi-linear, elliptic PDEs for 4 gravitational ”potentials” so that it makes both theoretical analysis and numerical approaches more tractable.
- Conformal decomposition:

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \tag{23}$$

$$A_{ij} = \psi^p \tilde{A}_{ij} (A^{ij} = \psi^{p-8} \tilde{A}^{ij}) \tag{24}$$

where p is integer and ψ is called conformal factor and will be determined by solving Hamiltonian constraint equation and A_{ij} is traceless part of extrinsic curvature, $K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij} K$.

ID problem

- $\tilde{\gamma}_{ij}$ is often called background or conformal metric.
- With this conformal decomposition of the 3-metirc, we get,

$$R = \psi^{-4}\tilde{R} - 8\psi^{-5}\tilde{\Delta}\psi \tag{25}$$

  where $\tilde{R}$ is a 3-Ricci scalar w.r.t to $\tilde{\gamma}_{ij}$ and $\tilde{\Delta} = \tilde{\gamma}^{ij}\tilde{D}_i\tilde{D}_j$ with $\tilde{D}_i$ is a covariant derivative associated with $\tilde{\gamma}_{ij}$.
- Hamiltonian constraint equation (in vacuum) becomes,

$$\tilde{\Delta}\psi - \frac{1}{8}\psi\tilde{R} - \frac{1}{8}\psi^5 K^2 + \frac{1}{8}\psi^5 K_{ij}K^{ij} = 0. \tag{26}$$

- Specify $\tilde{\gamma}_{ij}$ and solve HCE for ψ.
- Of course, we have to consider K_{ij} (or $K, \tilde{A}_{ij}$) together with ψ. Before we consider conformal decomposition of the K_{ij}, let's look at a simple exmaple, Brill wave.

Example: Brill wave

- Brill wave initial data represents a strong gravitaitonal wave solution in vacuum.
- Assume $K_{ij} = 0$, i.e., time-symmetric initial data. The momentum constraint equations are trivially satisfied.
- Consider the following form for the conformal 3-metric.

$$[e^{2q(\rho,z)}(d\rho^2 + dz^2) + \rho^2 d\phi^2] \tag{27}$$

where $q \equiv q(\rho, z)$, ($\rho^2 = x^2 + y^2$), can be arbitrarily chosen as long as some boundary conditions are met.

- Hamiltonian constraint equation for $\psi \equiv \psi(\rho, z)$ becomes

$$\tilde{\Delta}\psi - \frac{1}{8}\psi\tilde{R} = 0 \tag{28}$$

- Since $\tilde{R} = -2e^{-2q}(\frac{\partial^2 q}{\partial \rho^2} + \frac{\partial^2 q}{\partial z^2})$, and $\tilde{\Delta}\psi = e^{-2q}\tilde{\Delta}_\delta\psi$,

$$\tilde{\Delta}_\delta\psi + \frac{1}{4}(q_{,\rho\rho} + q_{,zz})\psi = 0 \tag{29}$$

where $\tilde{\Delta}_\delta$ represent *flat* Laplacian.

Brill Wave

- To verify the form of the HCE, eqn (29), in the previous slide, use maple script. As an exercise, do this in Cartesian coord.

```
##############################################################################
#
# (c) Dale Choi 2007 --
#
# Maple script to verify Brill wave solutions
#
##############################################################################
read ‘/root/maple/TensorV6/TENSOR‘;

alias(q=q);
alias(psi=psi);
laliasl([q,psi],[x,y,z]);

Coords[4] := [t,x,y,z];

Lmetric[4]:= makeTENSOR(g4,makeIS(4,L,L));
```

- Maple script (con't)

```
g4_LL[2,2]:= psi^4*( exp(2*q)*x^2/(x^2+y^2) + y^2/(x^2+y^2) );
g4_LL[2,3]:= psi^4*( (exp(2*q)-1)*x*y/(x^2+y^2)            );
g4_LL[3,2]:= g4_LL[2,3];
g4_LL[3,3]:= psi^4*( exp(2*q)*y^2/(x^2+y^2) + x^2/(x^2+y^2) );
g4_LL[4,4]:= psi^4*( exp(2*q)                                );

Umetric[4]:=makeINDEXEDinverse(Lmetric[4]);

Coords[3]:= [x,y,z];
Lmetric[3]:=projectN(Lmetric[4]);
Umetric[3]:=makeINDEXEDinverse(Lmetric[3]);

Chris1[3]:=makeChris1(3);
Chris2[3]:=raise(Chris1[3]);

Geometry3(on);
#RicciSc[3];
simplify(RicciSc[3]);
```

- Maple script (con't)

```
list1:=[diff(q,x) = dqdrho * x/sqrt(x^2+y^2), diff(q,y) = dqdrho * y/
   diff(q,x,x) = d2qdrho2 * x^2/(x^2+y^2)
                 +dqdrho *(1/sqrt(x^2+y^2)-x^2/(sqrt(x^2+y^2)^3)),
   diff(q,y,y) = d2qdrho2 * y^2/(x^2+y^2)
                 +dqdrho *(1/sqrt(x^2+y^2)-y^2/(sqrt(x^2+y^2)^3)),
   diff(q,x,y) = d2qdrho2 * x*y/(x^2+y^2)
                 -dqdrho *(x*y/(sqrt(x^2+y^2)^3)),
   diff(psi,x) = dpsidrho * x/sqrt(x^2+y^2),
   diff(psi,y) = dpsidrho * y/sqrt(x^2+y^2),
   diff(psi,x,x) = d2psidrho2 * x^2/(x^2+y^2)
                 +dpsidrho *(1/sqrt(x^2+y^2)-x^2/(sqrt(x^2+y^2)^3)),
   diff(psi,y,y) = d2psidrho2 * y^2/(x^2+y^2)
                 +dpsidrho *(1/sqrt(x^2+y^2)-y^2/(sqrt(x^2+y^2)^3)),
   diff(psi,x,y) = d2psidrho2 * x*y/(x^2+y^2)
                 -dpsidrho *(x*y/(sqrt(x^2+y^2)^3)) ];
simplify(subs(list1,RicciSc[3]));
```

- Maple output

```
> simplify(subs(list1,RicciSc[3]));
                       /                / 2      \
                       |                |d       |   2    2 1/2
[TENSOR, - 2 exp(-2 q) |4 dpsidrho + 4 |--- psi| (x  + y )
                       |                |  2     |
                       \                \dz      /

                        2    2 1/2                          2    2 1/2
     + psi d2qdrho2 (x  + y )    + 4 d2psidrho2 (x  + y )

           / 2   \             \
           |d    |   2    2 1/2|   /      5   2    2 1/2
     + psi |--- q| (x  + y )   |  /  (psi  (x  + y )   ), [IS, 3], -
           |  2  |             | /
           \dz   /             /

              /                / 2      \
              |                |d       |   2    2 1/2
    exp(-2 q) |4 dpsidrho + 4 |--- psi| (x  + y )
```

Brill Wave

- Use D. Holz *et al* form of $q(\rho, z)$:

$$q(\rho, z) = A\rho^2 e^{-r^2} \tag{30}$$

where $r^2 = \rho^2 + z^2$.

- Note that A is a *free* parameter which we can tune to adjust "strength" of the Brill wave solution.
- Note that the eqn (29) is an elliptic equation. We typically use multi-grid (MG) method to solve the constraints. I will discuss the MG method in coming weeks.
- If A is small, the spacetime is essentially perturbed flat space. Time evolution will result in the Minkowski space.
- On the other hand, if A is large, the spacetime is strongly curved. Time evolution will result in black hole formation.
- Is there a critical A^* between these two end states? Requires AMR and enormous resolution at least in 2D.
- Critical behavior at the threshold. This is open problem!

BW solutions

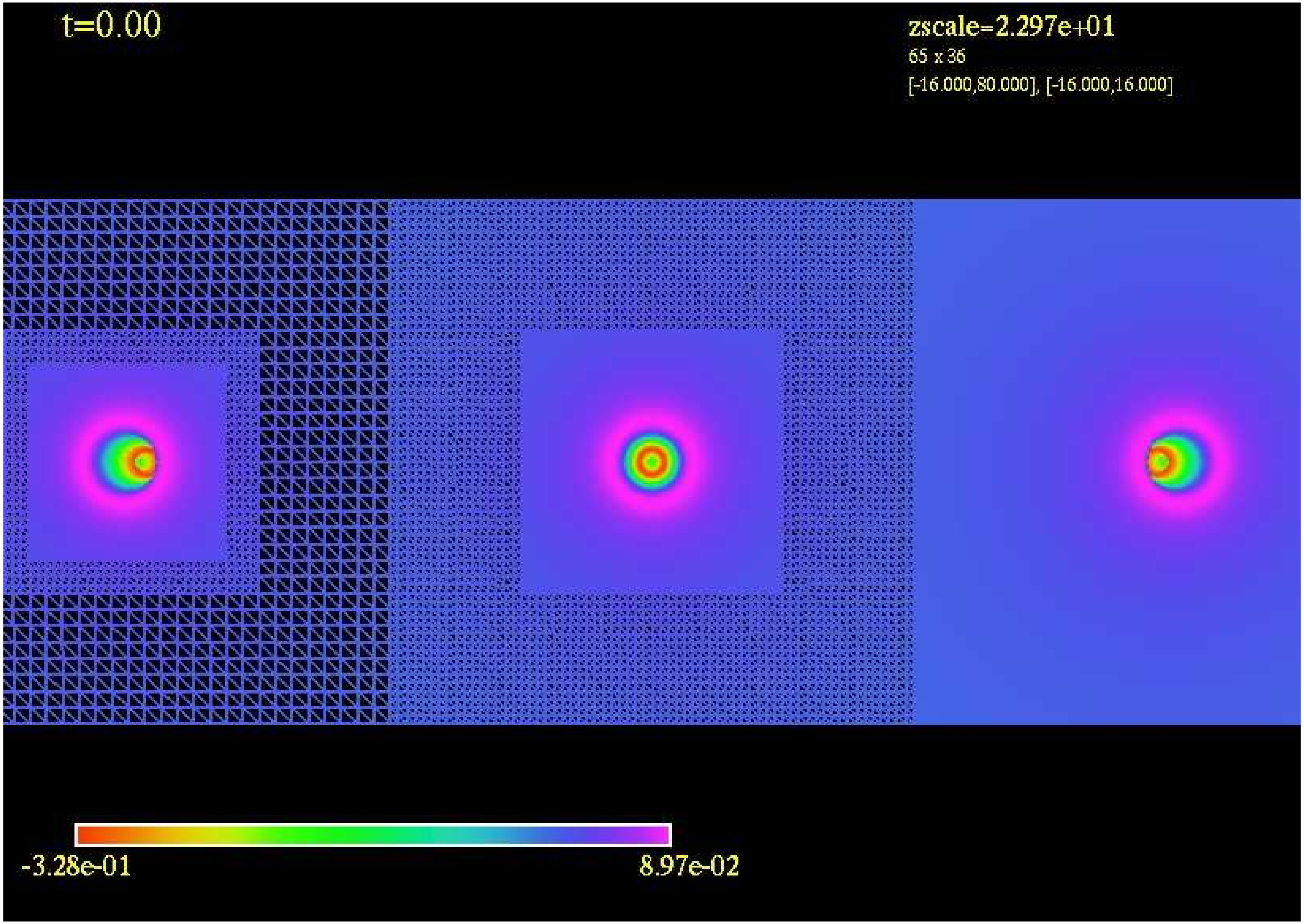
- Use multi-grid solver written in PAMR framework.
- Grid set-up: done in the context of adaptive meshes. In this example, I set up the grid structure by hand because we will only solve for constraint equations. Note that in general, evolution equations are solve at $t = 0$ to set up an initial grid hierarychy. More on this later.

```
base_shape  := [17 17 17]
base_bbox   := [-64 64 -64 64 -64 64]
init_depth  := 7
init_bbox_3 := [-32 32 -32 32 -32 32]
init_bbox_4 := [-16 16 -16 16 -16 16]
init_bbox_5 := [-8 8 -8 8 -8 8]
init_bbox_6 := [-6 6 -6 6 -6 6]
init_bbox_7 := [-4 4 -4 4 -4 4]
```

- (Note that I will show you later a simpler unigrid MG code that can run on a single CPU.)

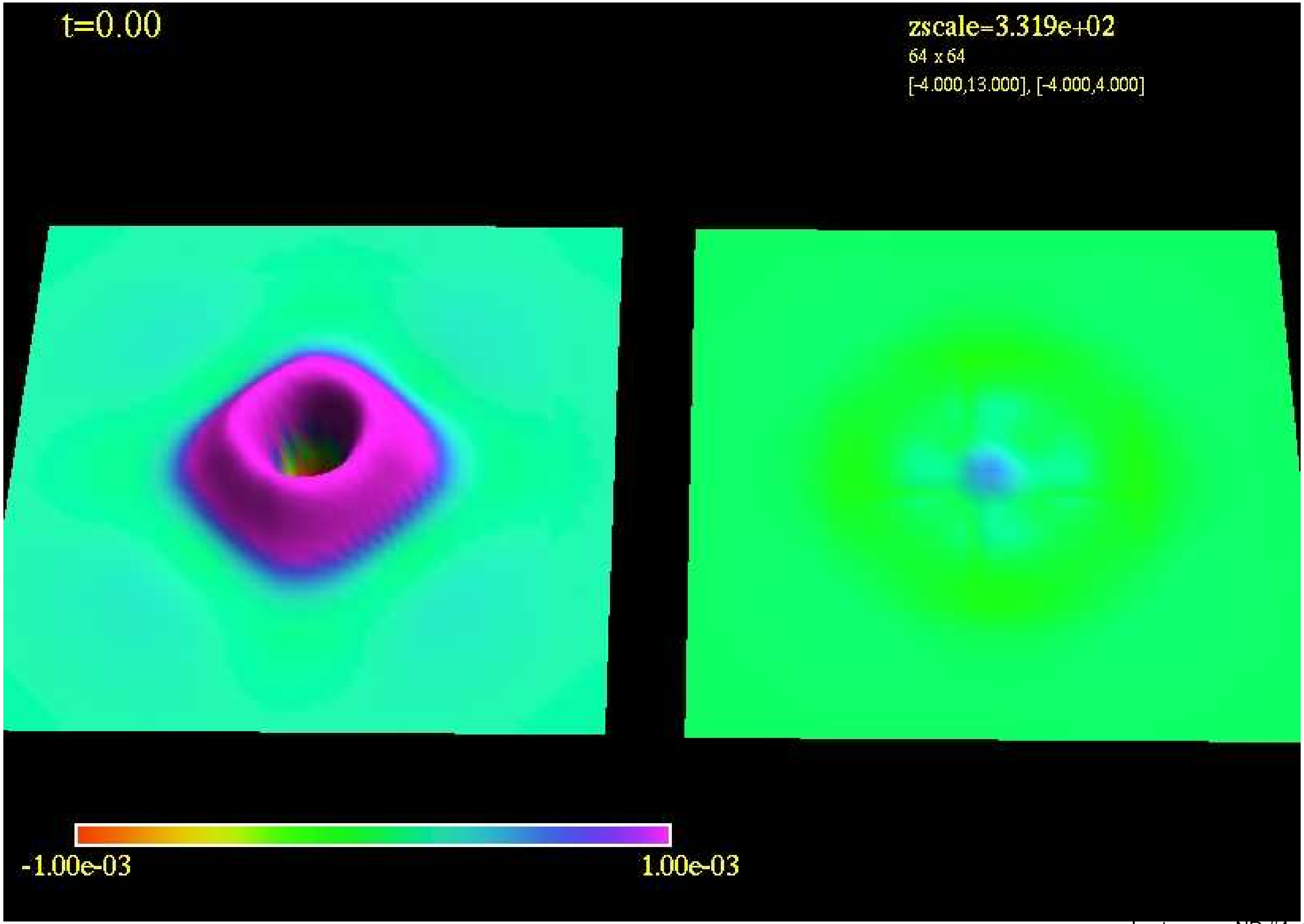
BW Solutions

- $\{A = 4\}$ solutions, ψ, at three different resolutions: $\psi^{4h}, \psi^{2h}, \psi^{h}$.



BW Solutions

- Differences between solutions: $(\psi^{4h} - \psi^{2h})$,$(\psi^{2h} - \psi^{h})$. Better than second order convergence!



Decomposition of Extrinsic Curvature

- York introduced decomposition of extrinsic curvature. We begin by splitting K_{ij} into its trace and traceless parts,

$$K_{ij} \equiv A_{ij} + \frac{1}{3}\gamma_{ij}K \quad (31)$$

- The decomposition proceeds by using the fact that we can covariantly split any symmetric tracefree tensors as follows:

$$\mathcal{S}^{ij} \equiv (\hat{l}X)^{ij} + T^{ij} \quad (32)$$

- T_{ij} present divergence-free and trace-free part,

$$\nabla_j T^{ij} = 0 \quad (33)$$
$$T^i_i = 0 \quad (34)$$

- The remaining part, called "longitudinal" is given by

$$(\hat{l}X)^{ij} = D^i X^j + D^j X^i - \frac{2}{3}\gamma^{ij} D_l X^l \quad (35)$$

- We want to apply this traceverse-traceless decomposition to the extrinsic curvature, A_{ij}.
- Conformal transverse-traceless decomposition: rather than decomposing A^{ij} directly, we introduce a base tensor, $\tilde{A}^{ij}$, which is related to the physical tensor via a conformal transformation:

$$A^{ij} \equiv \psi^{-10} \tilde{A}^{ij} \tag{36}$$

- Note that $A_{ij} = \gamma_{ik}\gamma_{jl}A^{ij} = \gamma_{ik}\gamma_{jl}\psi^{-10}\tilde{A}^{ij} = \psi^4\tilde{\gamma}_{ik}\psi^4\tilde{\gamma}_{jl}\psi^{-10}\tilde{A}^{ij} = \psi^{-2}\tilde{A}_{ij}$.

$$A_{ij} \equiv \psi^{-2} \tilde{A}_{ij} \tag{37}$$

- Note also the power “-10” was chosen principally because it results in the following property:

$$D_j A^{ij} = \psi^{-10} \tilde{D}_j \tilde{A}^{ij} \tag{38}$$

- Apply the transverse-traceless decomposition to $\tilde{A}^{ij}$:

$$\tilde{A}^{ij} \equiv (\tilde{l}X)^{ij} + \tilde{Q}^{ij} \tag{39}$$

- Note that $(\tilde{l}X)^{ij}$ and $\tilde{Q}^{ij}$ are both defined w.r.t *conformal* metric, $\tilde{\gamma}_{ij}$.
- Instead of treating $\tilde{Q}^{ij}$ as a freely specifiable part and solve for X^l, it is more convenient to take any trace-less, symmetric tensor $\tilde{M}^{ij}$ as a freely specifiable variable.
- In other words, given $\tilde{M}^{ij}$, we first obtain *its* transverse-trace part $\tilde{Q}^{ij}$ via

$$\tilde{Q}^{ij} \equiv \tilde{M}^{ij} - (\hat{l}Y)^{ij} \tag{40}$$

- Then, due to linearity of $\hat{l}$ operator,

$$\tilde{A}^{ij} = \tilde{M}^{ij} + (\hat{l}X)^{ij} - (\hat{l}Y)^{ij} \equiv \tilde{M}^{ij} + (\hat{l}V)^{ij} \tag{41}$$

and solve directly for V^i.

- Combining conformal decompositions and constrinat equations, we obtain, (for vacuum)

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \quad (42)$$

$$K^{ij} = \psi^{-10} \tilde{A}^{ij} + \frac{1}{3}\psi^{-4}\tilde{\gamma}^{ij} K \quad (43)$$

$$\tilde{A}^{ij} = (\hat{l}V)^{ij} + \tilde{M}^{ij} \quad (44)$$

$$\tilde{\Delta}_l V^i - \frac{2}{3}\psi^6 \tilde{D}^i K = -\tilde{D}_j \tilde{M}^{ij} \quad (45)$$

$$\tilde{D}^i \tilde{D}_i \psi - \frac{1}{8}\psi \tilde{R} - \frac{1}{12}\psi^5 K^2 + \frac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} = 0 \quad (46)$$

where

$$\tilde{\Delta}_l V^i \equiv \tilde{D}_j (\hat{l}V)^{ij} = \tilde{D}^j \tilde{D}_j V^i + \frac{1}{3}\tilde{D}^i(\tilde{D}_j V^j) + \tilde{R}^i_j V^j \quad (47)$$

- Freely specify $\tilde{\gamma}_{ij}$, $\tilde{M}^{ij}$ (with $\tilde{M}^i_i = 0$), and K.
- Solve constraint equations for ψ and V^i.

- Note if we choose maximal slicing $K = 0$ for initial data, momentum constraint equations fully decouple from Hamiltonian constraint equation.
- I know this is redundant but I write the same CTT equations with matter terms in them again here.

$$\tilde{\Delta}_l V^i - \frac{2}{3}\psi^6 \tilde{D}^i K = -\tilde{D}_j \tilde{M}^{ij} + \underline{8\pi\psi^{10} j^i} \quad (48)$$

$$\tilde{D}^i \tilde{D}_i \psi - \frac{1}{8}\psi \tilde{R} - \frac{1}{12}\psi^5 K^2 + \frac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} = \underline{-2\pi\psi^5 \rho} \quad (49)$$

Remember we have defined $\rho \equiv T_{ab} n^a n^b$ and $j^i \equiv -T^{ib} n_b$ for matter terms.

- Usually, matter terms are also conformally scaled and then only conformal part ($\tilde{\rho}$) is freely specified. E.g.

$$\rho = \psi^{-8} \tilde{\rho} \quad (50)$$

$$j^i = \psi^{-10} \tilde{j}^i \quad (51)$$

- Note that there is an alternative way to decompose extrinsic curvature, which is to take transverse-traceless decomposion of *physical* A^{ij} directly:
  physical transverse-traceless decomposition

$$A^{ij} \equiv (\bar{l}W)^{ij} + Q^{ij} \tag{52}$$

- Note that $(\bar{l}X)^{ij}$ and Q^{ij} are both defined w.r.t *physical* metric, γ_{ij}.
- As in the *conformal* transverse-traceless decomposition, we will freely specify symmetric traceless tensor $\tilde{M}^{ij}$. Then, symmetric transverse-traceless tensor Q^{ij} is given by

$$Q^{ij} \equiv \psi^{-10}\tilde{M}^{ij} - (\bar{l}Z)^{ij} \tag{53}$$

- Then, again, we have

$$A^{ij} = \psi^{-10}\tilde{M}^{ij} + (\bar{l}W)^{ij} - (\bar{l}Z)^{ij} = \psi^{-10}\tilde{M}^{ij} + (\bar{l}V)^{ij} \tag{54}$$

- Putting the definitions and constraint equations for *physical* transverse-traceless decomposition: (vaccum only here, matter terms are added as in CTT case)

$$
\begin{aligned}
\gamma_{ij} &= \psi^4 \tilde{\gamma}_{ij} && (55) \\
K^{ij} &= \psi^{-4}(\tilde{A}^{ij} + \frac{1}{3}\tilde{\gamma}^{ij} K) && (56) \\
\tilde{A}^{ij} &= (\hat{l}V)^{ij} + \psi^{-6}\tilde{M}^{ij} && (57) \\
\tilde{\Delta}_l V^i + 6(\tilde{l}V)^{ij}\tilde{D}_j \ln\psi &= \frac{2}{3}\tilde{D}^i K - \psi^{-6}\tilde{D}_j \tilde{M}^{ij} && (58) \\
\tilde{D}^i \tilde{D}_i \psi - \frac{1}{8}\psi\tilde{R} - \frac{1}{12}\psi^5 K^2 + \frac{1}{8}\psi^5 \tilde{A}_{ij}\tilde{A}^{ij} &= 0 && (59)
\end{aligned}
$$

where we used the fact

$$
(\bar{l}W)^{ij} = \psi^{-4}(\tilde{l}W)^{ij} \qquad (60)
$$

Note also a different comformal scaling exponent for traceless part of the extrinsic curvature $A^{ij} = \psi^{-4}\tilde{A}^{ij}$.

- Note again:(1) specify, $\tilde{M}^{ij}$ (w/ $\tilde{M}^i_i = 0$), $\tilde{\gamma}_{ij}$, and K and (2) solve for ψ and V^i.

Thin Sandwich Decomposition

- In the previous two decompositions (conformal TT decomp. and physical TT decomp.), one freely specify, $\{\tilde{\gamma}_{ij}, \tilde{M}^{ij}, K\}$.
- Decompositions themselves do NOT tell you directly how to choose those freely specifiable data.
- Also, it is concerned with a single spacelike hypersurface ($t = 0$) only, rightly so because we only need to specify $\{\gamma_{ij}, K_{ij}\}$. But this means that the initial data is completely independent of the kinematic variables, α and β^i, and so there is no connection to dynamics.
- But what if you need to consider a situation where some knowledge about the dynamics would be critical? E.g. quasi-circular orbits in compact binaries.
- York's thin-sandwich decomposition consideres the evolution of the metric between two neighboring hypersurfaces.

Thin Sandwich

- To include information about dynamics, we need to consider evolution equations as well.
- Thin-sandwich: consider two spacelike hypersurfaces labeled by $t = t_0$ and $t = t_0 + \delta t$. Then,

$$\gamma_{ij}(t_0 + \delta t) \sim \gamma_{ij}(t_0) + (\partial_t \gamma_{ij})|_{t_0} \delta t \tag{61}$$

- Make the following definitions:

$$u_{ij} \equiv \gamma^{\frac{1}{3}} \partial_t (\gamma^{-\frac{1}{3}} \gamma_{ij}) \tag{62}$$

$$\tilde{u}_{ij} \equiv \partial_t \tilde{\gamma}_{ij} \tag{63}$$

where again $\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij}$ as before.

- We also demand for convenience

$$\tilde{\gamma}^{ij} \tilde{u}_{ij} \equiv 0 \tag{64}$$

which implies $\tilde{\gamma}_{ij} \tilde{u}^{ij} \equiv 0$ so $\partial_t \tilde{\gamma} = 0$.

- Conformal scaling for u_{ij} then follows:

$$u_{ij} = \psi^4 \tilde{u}_{ij} \quad (u^{ij} = \psi^{-4} \tilde{u}^{ij}) \tag{65}$$

- From the evolution equation for $\tilde{\gamma}_{ij}$,

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i \tag{66}$$

we now get

$$u^{ij} = -2\alpha A^{ij} + (\bar{l}\beta)^{ij} \tag{67}$$

- Taking into account the following conformal scalings (as in CTT decomp. before),

$$A^{ij} \equiv \psi^{-10} \tilde{A}^{ij} \quad (A_{ij} \equiv \psi^{-2} \tilde{A}_{ij}) \tag{68}$$

$$(\bar{l}W)^{ij} = \psi^{-4} (\hat{l}W)^{ij} \tag{69}$$

we obtain,

$$\tilde{A}^{ij} = \frac{\psi^6}{2\alpha} \left((\hat{l}\beta)^{ij} - \tilde{u}^{ij} \right) \tag{70}$$

- Using the lapse scaling suggested by York,

$$\alpha = \psi^6 \tilde{\alpha} \tag{71}$$

we define, $\tilde{A}^{ij}$ as follows

$$\tilde{A}^{ij} \equiv \frac{1}{2\tilde{\alpha}} \left((\hat{l}\beta)^{ij} - \tilde{u}^{ij} \right) \tag{72}$$

- This condition is “natural” in that this definition preserves the scaling relation for A_{ij}, $A^{ij} = \psi^{-10} \tilde{A}^{ij}$.
- Because of this scaling of $\tilde{A}^{ij}$, the Hamiltonian constraint equation takes the same form as in CTT (conformal transverse traceless) decomposition.
- But for momentum constraint equations, it takes a different form.

- Grouping all the equations for thin-sandwich decomposition, we obtain

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \tag{73}$$

$$K^{ij} = \psi^{-10} \tilde{A}^{ij} + \frac{1}{3}\psi^{-4}\tilde{\gamma}^{ij} K \tag{74}$$

$$\tilde{A}^{ij} = \frac{1}{2\tilde{\alpha}}\left((\hat{l}\beta)^{ij} - \tilde{u}^{ij}\right) \tag{75}$$

$$\tilde{\Delta}_l \beta^i - (\hat{l}\beta)^{ij}\tilde{D}_j \ln\tilde{\alpha} + \frac{4}{3}\tilde{\alpha}\psi^6 \tilde{D}^i K = \tilde{\alpha}\tilde{D}_j(\frac{1}{\tilde{\alpha}}\tilde{u}^{ij}) + \underline{16\pi\tilde{\alpha}\psi^{10} j^i} \tag{76}$$

$$\tilde{D}^i\tilde{D}_i\psi - \frac{1}{8}\psi\tilde{R} - \frac{1}{12}\psi^5 K^2 + \frac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} = \underline{-2\pi\psi^5\rho} \tag{77}$$

- Freely specify, $\{\tilde{\gamma}_{ij}, \tilde{u}_{ij}, K, \tilde{\alpha}\}$ (and matter terms).
- Solve for $\{\psi, \beta^i\}$.
- We will later use thin sandwich framework to construct initial data that are in quasi-equilibrium.

Extended Thin Sandwich

- Pfeiffer & York (2003).
- Instead of choosing $\tilde{\alpha}$, compute it from the Einstein equations involving time derivative of K:

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_\beta\right) K = -\psi^{-4}(\tilde{D}^i\tilde{D}_i\alpha + 2\tilde{D}_i \ln\psi \tilde{D}^i\alpha) + \alpha(\tilde{A}^{ij}\tilde{A}_{ij} + \frac{K^2}{3}) + \frac{1}{2}\alpha(\rho + S) \quad (78)$$

where S is trace of $S_{ij} = T_{ij}$.

- Combining this with HCE, we obtain,

$$\tilde{D}^i\tilde{D}_i(\tilde{\alpha}\psi^7) - (\tilde{\alpha}\psi^7)(\frac{1}{8}\tilde{R} + \frac{5}{12}K^2\psi^4 + \frac{7}{8}\tilde{A}^{ij}\tilde{A}_{ij}\psi^{-8} + \underline{2\pi(\tilde{\rho} + 2\tilde{S})\psi^{-4}}) + (\dot{K} - \beta^i\tilde{D}_i K)\psi^5 = 0 \quad (79)$$

where we used $\rho = \psi^{-8}\tilde{\rho}$, $S = \psi^{-8}\tilde{S}$.

- Freely specify: $\{\tilde{\gamma}_{ij}, \tilde{u}_{ij}, K, \dot{K}, \tilde{\rho}, \tilde{j}^i, \tilde{S}\}$.
- Solve for $\{\psi, \beta^i, \alpha\}$.

Extended Thin Sandwich (con't)

- For quasi-equilibrium, choose coordinate (t, x^i) adapted to the helical killing vector, $\frac{\partial}{\partial t} = \xi$.
- So choose $\partial_t \tilde{\gamma}^{ij} = \tilde{u}^{ij} = 0$ and $\dot{K} = 0$.
- Choose in addition $K = 0$ (maximal slicing) and conformal flatness.
- Then equations simplify:

$$\tilde{D}^i \tilde{D}_i \psi + \frac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} = \underline{-2\pi\psi^{-3}\tilde{\rho}} \quad (80)$$

$$\tilde{\Delta}_l \beta^i - (\hat{l}\beta)^{ij}\tilde{D}_j \ln\tilde{\alpha} = \underline{16\pi\tilde{\alpha}\tilde{j}^i} \quad (81)$$

$$\tilde{D}^i \tilde{D}_i(\tilde{\alpha}\psi^7) - (\tilde{\alpha}\psi^7)(\frac{7}{8}\tilde{A}^{ij}\tilde{A}_{ij}\psi^{-8} + \underline{2\pi(\tilde{\rho} + 2\tilde{S})\psi^{-4}}) = 0 \quad (82)$$

where $\tilde{D}^i$ are flat operators.

- Once we solve for $\psi, \beta^i, \tilde{\alpha}$, we construct the initial data using

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} = \psi^4 f_{ij} \quad (83)$$

$$K^{ij} = \psi^{-10}\frac{1}{2\tilde{\alpha}}(\hat{l}\beta)^{ij} \quad (84)$$