

# Bäcklund transforms of chiral fields

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The action of the Bäcklund transformations on the axisymmetric chiral fields is investigated. In particular, it is calculated how the superpotential  $\Gamma$  associated with any chiral field is affected by the Bäcklund transformations.

## I. INTRODUCTION

The axisymmetric chiral field equations are derivable from a two-dimensional variational principle with a Lagrangian  $L$  of the particular structure

$$L = \nabla g \nabla g^{-1},$$

the chiral field  $g$  being an  $N \times N$  matrix. Examples are the stationary axisymmetric vacuum and electrovacuum fields in General Relativity, where  $g$  is an element of  $SU(1,1)$  and  $SU(2,1)$ , respectively.

The linear problem associated with the nonlinear chiral field equations<sup>1</sup> leads to the possibility to apply the polynomial method<sup>2</sup> in order to generate new solutions from known (seed) solutions of the chiral field equations. This generation method corresponds to the Bäcklund transformations used in the theory of nonlinear evolution equations.<sup>3,4</sup>

There is a superpotential  $\Gamma$  defined in Eq. (6) below. For stationary axisymmetric vacuum gravitational fields, in the Lewis-Papapetrou form of the metric,

$$ds^2 = (1/f) [e^{2\Gamma} (d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f(dt + \omega d\varphi)^2 \quad (1)$$

the superpotential  $\Gamma$  is one of the three gravitational potentials  $f = f(\rho, z)$ ,  $\omega = \omega(\rho, z)$ , and  $\Gamma = \Gamma(\rho, z)$ . Whereas the matrix  $g$  can be calculated from  $f$  and  $\omega$ , the superpotential  $\Gamma$  is related to the conformal factor of the two-metric orthogonal to the group orbits in space-time.

In this paper we will derive explicit expressions for the change of  $\Gamma$  and  $g$  under Bäcklund transformations.

All our calculations are valid for *principal* chiral fields, i.e., the chiral fields are not *a priori* subject to special constraints. In physical applications, such constraints have to be imposed on  $g$ .

## II. THE CHIRAL FIELD EQUATIONS AND THE ASSOCIATED LINEAR PROBLEM

By definition, a cylindrically symmetric or stationary axisymmetric chiral field is represented by an  $N \times N$  matrix  $g$  satisfying the nonlinear partial differential equation

$$(1/\rho)(\rho g_{,u} g^{-1})_{,v} + (1/\rho)(\rho g_{,v} g^{-1})_{,u} = 0. \quad (2)$$

The chiral field  $g$  depends on the two variables  $u$  and  $v$  which

can be either real ( $\bar{u} = u, \bar{v} = v$ ) for cylindrically symmetric fields or complex conjugate ( $v = \bar{u}$ ) for stationary axisymmetric fields. The radial coordinate  $\rho$  is defined by  $\rho = (u + v)/2$ .

The nonlinear equation (2) is implied by the linear equations

$$\begin{aligned} \Psi_{,u} &= \frac{1}{2}(1 + \lambda)A\Psi, & A &= g_{,u}g^{-1}, \\ \Psi_{,v} &= \frac{1}{2}(1 + \lambda^{-1})B\Psi, & B &= g_{,v}g^{-1}, \\ \lambda &= (K - iv)^{1/2}(K + iv)^{-1/2}, \end{aligned} \quad (3)$$

$K$  being a complex constant. For any given chiral field  $g$  one can calculate a corresponding  $N \times N$  matrix function  $\Psi(\lambda, u, v)$  with the normalization

$$\Psi(1) = g. \quad (4)$$

In this article, the dependence of  $\Psi$  on  $u$  and  $v$  is not explicitly indicated,  $\Psi(\lambda) \equiv \Psi(\lambda, u, v)$ .

Note that  $g$  and, consequently, the matrices  $A$  and  $B$  defined in (3) do not depend on  $\lambda$ .

To show that the linear problem (3) implies the chiral field equation (2), one has to consider the integrability condition  $\Psi_{,u,v} = \Psi_{,v,u}$  and to use the formulas

$$\lambda_{,u} = (1/4\rho)(\lambda^2 - 1)\lambda, \quad \lambda_{,v} = (1/4\rho)(\lambda^2 - 1)/\lambda, \quad (5)$$

which follow from the definition of  $\lambda$  as given in (3). The integrability condition, which must be true separately for the terms with different powers of  $\lambda$ , leads to (2), without any further restrictions.

As mentioned in the Introduction, for any chiral field  $g$ , there is an associated superpotential  $\Gamma$  defined by

$$\Gamma_{,u} = \frac{1}{4\rho} \text{Tr} A^2, \quad \Gamma_{,v} = \frac{1}{4\rho} \text{Tr} B^2. \quad (6)$$

One can easily verify that the integrability condition  $\Gamma_{,u,v} = \Gamma_{,v,u}$  holds as a consequence of the chiral field equation (2).

Starting with the linear equations (3) and using (5) one derives the expressions<sup>5</sup>

$$\begin{aligned} A &= (1/2\rho) \lim_{\lambda \rightarrow \infty} (\lambda^2 \Psi' \Psi^{-1}), \\ B &= -\frac{1}{2\rho} \lim_{\lambda \rightarrow 0} (\Psi' \Psi^{-1}), \quad \Psi' = \frac{\partial \Psi}{\partial \lambda}. \end{aligned} \quad (7)$$

It is the aim of this paper to calculate  $\Gamma$ , for a special class of solutions to (2), from (6) and (7).

### III. THE POLYNOMIAL METHOD

Suppose we have at hand a seed solution  $g_0$  to (2) and the associated solution  $\Psi_0$  to the linear problem (2). Then we try to generate a new solution  $\Psi$  by means of the ansatz<sup>2</sup>

$$\Psi = T\Psi_0, \quad T = q_n P_n(\lambda), \quad (8)$$

where  $P_n(\lambda)$  is a matrix polynomial in  $\lambda$  of degree  $n$ ,

$$P_n(\lambda) = \sum_{s=0}^n a_s \lambda^s, \quad (9)$$

with  $\lambda$ -independent  $N \times N$  matrix coefficients  $a_s$ . The scalar term  $q_n$  can be appropriately chosen as

$$q_n = \kappa(K)(K + iu)^{n/2}. \quad (10)$$

The matrix  $\Psi$  constructed in this way must again satisfy linear equations of the form (3). This behavior is guaranteed if the coefficients  $a_s$  in (9) are determined from the following set of algebraic equations:

$$P_n(\lambda_i)\Psi_0(\lambda_i)C_i = 0, \quad (11a)$$

$$P_n(-1) = I, \quad (11b)$$

$C_i$  being constant  $N$ -dimensional column eigenvectors. The operation (8)–(11) is called the  $n$ -fold Bäcklund transformation (BT) because it corresponds to the BT in the theory of nonlinear evolution equations.<sup>4</sup>

Equation (11a) means that the  $\lambda_i$ 's are the  $Nn$  zeros of  $\det P_n(\lambda)$  provided that  $\Psi_0(\lambda_i)$  is regular and  $\det \Psi_0(\lambda_i) \neq 0$ . (The zeros  $\lambda_i$  have to be prescribed.) Under these conditions one infers that the new  $\Psi$  generated from  $\Psi_0$  by the BT (8)–(11) satisfies (3). The proof runs as follows.<sup>6</sup>

The expression

$$Y(\lambda) := \lambda \Psi_{,\nu} \Psi^{-1} \det P_n \\ = \lambda [P_{n,\nu} + \frac{1}{2}(1 + \lambda^{-1})P_n B_0] (P_n^{-1} \det P_n) \quad (12)$$

is a matrix polynomial in  $\lambda$  of degree  $Nn + 1$ . From (11a) and (8) one gets

$$\kappa_i^{-1} \Psi(\lambda_i) C_i = 0, \quad i = 1 \cdots Nn, \quad (13)$$

$$\lambda_i := (K_i - iu)^{1/2} (K_i + iu)^{-1/2}, \quad \kappa_i := \kappa_i(K_i).$$

The comparison of (13) with the identity

$$\kappa_i^{-1} \Psi(\lambda_i) \kappa_i \Psi^{-1}(\lambda_i) \det P_n(\lambda_i) = 0 \quad (14)$$

shows that the columns of  $\kappa_i \Psi^{-1}(\lambda_i) \det P_n(\lambda_i)$  are proportional to  $C_i$ . Using this fact one obtains from (13) after differentiation with respect to  $\nu$  (the  $C_i$ 's are constants)

$$[\kappa_i^{-1} \Psi(\lambda_i)]_{,\nu} \kappa_i \Psi^{-1}(\lambda_i) \det P_n(\lambda_i) = 0, \quad (15)$$

which yields  $Y(\lambda_i) = 0$ , i.e., the polynomials  $Y(\lambda)$  and  $\det P_n(\lambda)$  have the same zeros  $\lambda_i, i = 1 \cdots Nn$ . Hence, the polynomial  $Y(\lambda)/\det P_n(\lambda)$  is linear in  $\lambda$ , and  $\Psi_{,\nu} \Psi^{-1}$  is linear in  $\lambda^{-1}$ . Similarly, it turns out that  $\Psi_{,\mu} \Psi^{-1}$  is linear in  $\lambda$ :

$$\Psi_{,\mu} \Psi^{-1} = \alpha + \beta \lambda, \\ \Psi_{,\nu} \Psi^{-1} = \gamma + \delta \lambda^{-1}. \quad (16)$$

Finally, from (11b) one finds  $\alpha = \beta \equiv \frac{1}{2}A$  and  $\gamma = \delta \equiv \frac{1}{2}B$ , and we conclude that  $\Psi$  satisfies the linear problem (3). According to (4) one can assign to  $\Psi$  the new chiral field  $g$  generated from  $g_0$  by the BT (8)–(11).

The BT method described in this section generates from any seed  $g_0$  a new chiral field  $g$  which contains the additional parameters  $K_i$  corresponding to the zeros  $\lambda_i$  of  $\det P_n(\lambda)$ , and  $C_i$ .

### IV. THE CALCULATION OF THE SUPERPOTENTIAL $\Gamma$

Now we will derive an expression for the superpotential  $\Gamma$  defined in (6) in terms of the seed potential  $\Gamma_0$ , and the constants  $K_i$  and  $C_i$  arising from the BT (8)–(11). For the class of chiral fields generated in this way, one can calculate the corresponding superpotential  $\Gamma$  algebraically whereas otherwise the calculation of  $\Gamma$  leads to line integrals, see (6).

First we determine  $A$  and  $B$  according to (7) from the seed quantities  $A_0$  and  $B_0$ ,

$$A_0 = (1/2\rho) \lim_{\lambda \rightarrow \infty} (\lambda^2 \Psi'_0 \Psi_0^{-1}), \\ B_0 = - (1/2\rho) \lim_{\lambda \rightarrow 0} (\Psi'_0 \Psi_0^{-1}). \quad (17)$$

The relations (8)–(10) link the new  $\Psi$  with the original  $\Psi_0$  and the term  $q_n$  in (10) can be chosen in the form

$$q_n = \prod_{i=1}^n \left( \frac{1 - \lambda_i^2}{\lambda^2 - \lambda_i^2} \right)^{1/2}, \quad (18)$$

where the product is formed with  $n$  of the  $Nn$  zeros  $\lambda_i$ . From (8) it follows that the quantity  $\Psi' \Psi^{-1}$  in (7) reads

$$\Psi' \Psi^{-1} = P'_n P_n^{-1} + P_n \Psi'_0 \Psi_0^{-1} P_n^{-1} + q'_n q_n^{-1}. \quad (19)$$

To calculate  $A$  from (7) it is convenient to introduce  $\tilde{q}_n := \lambda^n q_n$  and the polynomial (in  $\lambda^{-1}$ )  $\tilde{P}_n := \lambda^{-n} P_n$  with the limits

$$\lim_{\lambda \rightarrow \infty} \tilde{P}_n = a_n, \quad \lim_{\lambda \rightarrow \infty} \lambda^2 \tilde{P}'_n \tilde{P}_n^{-1} = -a_{n-1} a_n^{-1}, \\ \lim_{\lambda \rightarrow \infty} \lambda^2 \tilde{q}'_n \tilde{q}_n^{-1} = 0. \quad (20)$$

On the other hand, the calculation of  $B$  from (7) requires the limit  $\lambda \rightarrow 0$  of (19); one obtains

$$\lim_{\lambda \rightarrow 0} P_n = a_0, \quad \lim_{\lambda \rightarrow 0} P'_n P_n^{-1} = a_1 a_0^{-1}, \quad \lim_{\lambda \rightarrow 0} q'_n q_n^{-1} = 0. \quad (21)$$

Using the limits (20), (21) one derives from (7) and (19) the relations

$$A = a_n A_0 a_n^{-1} - (1/2\rho) a_{n-1} a_n^{-1}, \\ B = a_0 B_0 a_0^{-1} - (1/2\rho) a_1 a_0^{-1}, \quad (22)$$

and the terms  $\rho \text{Tr} A^2$  and  $\rho \text{Tr} B^2$ , which occur in the definition (6) of the superpotential  $\Gamma$ , are given by

$$\rho \text{Tr} A^2 = \rho \text{Tr} A_0^2 - \text{Tr}(A_0 a_n^{-1} a_{n-1}) \\ + (1/4\rho) \text{Tr}(a_n^{-1} a_{n-1})^2, \\ \rho \text{Tr} B^2 = \rho \text{Tr} B_0^2 - \text{Tr}(B_0 a_0^{-1} a_1) \\ + (1/4\rho) \text{Tr}(a_0^{-1} a_1)^2. \quad (23)$$

The linear algebraic equations (11a) for the matrices  $a_0, \dots, a_n$  can be written in the form

$$a_n^{-1} (a_0, a_1, \dots, a_{n-1}) \mathbb{D} = -(\Psi_1, \Psi_2, \dots, \Psi_n) \Lambda^n, \quad (24)$$

where the following definitions are used. The elements of the

$nN \times nN$  block matrices  $\mathbb{D}$  and  $\Lambda$ ,

$$\mathbb{D} := \begin{pmatrix} \Psi_1 & \cdots & \Psi_n \\ \Psi_1 \Lambda_1 & \cdots & \Psi_n \Lambda_n \\ \vdots & \ddots & \vdots \\ \Psi_1 \Lambda_1^{n-1} & \cdots & \Psi_n \Lambda_n^{n-1} \end{pmatrix}, \quad (25)$$

$$\Lambda := \begin{pmatrix} \Lambda_1 & 0 & \cdots & 0 \\ 0 & \Lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda_n \end{pmatrix} \quad (26)$$

are the  $N \times N$  matrices  $\Lambda_k$  and  $\Psi_k$  ( $k = 1 \cdots n$ ) defined by

$$\Lambda_k := \begin{pmatrix} \lambda_{(k-1)N+1} & 0 & \cdots & 0 \\ 0 & \lambda_{(k-1)N+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{kN} \end{pmatrix}, \quad (27)$$

$$\Psi_k := (\Psi_0(\lambda_{(k-1)N+1})C_{(k-1)N+1}, \dots, \Psi_0(\lambda_{kN})C_{kN}). \quad (28)$$

Note that the  $C_i$ 's are column vectors.

The  $nN$  zeros  $\lambda_i$ , and the corresponding eigenvectors  $C_i$ , are arbitrarily ordered into  $n$  sets of  $N$  elements each. The brackets in (24) denote block row vectors.

Because of (5) and

$$\begin{aligned} \Psi_{0,u} &= \frac{1}{2}(1 + \lambda)A_0\Psi_0, \\ \Psi_{0,v} &= \frac{1}{2}(1 + 1/\lambda)B_0\Psi_0, \end{aligned} \quad (29)$$

the block matrix  $\mathbb{D}$  satisfies the relations

$$\begin{aligned} \mathbb{D}_{,u} &= \frac{1}{2}A_0\mathbb{D}(\Lambda + \mathbb{I}) + (1/4\rho)N\mathbb{D}(\Lambda^2 - \mathbb{I}), \\ \mathbb{D}_{,v} &= \frac{1}{2}B_0\mathbb{D}(\Lambda^{-1} + \mathbb{I}) - (1/4\rho)N\mathbb{D}(\Lambda^{-2} - \mathbb{I}), \end{aligned} \quad (30)$$

where the block diagonal matrices  $A_0, B_0, N, \mathbb{I}$  are defined by

$$\begin{aligned} A_0 &:= \begin{pmatrix} A_0 & 0 & \cdots & 0 \\ 0 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{pmatrix}, \\ B_0 &:= \begin{pmatrix} B_0 & 0 & \cdots & 0 \\ 0 & B_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_0 \end{pmatrix}, \\ N &:= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ 0 & 0 & 2I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (n-1)I \end{pmatrix}, \\ \mathbb{I} &:= \begin{pmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{pmatrix}, \end{aligned} \quad (31)$$

$I$  being the  $N \times N$  unit matrix.

From the first part of (30) we conclude

$$\begin{aligned} (\ln \det \mathbb{D})_{,u} &= \text{Tr}(\mathbb{D}_{,u}\mathbb{D}^{-1}) \\ &= \frac{1}{2} \text{Tr}(A_0\mathbb{D}\mathbb{A}\mathbb{D}^{-1} + A_0) \\ &\quad + (1/4\rho) \text{Tr}(N\mathbb{D}\Lambda^2\mathbb{D}^{-1} - N). \end{aligned} \quad (32)$$

The definitions (31) and  $A_0 = g_{0,u}g_0^{-1}$  imply

$$\begin{aligned} \text{Tr} A_0 &= n \text{Tr} A_0 = n (\ln \det g_0)_{,u} \\ \text{Tr} N &= (Nn/2)(n-1). \end{aligned} \quad (33)$$

To calculate the remaining terms in (32) one needs the diagonal elements of the block matrices  $\mathbb{R} := \mathbb{D}\mathbb{A}\mathbb{D}^{-1}$  and  $\mathbb{R}^2$ . In the block matrix formulation, Eq. (24) can be rewritten as

$$\begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ x_1 & x_2 & x_3 & \cdots & x_n \end{pmatrix} \begin{pmatrix} \Psi_1 & \cdots & \Psi_n \\ \Psi_1 \Lambda_1 & \cdots & \Psi_n \Lambda_n \\ \vdots & \ddots & \vdots \\ \Psi_1 \Lambda_1^{n-2} & \cdots & \Psi_n \Lambda_n^{n-2} \\ \Psi_1 \Lambda_1^{n-1} & \cdots & \Psi_n \Lambda_n^{n-1} \end{pmatrix} \\ = \begin{pmatrix} \Psi_1 \Lambda_1 & \cdots & \Psi_n \Lambda_n \\ \Psi_1 \Lambda_1^2 & \cdots & \Psi_n \Lambda_n^2 \\ \vdots & \ddots & \vdots \\ \Psi_1 \Lambda_1^{n-1} & \cdots & \Psi_n \Lambda_n^{n-1} \\ \Psi_1 \Lambda_1^n & \cdots & \Psi_n \Lambda_n^n \end{pmatrix}, \quad (34)$$

with

$$(x_1, x_2, \dots, x_n) := -a_n^{-1}(a_0, a_1, \dots, a_{n-1}). \quad (35)$$

The second matrix in (34) is  $\mathbb{D}$  as defined in (25) and the matrix on the right-hand side of (34) is  $\mathbb{D}\Lambda$ . Hence, one infers from (34)

$$\mathbb{R} = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ x_1 & x_2 & x_3 & \cdots & x_n \end{pmatrix} \quad (36)$$

and the diagonal elements of  $\mathbb{R}^2$  are found to be

$$\begin{aligned} \mathbb{R}^2 &= \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & x_{n-1} \\ & & & & & y_n \end{pmatrix}, \\ x_{n-1} &= -a_n^{-1}a_{n-2}, \\ x_n &= -a_n^{-1}a_{n-1}, \\ y_n &= x_n^2 + x_{n-1}. \end{aligned} \quad (37)$$

(The irrelevant off-diagonal elements have been omitted.) From (37) one gets

$$\begin{aligned} \text{Tr} \mathbb{R}^2 &= \text{Tr} \Lambda^2 = \sum_{i=1}^{Nn} \lambda_i^2 \\ &= \text{Tr}(a_n^{-1}a_{n-1})^2 - 2 \text{Tr}(a_n^{-1}a_{n-2}). \end{aligned} \quad (38)$$

With (36), (37) the terms  $\text{Tr}(A_0\mathbb{R})$  and  $\text{Tr}(N\mathbb{R}^2)$  in (32) can be written as

$$\begin{aligned} \text{Tr}(A_0 \mathbf{R}) &= -\text{Tr}(A_0 a_n^{-1} a_{n-1}), \\ \text{Tr}(N \mathbf{R}^2) &= -(2n-3) \text{Tr}(a_n^{-1} a_{n-2}) \\ &\quad + (n-1) \text{Tr}(a_n^{-1} a_{n-1})^2, \end{aligned} \quad (39)$$

and one obtains from (32), (33), and (39)

$$\begin{aligned} (\ln \det \mathbb{D})_{,u} &= -\frac{1}{2} \text{Tr}(A_0 a_n^{-1} a_{n-1}) + \frac{n}{2} (\ln \det g_0)_{,u} \\ &\quad - \frac{(2n-3)}{4\rho} \text{Tr}(a_n^{-1} a_{n-2}) + \frac{n-1}{4\rho} \\ &\quad \times \text{Tr}(a_n^{-1} a_{n-1})^2 - \frac{Nn}{8\rho} (n-1). \end{aligned} \quad (40)$$

Inserting (38) into (41) one finds the relation

$$\begin{aligned} (\ln \det \mathbb{D})_{,u} &= -\frac{1}{2} \text{Tr}(A_0 a_n^{-1} a_{n-1}) + \frac{n}{2} (\ln \det g_0)_{,u} \\ &\quad + \frac{1}{8\rho} \text{Tr}(a_n^{-1} a_{n-1})^2 \\ &\quad + \frac{2n-3}{8\rho} \sum_{i=1}^{Nn} \lambda_i^2 - \frac{Nn}{8\rho} (n-1). \end{aligned} \quad (41)$$

Taking into account Eq. (23) and the definitions (6) together with the corresponding relations

$$\Gamma_{0,u} = \frac{1}{2} \rho \text{Tr} A_0^2, \quad \Gamma_{0,v} = \frac{1}{2} \rho \text{Tr} B_0^2 \quad (42)$$

for the seed solution, Eq. (41) reads

$$\begin{aligned} (\ln \det \mathbb{D})_{,u} &= 2(\Gamma - \Gamma_0)_{,u} + \frac{n}{2} (\ln \det g_0)_{,u} \\ &\quad + \frac{2n-3}{8\rho} \sum_{i=1}^{Nn} \lambda_i^2 - \frac{Nn}{8\rho} (n-1). \end{aligned} \quad (43)$$

A similar consideration yields

$$\begin{aligned} (\ln \det \mathbb{D})_{,v} &= 2(\Gamma - \Gamma_0)_{,v} + \frac{n}{2} (\ln \det g_0)_{,v} \\ &\quad - \frac{1}{8\rho} \sum_{i=1}^{Nn} \lambda_i^{-2} + \frac{Nn}{8\rho} (n-1). \end{aligned} \quad (44)$$

Finally, the integration of (43) and (44) leads to the expression

$$e^{2\Gamma} = M e^{2\Gamma_0} \det \mathbb{D} \frac{\rho^{Nn(n-2)/4}}{(\det g_0)^{n/2}} \prod_{i=1}^{Nn} \frac{(\lambda_i^2 - 1)^{1-n/2}}{\lambda_i^{1/2}} \quad (45)$$

for the Bäcklund transform of the superpotential  $\Gamma$ . ( $M$  is an arbitrary constant of integration.) This formula contains as a particular case (for  $N=2$ ) the results derived in Refs. 1 and 7 for stationary axisymmetric vacuum fields.

## V. THE CALCULATION OF THE CHIRAL FIELD $g$

The new chiral field  $g$  generated from  $g_0$  by means of a BT is given by

$$g = P_n(1) g_0, \quad P_n(1) = \sum_{s=0}^n a_s. \quad (46)$$

The matrices  $a_0, \dots, a_n$  are determined from the algebraic equations (11) which are equivalent to

$$P_n(-1) = \sum_{s=0}^n (-1)^s a_s = I \quad (47)$$

and (24),

$$a_n^{-1}(a_0, a_1, \dots, a_{n-1}) = -(\Psi_1 \Lambda_1^n, \dots, \Psi_n \Lambda_n^n) \mathbb{D}^{-1}, \quad (48)$$

where the definitions (25)–(28) have been used.

From (46) one gets

$$a_n^{-1}(P_n(1) - a_n) = a_n^{-1} P_n(1) - I$$

$$= a_n^{-1}(a_0, a_1, \dots, a_{n-1}) \begin{pmatrix} I \\ I \\ \vdots \\ I \\ I \end{pmatrix}, \quad (49)$$

whereas (47) implies

$$\begin{aligned} a_n^{-1}(P_n(-1) - a_n) &= a_n^{-1} - I \\ &= a_n^{-1}(a_0, a_1, \dots, a_{n-1}) \begin{pmatrix} I \\ -I \\ \vdots \\ I \\ -I \end{pmatrix}. \end{aligned} \quad (50)$$

From (48), and the last two equations, one obtains the final result:

$$\begin{aligned} g &= P_n(1) g_0, \\ P_n(1) &= \left[ I - (\Psi_1 \Lambda_1^n, \dots, \Psi_n \Lambda_n^n) \mathbb{D}^{-1} \begin{pmatrix} \Lambda \\ -I \\ \vdots \\ I \\ -I \end{pmatrix} \right]^{-1} \\ &\quad \times \left[ I - (\Psi_1 \Lambda_1^n, \dots, \Psi_n \Lambda_n^n) \mathbb{D}^{-1} \begin{pmatrix} I \\ I \\ \vdots \\ I \\ I \end{pmatrix} \right]. \end{aligned} \quad (51)$$

Summarizing our results, the Bäcklund transforms of  $\Gamma$  and  $g$  are given by the formulas (45) and (51), respectively.

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