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New cosmological tracker solution for quintessence

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In this paper we propose a quintessence model with the potential $V(\Phi)=V_0[\sinh(\alpha\sqrt{\kappa_0}\Delta\Phi)]^\beta$ with $\beta<0$, whose asymptotic behavior corresponds to an inverse power-law potential at early times and to an exponential one at late times. We demonstrate that this is a tracker solution and that it could have driven the Universe into its current inflationary stage. The exact solutions and the description for a complete evolution of the Universe are also given. We compare such a model with current cosmological observations.

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Recent observations of type Ia supernovae (SNIa) revealed the existence of a certain vacuum energy with a negative equation of state, $\omega<0$, where $p=\omega\rho$. It seems that the contribution of this vacuum energy, also called dark energy, leads the Universe to reach a critical energy density and to accelerate its expansion [1]. Furthermore, if these observations can be confirmed, they indicate that this missing energy will dominate the future evolution of the cosmos. The most reliable models for such missing energy are the cosmological constant (Λ) [2] and a fluctuating, inhomogeneous scalar field, rolling down a scalar potential, called quintessence (Q) [3,4]. For the latter case, a great deal of effort has been made to determine the appropriate scalar potential $V(Q)$ that could explain current cosmological observations [3–7]. One example is the pure exponential potential [5,6]. It has the advantage that it mimics the dominant density background, but nucleosynthesis constraints require the scalar field contribution to be $\Omega_Q\leqslant 0.2$, which indicates that the scalar field would never dominate the Universe [5]. However, a sum of exponential potentials could avoid this last constraint [6]. On the other hand, a special group of scalar potentials has been proposed in order to have $\rho_Q\sim\rho_M$ at the present epoch: the tracker solutions [4]. In these solutions, the cosmology at late times is extremely insensitive to initial conditions, reducing the fine-tuning problem. One of these potentials is the pure inverse power-law one, $V(Q)\sim Q^{-\alpha}$ [4,7,8]. Although it reduces the fine-tuning and the cosmic coincidence problems, the predicted value for the current equation of state for quintessence is not in good agreement with supenovae results [4]. The same problem arises with the inverse power-law-like potentials. Another example are the new potentials proposed in [9]. They avoid efficiently the troubles stated above, but it is not possible to determine unambiguously their parameters.

In a previous work, we showed that if the Universe is completely dominated by the scalar energy density with a constant equation of state, then it naturally arises that the effective scalar field potential is an exponential one [10]. In this Rapid Communication, we combine this fact of the late Universe with a power-law-like potential, which seems to be a good candidate for the early Universe. As we will see, this combination avoids the problems of the exponential potential and coincides very well with observations. In order to do so, we propose a new cosmological tracker solution with the scalar field potential

$$\tilde{V}(\Phi)=\tilde{V}_0[\sinh(\alpha\sqrt{\kappa_0}\Phi)]^\beta \quad (\beta<0),$$
$$=\begin{cases}\tilde{V}_0(\alpha\sqrt{\kappa_0}\Phi)^\beta & |\alpha\sqrt{\kappa_0}\Phi|\ll 1,\\ (\tilde{V}_0/2^\beta)\exp(\alpha\beta\sqrt{\kappa_0}\Phi) & |\alpha\sqrt{\kappa_0}\Phi|\gg 1,\end{cases} \tag{1}$$

whose asymptotic behavior corresponds to an inverse power-law-like potential at early times and to an exponential one at later times. We will analyze the exact solutions of the evolution equations for this case and we will determine all the parameters in order to have a tracker solution and an accelerated expansion at the present time.

We start with the flat Friedmann-Robertson-Walker (FRW) metric

$$ds^2=-dt^2+a^2(t)[dr^2+r^2(d\theta^2+\sin^2(\theta)d\phi^2)]. \tag{2}$$

Thus, the equation of evolution for the scalar field Φ is

$$\ddot{\Phi}+3\frac{\dot{a}}{a}\dot{\Phi}+\frac{dV}{d\Phi}=0, \tag{3}$$

and the Hubble parameter satisfies the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2=\frac{\kappa_0}{3}(\rho+\rho_\Phi) \tag{4}$$

where $\rho_\Phi=\frac{1}{2}\dot{\Phi}^2+V(\Phi)$ is the density of the scalar field, ρ is the density of the baryons, plus neutrinos, plus radiation, etc., and $\kappa_0=8\pi G=M_P^{-2}$. The equations (3) and (4) can be written in a more convenient form if we define the function $F(a)$ such that $V(\Phi(a))=F(a)/a^6$ [11]. A first integral of the field equation (3) can be found

$$\frac{1}{2}\dot{\Phi}^2+V(\Phi)=\frac{6}{a^6}\int da\frac{F}{a}+\frac{C}{a^6}=\rho_\Phi, \tag{5}$$

C being an integration constant. If the scale factor is considered as the independent variable, the field equations can be integrated up to quadratures [11]

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$$\Delta t=\sqrt{3}\int \frac{da}{a\sqrt{\kappa_0(\rho_\Phi+\rho)}} \tag{6}$$

$$\Delta\Phi=\sqrt{6}\int \frac{da}{a}\left[\frac{\rho_\Phi-F/a^6}{\rho_\Phi+\rho}\right]^{1/2}, \tag{7}$$

where we have normalized the scalar field in units of M_P. The exact solution for the potential (1) can be found if we choose $F(a)=Ba^s$ and $C=0$, where $6-s=3(1+\omega^*)$ and s being a constant [we shall denote the quantities in the solution with an asterisk (*)]. Then, the scalar energy density scales as $\rho^*=\rho_0^* a^{-3(1+\omega^*)}$, being $\rho_0^*=2B/(1-\omega^*)$. The complete solution is [11]

$$\alpha\Delta\Phi=\mathrm{arccoth}\left(1+\frac{\Omega_{0D}}{\Omega_0^*}a^{3(\omega^*-\omega_D)}\right)^{1/2}, \tag{8}$$

$$\alpha=-\frac{3(\omega^*-\omega_D)}{2\sqrt{3(1+\omega^*)}}, \tag{9}$$

$$\beta=\frac{2(1+\omega^*)}{\omega^*-\omega_D}, \tag{10}$$

$$\rho_0^*=\left(\frac{2V_0}{1-\omega^*}\rho_{0D}^{-\beta/2}\right)^{1/(1-\beta/2)}, \tag{11}$$

$$H_0\Delta t=\frac{2}{\sqrt{\Omega_0^*}}\frac{a^{3(1+\omega^*)/2}}{3(1+\omega^*)} \times {}_2F_1\left(\frac{1}{2},\frac{\beta}{4},\frac{\beta}{4}+1,-\frac{\Omega_{0D}}{\Omega_0^*}a^{3(\omega^*-\omega_D)}\right), \tag{12}$$

$$V'(\Phi)=\alpha\beta\tanh^{-1}(\alpha\Delta\Phi)V(\Phi), \tag{13}$$

$$V''(\Psi)=[(\beta-1)\tanh^{-2}(\alpha\Delta\Phi)+1]\alpha^2\beta V(\Phi), \tag{14}$$

where $\rho_D=\rho_{0D}a^{-3(1+\omega_D)}$ and ω_D respectively are the energy density and the equation state for the dominant component in the Universe and ${}_2F_1$ is the hypergeometric function. For given values of α and β we will have this solution only when Eqs. (9),(10) are simultaneously satisfied and $\rho^*\sim\rho_D$. As stated above, the important feature of this potential is its asymptotic behaviors. For $|\alpha\Delta\Phi|\ll 1$, it behaves as an inverse power-law potential if $\omega^*<\omega_D$ [Eq. 1)]. Using Eqs. (6),(7), this form of the potential is a solution when the scalar energy is strongly subdominant ($\rho^*\ll\rho_D$) with the same conditions given above for $F(a)$.

In [4], it is demonstrated that an inverse power-law scalar potential has a cosmological tracker solution, and then avoids the fine-tuning and the cosmic coincidence problems of current cosmology. Therefore, our solution is as follows: We will choose the appropriate values for the parameters in the potential in such a way that the solution for Eqs. (8)–(14) can only be reached until the matter and the scalar energies are of the same order, as it happens today. From Eqs. (9)–(11), we can see that $\omega_D=0$ and $\rho_D=\rho_M$, and that ω^* and ρ_0^* will become the current equation of state and the current energy density for the scalar field, respectively. Thus, all parameters can be determined and the solution becomes directly a tracker one. One can check this last fact by observing that function $\Gamma=(V''V)/(V')^2>1$ and nearly constant over the initial conditions $0<\Phi<M_P$ [4]. We can now draw a complete history for the Universe modeled by potential (1). The behavior of the scalar field for the completely radiation- and matter-dominated epochs is exactly the same as that found for an inverse power-law potential in [4], because at those epochs the scalar energy is subdominant. For instance, say that $\rho_{i\Phi}\leqslant\rho_{i\gamma}$ at the end of inflation. At the beginning, the kinetic energy dominates and $p_\Phi\simeq\rho_\Phi$, and eventually the evolution of the scalar energy stops, $p_\Phi\simeq-\rho_\Phi$, mimics a ''cosmological constant'' and remains frozen until the time when the tracker solution is satisfied. This is well described by the evolution of the equation of state $\omega_\Phi=p_\Phi/\rho_\Phi$ (see Fig. 1). However, the case $\rho_{i\Phi}>\rho_{i\gamma}$ continues being troublesome. Another similar aspect is that potential (1) reaches its tracker solution when Φ is of order M_P, as can be seen from Eq. (14).

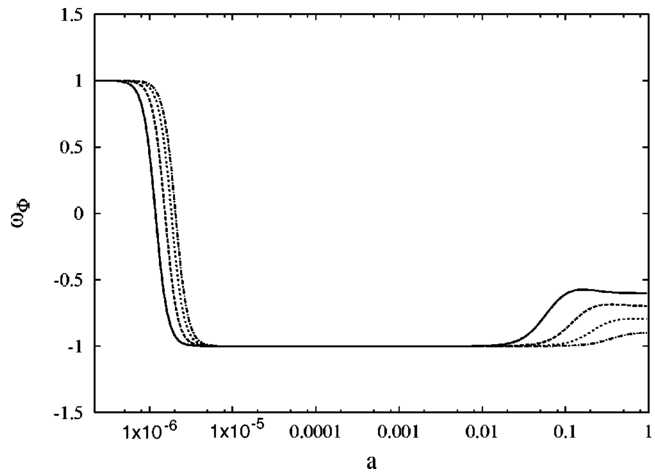

FIG. 1. Evolution of the equation of state ω_Φ vs the scale factor $a=(1+z)^{-1}$, z being the redshift. The models shown are $\omega_\Phi=\{-0.6$ (solid line), -0.7 (dashed line), -0.8 (dotted line), -0.9 (dash-dotted line)$\}$.

If β is a negative integer, we obtain that $\omega_\Phi\geqslant-0.6$ for $\beta\leqslant-1$. Then, we would not be able to fit SNIa observations as it was said above for potential $Q^{-\alpha}$. At this point, one can ask what would happen if $\omega_D=1/3$ (radiation). The tracker solution could have been reached during the radiation-dominated (RD) era. But matter would have dominated then (provided that $\omega_\Phi<0$) and the parameters α, β, and V_0 would not have matched the solution for matter dominance. In general, the tracker solution would only be reached completely until far after the matter dominance.

There are some key differences when the scalar field completely dominates the evolution ($\rho_\Phi>\rho_M$). First, when passing from a matter-dominated epoch to a scalar-field-dominated one, the Universe is driven naturally into a

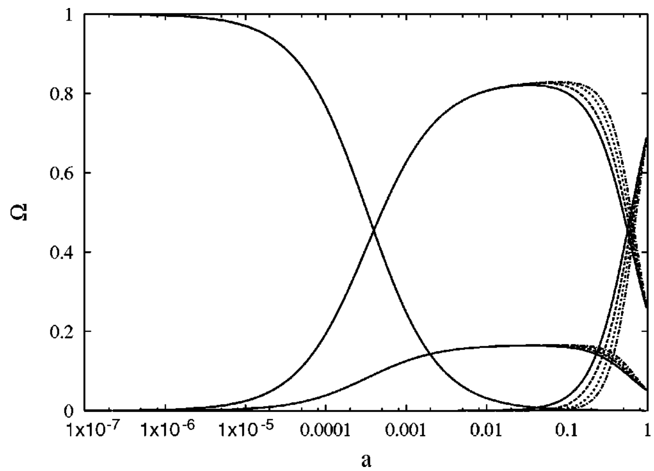

FIG. 2. Evolution of the dimensionless density parameters taking $\Omega_{0M}=0.30$ and $\Omega_{0\Phi}=0.70$ for the equations of state shown in Fig. 1.

power-law inflationary stage ($a\sim t^p$, $p>1$) due to the exponential late-time behavior [11]. Second, note that the equation of state ω_Φ will maintain the value it acquired in the matter-dominated epoch, that is, it will never switch to -1 (Fig. 1). Then, the scalar energy density ρ_Φ will continue evolving and decreasing with a constant equation of state, contrary to the late cosmological constant behavior (the energy would be frozen again) of some tracker potentials. Third, because of this late-time exponential behavior, the parameter V_0 must be set to an appropiate value in order to obtain the observed scalar energy density: a fine-tuning problem arises on the parameters of potential (1).

There is an almost generalized agreement in the current amounts for the different components in the Universe [1–3], thus, we will take $\Omega_{0M}=0.30$ and $\Omega_{0\Phi}=0.70$. First of all, in Fig. 2 is shown the evolution of the density parameters for each component of the Universe as functions of the scale factor a, for the values $\omega_\Phi=\{-0.6,-0.7,-0.8,-0.9\}$. The scalar energy remains subdominant until the epoch of matter domination and then the tracker solution is reached. This must have happened at a redshift $6<z<19$. Observe that a more negative ω causes a later scalar dominance. On the other hand, the matter-scalar equality must have occurred at a redshift $0.3<z<0.6$.

Now, we will explore the consequences of the models for the different observational constraints. The luminosity distance for a flat Universe where the scalar field has a constant equation of state can be easily found to be

$$d_L=\frac{cH_0^{-1}(1+z)}{(1+3\omega_\Phi)\sqrt{\Omega_{0\Phi}}}\Bigg\{-(1+z)^{-(1+3\omega_\Phi)/2}\times{}_2F_1\left[\frac{1}{2},\frac{\beta+4}{12},\frac{\beta+4}{12}+1;-\frac{\Omega_{0M}}{\Omega_{0\Phi}}(1+z)^{-3\omega_\Phi}\right]+{}_2F_1\left[\frac{1}{2},\frac{\beta+4}{12},\frac{\beta+4}{12}+1;-\frac{\Omega_{0M}}{\Omega_{0\Phi}}\right]\Bigg\}, \qquad (15)$$

${}_2F_1$ being again the hypergeometric function. The magnitude $m(z)$ of SNIa is related to d_L [1] and the results for different values of the scalar equation of state are shown in Fig. 3.

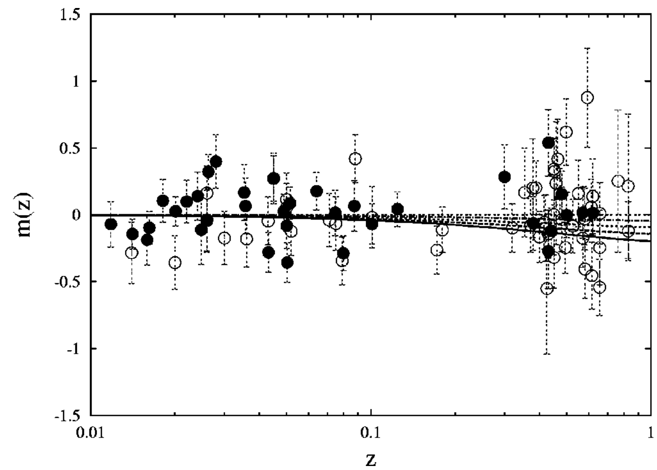

FIG. 3. The magnitude-redshift relation for the models shown in Fig. 2 compared to type Ia supernovae (SNIa) observations. The horizontal line corresponds to Λ CDM. The open circles represent the observational results from SCP and the opaque circles are from HZS [1].''

We used an amended version of CMBFAST [12] to compute the angular power spectrums for the different models in comparison with observational data and the cold dark matter model with a cosmological constant (Λ CDM) (Fig. 4). The processed spectra are very similar.

Note that even for more negative ω_Φ, the more similar the model is to the Λ CDM one, there is no solution for the potential (1) with $\omega_\Phi=-1$ [see Eq. (9)]. Following [14], we also computed the mass power spectrums (Fig. 5).

We can observe the same similarity among the models, as it was expected from the angular spectrum. The deceleration parameter for this model can be written in terms of the present energy densities Ω_{0M}, $\Omega_{0\Phi}$ and the scalar equation of state ω_Φ

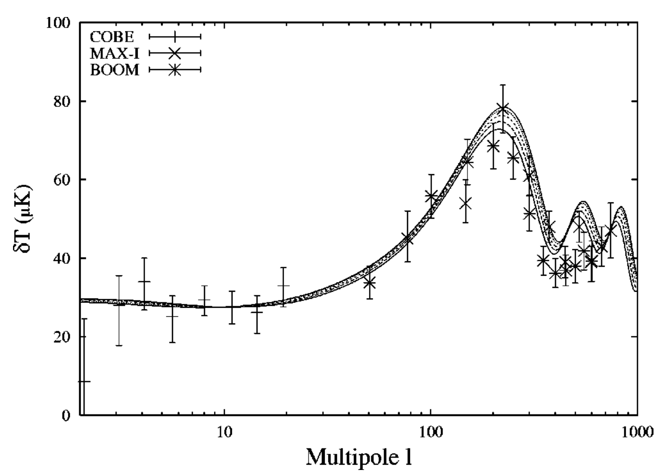

FIG. 4. Angular power spectrum for the models considered in Fig. 2 and Λ CDM (first line on top). The data were taken from [13].

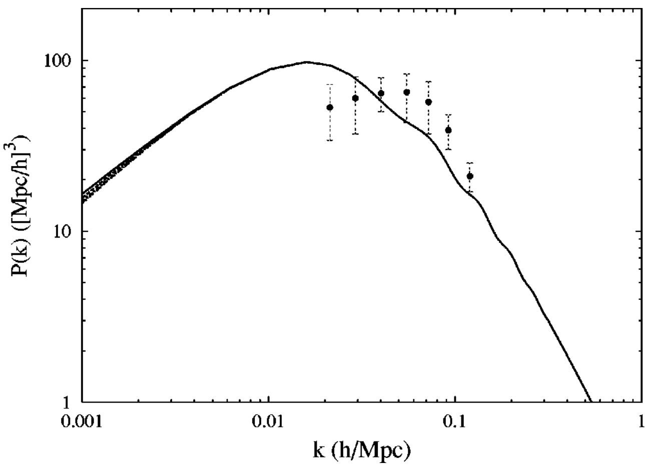

FIG. 5. Mass power spectrum for the models shown in Fig. 2. All models with spectral index $n_s=1$. The data points are from [15].

$$q_0 \equiv -H_0^2(\ddot{a}/a)_0 = \frac{1}{2}\Omega_{0M} + \frac{1+3\omega_\Phi}{2}\Omega_{0\Phi}. \tag{16}$$

Thus, $-0.13>q_0>-0.4$. The age of the Universe in units of the Hubble parameter is $1.0<H_0\Delta t_0<4.1$. Note that the best value $\omega_\Phi=-0.65$ for quintessence potentials [3] corresponds to $H_0\Delta t_0=1.22$.

Summarizing, we have found that the potential

$$V(\Phi) = \frac{1-\omega_\Phi}{2}\rho_{0\Phi}\left(\frac{\Omega_{0M}}{\Omega_{0\Phi}}\right)^{(1+\omega_\Phi)/\omega_\Phi} \times\left[\sinh\left(\frac{-3\omega_\Phi}{2\sqrt{3(1+\omega_\Phi)}}\Delta\Phi\right)\right]^{2(1+\omega_\Phi)/\omega_\Phi} \tag{17}$$

is a good quintessential candidate to be the missing energy in the Universe. Its behavior as an inverse power-law potential at early times allows us to avoid the fine-tuning problem under the initial condition (say, at the end of inflation) $\rho_{i\Phi} \leqslant \rho_{i\gamma}$. Its exponential-like behavior would drive the Universe into a power-law inflationary stage, which is in accord with recent observations. The three parameters [V_0, α, and β in Eq. (1)] can be determined uniquely by the measured values for the equation of state and the amount of vacuum energy to obtain a tracker solution. For these values of the parameters, the solution can only be reached during a matter-dominated epoch, followed by a scalar-dominated one. Then, we can avoid the coincidence problem. Actually, the current value of the scalar field can be found by setting $a=1$ in Eq. (8). Thus, $1.47 \leqslant (\Delta\Phi/M_P) \leqslant 0.49$ for $-0.6 \leqslant \omega_\Phi \leqslant -0.9$ ($\Omega_M=0.3$), that is, the scalar field is of order of the Planck mass today. Also, we found a good agreement with the current observations for SNIa, angular, and mass power spectrums. It must be said that the sinh-like potential naturally arises just under the assumption that $\omega_\Phi=$const nowadays and it is an exact solution [2]. Moreover, we have shown that it is indeed a tracker solution. Therefore, this is the simplest solution (constant equation of state) after the cosmological constant case (constant energy density).

However, even if the fine-tuning and cosmic coincidence problems have been ameliorated, a fine-tuning appears now in determining the values of the potential parameters in order to match the observed values for ω_Φ and Ω_Φ. This is a consequence of its exponential-like behavior at late times. Also, we do not already know about a theory that could predict these kinds of potentials, but potentials whose tracker solutions are reached when $\Phi \sim M_P$ must be treated in supergravity theory [16,17]. Therefore, we think that the attractive features of potential (1) make it worthwhile to study as an effective model for dark energy.

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