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Inside oscillatons

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Abstract

Non-singular self-gravitating objects can be found by solving the coupled Einstein–Klein–Gordon (EKG) equations for a real scalar field. Such objects are generically known as *oscillatons*, in which the scalar field and the metric are fully time-dependent. In this paper, we describe a numerical procedure to minimize the nonlinearities present in the EKG equations, in the case of spherical symmetry, which permits us to find accurate numerical solutions. In order to gain physical insight of relativistic oscillatons, we study oscillatons in flat space, in the weak field limit, the so-called Newtonian oscillatons, using a fixed Schwarzschild background. This last case may be related to the ejected scalar field during a gravitational collapse of scalar field configurations.

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1. Introduction

The Einstein–Klein–Gordon (EKG) equations with spherical symmetry have been numerically solved to find the so-called oscillatons [1, 2]. Oscillatons are non-singular and asymptotically flat solutions for which the metric and the scalar field are periodically time-dependent. This feature makes oscillatons totally different from what is known about static, or even stationary solutions of the Einstein equations with scalar fields.

Even though the existence and some semi-analytical properties of oscillatons have been shown before [1, 2], the solutions of the EKG equations by means of a Fourier expansion were limited by the nonlinearities present in the differential equations. This resulted in the study of very few Fourier modes. It is important to find other procedures in which nonlinearities could be minimized as much as possible [2]. In addition, it is difficult to explore inside oscillatons because of the lack of a global definition of gravitational energy in general relativity. Another question that remains open is the Newtonian-like limit of oscillatons. Since these objects are strongly time-dependent, it would be interesting to know whether such a limit exists and if it provides new features yet unknown.

An additional motivation for studying oscillatons arises from models of scalar fields as dark matter in the universe [3–16]. A central issue in these models is the formation of the cosmological structure using scalar fields. The most studied models are those with complex scalar fields which form objects called boson stars [1, 11, 17–19]. Boson stars are so simple that it is possible to find well-behaved static solutions, in both relativistic and Newtonian regimes, with nice features to be the dark matter galactic halos [9–11, 17, 20]. However, the cosmological evolution of such models seems to be at variance with cosmological observations [11, 15].

A similar picture arises by considering real scalar fields. The cosmological evolution has been addressed successfully in many recent works [3, 7, 8, 16], but the point about structure formation has not been completely worked out [3–6]. The main idea is to provide a hierarchical model of structure formation by taking into account the oscillatons that would be formed during the gravitational collapse of scalar fluctuations [12–14]. Unlike previous ideas with complex scalar fields (see for instance [11]), the picture would be to have a central object (the oscillaton) surrounded by a cloud of scalar field which was ejected during the gravitational collapse [8, 14], a process known as gravitational cooling [1]. Albeit that this picture should be worked out using the full machinery of numerical relativity [1, 8], it is desirable to find semi-analytic results in order to have a better control on the physics involved.

The main aim of this paper is to work out and clarify the points drawn above when applied to oscillatons. A summary of the paper is as follows. In section 2, we briefly summarize the procedure for solving numerically the coupled Einstein–Klein–Gordon equations for the case of a quadratic scalar potential solely since this case is representative of all oscillatons. The resulting differential equations are arranged such that the nonlinearities present are minimized and then we give solutions up to ten Fourier modes. All interesting features of oscillatons (total mass, fundamental frequency, etc) are shown, including a brief description of what we shall call the stable and unstable configurations. Also, typical solutions for the energy density and the metric coefficients are presented in turn.

In section 3, we solve the Klein–Gordon equation in flat space, which we shall call flat oscillaton, whose solution will be compared with the gravitationally bounded oscillaton described in section 2. This will allow us to gain some physical insight into the energy transformations inside oscillatons. In doing this, we will also make use of the local conservation equation that the scalar field obeys, the well-known Klein–Gordon equation.

In section 4, we also present the Newtonian-like limit of oscillatons, called Newtonian oscillaton, in which we find many similarities with the Newtonian limit of boson stars and the so called Schrödinger–Newton (SN) equations. As in the relativistic case, we will find a non-vanishing time-dependent part of the solution. Using the simplicity of Newtonian oscillatons, we find that such time dependence is due to the non-vanishing and fully time-dependent scalar pressure.

In section 5, we give semi-analytical solutions for a scalar field in a Schwarzschild background in order to find, in the first approximation, the behaviour of a scalar field *cloud* around a central object that may be identified with an oscillaton. Taking into account the properties of the differential equations, some semi-analytical results are given.

Finally, we give a brief summary of the results and some issues left for future investigation.

2. Oscillatons

The most general spherically-symmetric metric is written as

$$\mathrm{d}s^2 = g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta = -\mathrm{e}^{\nu-\mu}\,\mathrm{d}t^2 + \mathrm{e}^{\nu+\mu}\,\mathrm{d}r^2 + r^2\,(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\varphi^2), \tag{1}$$

where $\nu = \nu(t,r)$, $\mu = \mu(t,r)$. The energy–momentum tensor for the scalar field $\Phi(t,r)$ endowed with a scalar field potential $V(\Phi)$ is defined as [21]

$$T_{\alpha\beta} = \Phi_{,\alpha}\Phi_{,\beta} - \tfrac{1}{2}g_{\alpha\beta}[\Phi^{,\lambda}\Phi_{,\lambda} + 2V(\Phi)], \tag{2}$$

whose non-vanishing components are

$$-T^0{}_0 = \rho_\Phi = \tfrac{1}{2}[\mathrm{e}^{-(\nu-\mu)}\dot{\Phi}^2 + \mathrm{e}^{-(\nu+\mu)}\Phi'^2 + 2V(\Phi)], \tag{3a}$$

$$T_{01} = \mathcal{P}_\Phi = \dot{\Phi}\Phi', \tag{3b}$$

$$T^1{}_1 = p_r = \tfrac{1}{2}[\mathrm{e}^{-(\nu-\mu)}\dot{\Phi}^2 + \mathrm{e}^{-(\nu+\mu)}\Phi'^2 - 2V(\Phi)], \tag{3c}$$

$$T^2{}_2 = p_\perp = \tfrac{1}{2}[\mathrm{e}^{-(\nu-\mu)}\dot{\Phi}^2 - \mathrm{e}^{-(\nu+\mu)}\Phi'^2 - 2V(\Phi)] \tag{3d}$$

and also $T^3{}_3 = T^2{}_2$. Overdots denote $\partial/\partial t$ and primes denote $\partial/\partial r$. These different components are identified as the energy density ρ_Φ, momentum density $\mathcal{P}_\Phi$, radial pressure p_r and the angular pressure $p_\perp$.

The Einstein equations, $G_{\alpha\beta} = \kappa_0 T_{\alpha\beta}$, read

$$(\nu+\mu)^{\cdot} = \kappa_0 r\dot{\Phi}\Phi', \tag{4a}$$

$$\nu' = \frac{\kappa_0 r}{2}(\mathrm{e}^{2\mu}\dot{\Phi}^2 + \Phi'^2), \tag{4b}$$

$$\mu' = \frac{1}{r}[1 + \mathrm{e}^{\nu+\mu}(\kappa_0 r^2 V(\Phi) - 1)], \tag{4c}$$

where $\kappa_0 = 8\pi G = 8\pi m_{\mathrm{Pl}}^{-2}$ is the inverse of the reduced Planck mass (m_{Pl}) squared. The conservation equations for the scalar field energy–momentum tensor (2) are written as

$$T^{\alpha\beta}{}_{;\beta} = \Phi^{,\alpha}\left(\Box\Phi - \frac{\mathrm{d}V}{\mathrm{d}\Phi}\right) = 0, \tag{5}$$

where $\Box = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ is the d'Alembertian operator. Therefore, we obtain the Klein–Gordon (KG) equation for the scalar field Φ that is

$$\Phi'' + \Phi'\left(\frac{2}{r} - \mu'\right) - \mathrm{e}^{\nu+\mu}\frac{\mathrm{d}V}{\mathrm{d}\Phi} = \mathrm{e}^{2\mu}(\ddot{\Phi} + \dot{\mu}\dot{\Phi}). \tag{6}$$

2.1. The Φ^2-oscillaton

Equations (4*a*)–(6) are valid for any scalar potential, but in this section we will restrict ourselves solely to the quadratic case $V(\Phi) = m_\Phi^2\Phi^2/2$. This case is the simplest one and shares all important properties with other oscillatons [2]. Here we briefly describe the procedure to solve the coupled Einstein–Klein–Gordon equations.

In previous works, equations (4*a*)–(6) were solved using Fourier expansions in both the metric and scalar field functions [1, 2], but the inclusion of higher modes was difficult because of the nonlinearities present in the differential equations. These nonlinearities can be minimized introducing two variables $A(t,r) = \mathrm{e}^{\nu+\mu}$, $C(t,r) = \mathrm{e}^{2\mu}$, for which equations (4*a*)–(6) are rewritten as

$$\dot{A} = \kappa_0 r A \dot{\Phi} \Phi', \tag{7a}$$

$$A' = \frac{\kappa_0 A r}{2}\left(C\dot{\Phi}^2 + \Phi'^2 + A m_\Phi^2 \Phi^2\right) + \frac{A}{r}(1 - A), \tag{7b}$$

$$C' = \frac{2C}{r}\left[1 + A\left(\frac{1}{2}\kappa_0 r^2 m_\Phi^2 \Phi^2 - 1\right)\right], \tag{7c}$$

$$C\ddot{\Phi} = -\frac{1}{2}\dot{C}\dot{\Phi} + \Phi'' + \Phi'\left(\frac{2}{r} - \frac{C'}{2C}\right) - A m_\Phi^2 \Phi \tag{7d}$$

and we then calculate the metric functions $g_{tt} = -A(t,r)/C(t,r)$, $g_{rr} = A(t,r)$.

We will consider the following Fourier expansions

$$\begin{aligned}
\sqrt{\kappa_0}\Phi(t,r) &= \sum_{j=1}^{j_{\max}} \phi_j(r)\cos(j\omega t), \\
A(t,r) &= \sum_{j=0}^{j_{\max}} A_j(r)\cos(j\omega t), \\
C(t,r) &= \sum_{j=0}^{j_{\max}} C_j(r)\cos(j\omega t),
\end{aligned} \tag{8}$$

where ω is the fundamental frequency and $j_{\max}$ is the Fourier mode at which the series are truncated. Note that there are only direct products among the Fourier expansions (8) in equations (7*a*)–(7*d*); it is in this sense that we say that the nonlinearities were minimized. For comparison, see the corresponding equations in [1, 2].

Observe that there is no time-independent term in the scalar field expansion. The reason for this lies in the form of the Klein–Gordon equation ((6) and (7*d*)), which suggests that the scalar field should be of the form $\sqrt{\kappa_0}\Phi \sim \phi(r)f(t)$, mainly because of the quadratic potential (see also the flat oscillaton below) [2]. For a different scalar potential, we may have a time-independent term, as in oscillons [22].

For numerical computation, we take the change of variables [2]

$$x = m_\Phi r, \qquad C \to C\Omega^{-2}, \tag{9}$$

where $\Omega \equiv (\omega/m_\Phi)$, and which also implies a redefinition of the temporal metric coefficient $g_{tt} = -\Omega^2 A/C$.

Solutions are obtained by introducing the Fourier expansions (8) in equations (7*b*)–(7*d*). The resulting equations appear by setting each Fourier coefficient to zero. Because of the form of equations (7*b*)–(7*d*) the series on their right-hand sides have to be truncated [1], and then higher Fourier modes are eliminated by hand.

However, we can take advantage of equation (7*a*), which involves only *algebraic* relations among the coefficients A_j, ϕ_j [2]. In fact, we only need to solve equation (7*a*) for the first coefficient A_0 and then calculate A_n in terms of $A_{j<n}, \phi_{j<n}$ for $n = 1, 2, \ldots$. This procedure gives more precise results for the different Fourier coefficients [2].

2.2. Boundary conditions

The boundary conditions are determined by requiring non-singular and asymptotically flat solutions. Regularity at $x = 0$ requires $\phi_j' = 0$ and $A(t, x = 0) = 1$. The latter condition implies $A_0(0) = 1$, $A_{j\geqslant 1}(0) = 0$. Asymptotic flatness requires $\phi_j(\infty) = 0$ and $A(t, x = \infty) = 1$, which in turn means $A_0(\infty) = 1$, $A_{j\geqslant 1}(\infty) = 0$.

On the other hand, we also need $C_{j\geqslant 1}(\infty) = 0$, but $C_0(\infty) \neq 1$ because this variable gives us the value of the fundamental frequency $\Omega = \sqrt{C_0(\infty)}$. Therefore, the value of the fundamental frequency is an *output* value obtained after we solve equations (7*a*)–(7*d*). This is the reason behind the change of variables (9) [19].

Taking a fixed value for the first scalar Fourier coefficient $\phi_1(0)$, we only need to adjust the remaining values $\phi_{j\geqslant 2}$, $C_{j\geqslant 0}$ until we satisfy the boundary conditions. What we finally obtain is a unique set of eigenvalues for each value $\phi_1(0)$, which also labels the particular configuration.

In practice, only the odd coefficients in the scalar field expansion and the even coefficients in the metric functions are required. There is not reason to do that *a priori*, but the numerical solutions straightforwardly point out that. If we turn off the terms $\phi_{j\geqslant 2}(0) = 0$, the even $\phi_{2l}(x)$ and the odd $A_{2l-1}(x)$, $C_{2l-1}(x)$ $(l \geqslant 1)$ coefficients remain off all along the x-range. This does not happen to the other Fourier coefficients first mentioned in this paragraph. Therefore, we will only take the odd (even) coefficients of the scalar field (metric functions). This is another reason to justify the expansions used in [1, 2].

2.3. Numerical solutions and general features

Typical results of the metric functions $g_{rr} - 1$, $C - 1$ and the scalar field energy density for a 0-node oscillaton are shown in figures 1 and 2, respectively. We took $j_{\max} = 10$ in both figures, but found that good results are obtained from $j_{\max} = 6$. We also observed that solutions for different Fourier coefficients converge more rapidly because of the algebraic relations as compared with the solutions obtained by integrating together equations (7*b*)–(7*d*).

It should be noted that the higher the Fourier coefficient the smaller its contribution, which simply points out that the solution converges. Also, it is worth remarking that the weaker the central value $\phi_1(0)$, the smaller the contribution of higher-order modes $(n > 2)$, and then less terms are necessary in the series (8). These characteristics will be exploited in section 4.

Another general feature of oscillatons is that more massive oscillatons are more compact, in the sense that their spatial extension is smaller than that of weak oscillatons [1].

The central energy density oscillates, and its behaviour suggests that it may be expanded in a Fourier series as

$$\rho_\Phi(t, x) = \sum_{j=0} \rho_{2j} \cos 2j\omega t, \tag{10}$$

taking only the even coefficients. An example of the energy density behaviour is shown in figure 2 for two different times.

The oscillations of the metric coefficients and the energy density are confined to the (finite) region of the oscillaton, and the dominant component of these quantities is the first term in their Fourier expansion, as can be seen in figures 1 and 2 where we plot the behaviour of the expansion coefficients A_j. A similar behaviour is found for the coefficients C_j. Thus, the time-independent components $A_0(x)$, $C_0(x)$ determine the behaviour of the metric outside the oscillaton. In this sense, we say that the metric is also asymptotically static.

Moreover, $A_0(x)$, $C_0(x)$ also match the Schwarzschild metric with the same mass at large distances from the centre. This is not surprising, because we search for bounded solutions and then outside oscillatons there is no scalar field, only vacuum. In this sense, we are just rephrasing Birkhoff's theorem [21].

This property of the metric can be used to calculate the mass of the oscillaton as measured by an observer at infinity, that is, using the so-called Schwarzschild mass

$$M_\Phi = \frac{m_{\mathrm{Pl}}^2}{m_\Phi} \lim_{x\to\infty} \frac{x}{2}[1 - A^{-1}(t, x)]. \tag{11}$$

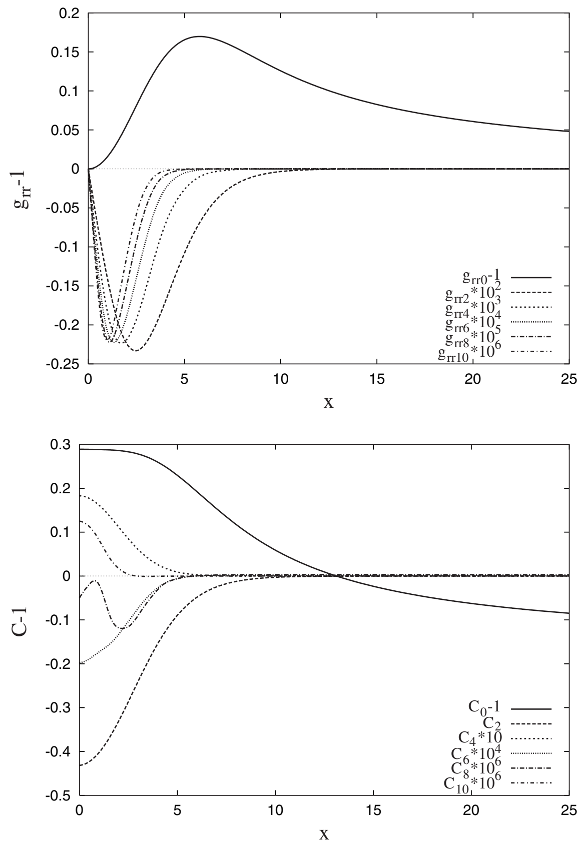

Figure 1. Plots of the even Fourier coefficients of the metric functions $g_{rr} - 1, C - 1$. The total mass of this configuration is $M = 0.5725\, m_{\mathrm{Pl}}^2/m_\Phi$ and its fundamental frequency is $\Omega = 0.9128$, see text below. This solution was obtained with $\phi_1(0) = 0.2828$ and $j_{\max} = 10$. The coefficients were scaled appropriately to show them in the same plot. Observe that the sixth coefficient is four orders of magnitude smaller than the first one. Then, the convergence of the solution is manifest.

This function converges very rapidly and is more convenient from the numerical point of view than, for instance, the ADM mass [2].

As we said before, the fundamental frequency Ω can be calculated from $C_0(\infty)$. But, there is another function with a faster convergence, and then we calculate the fundamental frequency from [2]

$$\Omega^{-1} = \mathrm{e}^{-\nu(\infty)} = \lim_{x\to\infty} \frac{A}{\sqrt{C}}. \tag{12}$$

In figure 3, we show the calculated mass and fundamental frequencies for some oscillatons. We were able to determine the maximum mass (also known as the critical mass, see below) $M_{\mathrm{c}} = 0.607 m_{\mathrm{Pl}}^2 \big/ m_\Phi$ reached at a central value $\phi_{1\mathrm{c}}(0) = 0.48$, to which also corresponds $\Omega = 0.864$. The fundamental frequency is always such that $\Omega \leqslant 1$ for all oscillatons, being $\Omega = 1$ for the trivial solution. In general, more massive oscillatons oscillate with a smaller fundamental frequency.

As a final point for this section, we would like to comment on the solutions of oscillatons in Fourier series. Although we could solve the EKG equations for an arbitrary number of

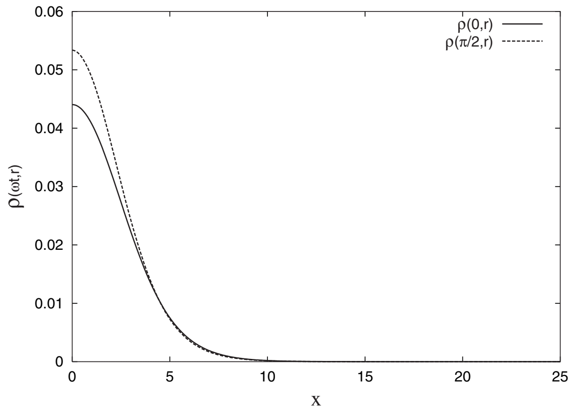

Figure 2. The scalar field energy density ρ_Φ for a 0-node oscillaton shown in figure 1. ρ_Φ is plotted for two different times $\omega t = 0, \pi/2$. Actually, $\rho_\Phi(\pi, x) = \rho_\Phi(0, x)$ (all in units of $m_{\mathrm{Pl}}^2 m_\Phi^2/(4\pi)$).

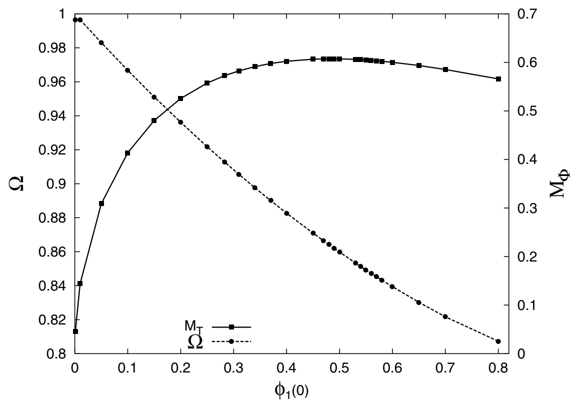

Figure 3. Total masses (11) and frequencies (12) of different oscillatons. The latter are labelled by the first Fourier coefficient of the scalar field expansion (8). The critical (maximum) mass is $M_{\mathrm{c}} = 0.607(m_{\mathrm{Pl}}^2/m_\Phi)$ for a configuration with a central value $\phi_{1\mathrm{c}}(0) = 0.48$; its corresponding frequency is $\Omega = 0.864$.

Fourier modes, the right-hand sides of equations (7*b*)–(7*d*) contain temporal derivatives which provoke the well-known loss of convergence in the Fourier expansions. Hence, the solutions of higher modes of functions Φ, A, C would not be smaller that the first ones in general. In fact, we have observed loss of convergence when we solve up to the ninth (tenth) mode of Φ (A, C). However, this loss of convergence is also minimized by using equation (7*a*), because higher time derivatives are cancelled on both sides.

2.4. Stable (S-branch) and unstable (U-branch) configurations

An important question about oscillatons is whether they are stable configurations against perturbations, and a full answer requires a numerical evolution of the EKG equations.

However, one can make physical arguments about the stability of the different configurations, depending on which side of the plot M versus $\phi_1(0)$ (see figure 3) they are located with respect to the critical configuration, i.e., the configuration with the maximum mass.

Let us imagine a perturbed configuration with a definite value $\phi_1(0)$ on the *right-hand* side of the critical one in figure 3, in such a way that it is more massive than the corresponding solution to the same $\phi_1(0)$. In consequence, there is more gravitational energy on the system and then the perturbed configuration will begin to collapse. But the collapse will *raise* the central value $\Phi(t, 0)$ and then the configuration must move to the *right*. But configurations with a stronger central field are in general *less massive* and then the oscillaton will not be able to reach another configuration of those shown in figure 3. The picture becomes worse if the oscillaton continues collapsing and raising the central value $\Phi(t, 0)$. The only possibility for the perturbed oscillaton is to never stop collapsing and then to finish as a black hole.

Now imagine a perturbed configuration also on the *right-hand* side, but with less mass than the corresponding solution with the same $\phi_1(0)$. There is less gravitational energy and then the oscillaton begins to expand. The central value $\Phi(t, 0)$ *diminishes* and then the configuration moves to the *left* in the plot 3. Since the configuration was at the right of the maximum mass, the oscillaton can find an equilibrium configuration to stop at. In principle, it would neither finish as a black hole nor disperse away.

The arguments can be inverted for perturbed configurations initially on the *left-hand* side from the critical configuration. It is concluded that they can always find another equilibrium configuration which is also on the *left-hand* side.

Similar arguments demonstrate that an initial configuration with a mass larger than the critical one will collapse in general into a black hole. The perturbed configuration begins to collapse and then moves to the right. Since by hypothesis it is more massive than the critical configuration, it would not be able to settle down into another equilibrium configuration.

The imaginary situations considered above have been observed for the close relatives of oscillatons: boson stars [24]. Therefore, we shall call those solutions on the *left* of the critical one, as seen in a plot M_Φ versus $\phi_1(0)$ (figure 3) stable configurations (S-branch), while those on the *right* shall be called unstable configurations (U-branch), in the same way as for boson stars.

At this point, we should recall that, as in the case of boson stars, oscillatons are prevented from collapsing because of the uncertainty principle. This fact is easily seen by calculating the Schwarzschild radius of an oscillaton which is

$$r_S = 2GM_\Phi = 2M_0 m_\Phi^{-1}, \tag{13}$$

where M_0 is a number obtained from the numerical solution, and then $M_\Phi = M_0 m_{\mathrm{Pl}}^2 \big/ m_\Phi$.

The maximum mass corresponds to a numerical factor of $M_0 = 0.607\,37$, which means $r_S \simeq 1.2\lambda_C$ with $\lambda_C = m_\Phi^{-1}$ being the Compton length of the boson particles. The S-configurations have a smaller associated Schwarzschild radius and they can 'escape' from the gravitational collapse to a black hole and then form equilibrium configurations. More massive configurations than the critical one would have a larger Schwarzschild radius, and then the boson particles can 'fit' in it; once the collapse starts the bosons remain inside the appearing event horizon from the very beginning.

The arguments given above consider no loss of mass during the evolution of the perturbed configurations. The picture becomes more interesting once we take into account the gravitational cooling process [1, 24], in which the initial configuration can eject part of the scalar field mass and then can migrate to a stable equilibrium configuration, even if the initial mass is larger than the critical value.

In this section, we have focused our attention only on the 0-node relativistic solutions. Solutions with nodes can be obtained, which in turn are much more involved than in the case of boson stars: each Fourier coefficient of the field (8) could have an arbitrary number of nodes once the number of nodes of $\phi_1(x)$ is fixed. We think they will be unstable and then ultimately decay into the ground solution by means of gravitational cooling, as happens for excited boson stars [1, 24].

2.5. Looking inside an oscillaton

Since it is not possible to have a precise definition of a gravitational energy density in general relativity, we do not have control of the internal processes of oscillatons, such as the interchange of energy and the appearance of possible scalar currents. An oscillaton can appear strange at first sight, because the scalar energy density (3*a*) is not constant on time and one may think that there must exist creation and annihilation of matter inside these objects.

In the following sections, we revise the solutions of scalar fields in different simple situations. In the first case, we try a heuristic procedure to look inside oscillatons: we compare the bound oscillaton with its counterpart in flat space, which we shall call flat oscillaton. The latter has an analytic solution and all quantities have an easy interpretation.

Another procedure we will follow is to find a Newtonian-like limit for relativistic oscillatons, what we shall call Newtonian oscillatons. In this case we can give a series of semi-analytical results that can be compared to the classical case in which the physical interpretation is much simpler. Here the scalar field is permitted to modify the spacetime and then Newtonian oscillatons are also bounded by their own gravity.

The last procedure we explore is the asymptotic solutions of a scalar field in a fixed Schwarzschild background. This is an intermediate state between the previous cases and is interesting because it reveals the behaviour of the scalar field under the gravity produced outside a (spherical) body, which could be a first approximation to the ejected scalar field during the gravitational cooling process.

All these different procedures will help us to find out what roles gravity and scalar fields play in forming bound oscillatons.

3. The flat oscillaton

In order to have some physical insight into what happens inside an oscillaton, we go first to the flat case in which $-g_{tt} = g_{rr} = 1$ (Minkowski background). Even though it is not a solution to the Einstein equations in a curved spacetime, the solution is analytic and it helps to understand some features that appear in oscillatons.

The Klein–Gordon equation (6) now reads

$$\Phi'' + \frac{2}{r}\Phi' - m_\Phi^2\Phi = \ddot{\Phi}. \tag{14}$$

The exact general solution for the scalar field Φ is

$$\Phi(t, r) = \frac{e^{\pm ikr}}{r} e^{\pm i\omega t}, \tag{15}$$

where we obtain the dispersion relation $k^2 = \omega^2 - m_\Phi^2$.

The physical properties of the solution depend on the ratio $\Omega = \omega/m_\Phi$. For $\Omega < 1$, the solution decays exponentially but they are singular at $r = 0$. On the other hand, $\Omega > 1$ allows for non-singular solutions which vanish at infinity and we will restrict ourselves to this case. Then, it is more convenient to use trigonometric functions and to write the particular solution in the form

$$\Phi(t, x) = \Phi_0 \frac{\sin(x)}{x} \cos(\omega t) \tag{16}$$

with $x = kr$.

The scalar field spreads over all space, i.e., it is not confined (at least not exponentially) to a finite region, and oscillates harmonically in time. The analytic expression for the scalar energy density (3*a*) is written in the form (10) but where ρ_1 and $\rho_{j\geqslant 3}$ vanish, and then the energy density oscillates with a frequency $2\omega t$.

The asymptotic behaviour when $x \to 0$ is

$$\rho_\Phi(t,x) = \frac{\Phi_0^2 m_\Phi^2}{4}[(\Omega^2+1) - (\Omega^2-1)\cos(2\omega t)] + \mathcal{O}(x^2), \tag{17}$$

where it is seen that the central density oscillates around a fixed value. In the opposite side, when $x \to \infty$

$$\rho_\Phi(t,x) = \frac{\Phi_0^2 m_\Phi^2}{4x^2}[(\Omega^2+1)\sin^2(x) + (\Omega^2-1)\cos^2(x)] + \frac{\Phi_0^2 m_\Phi^2}{4x^2}(\Omega^2-1)\cos(2x)\cos(2\omega t) + \mathcal{O}(x^{-3}), \tag{18}$$

and then $\rho_\Phi \sim \mathcal{O}(x^{-2})$. Hence, the integrated mass of the scalar field gives an infinite value and then we cannot talk of a localized configuration. Also, observe that the energy density is fully time-dependent even at large distances from the origin of coordinates.

We are working in flat space, so the KG equation can be written in a more convenient form in terms of the energy density, that is,

$$\frac{\partial \rho_\Phi}{\partial t} - \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \mathcal{P}_\Phi) = 0. \tag{19}$$

This last equation has an easy interpretation: since its form looks like the conservation of charge in electrodynamics, $\dot{\rho} + \nabla \cdot \vec{J} = 0$, it is the conservation of the scalar field energy. It also tells us that there is a scalar field current given by

$$\vec{J}_\Phi = -\mathcal{P}_\Phi \vec{r} = -\Phi_0^2 \frac{k\omega}{4}\left[\frac{\cos(2x)}{x} - 2\frac{\sin^2(x)}{x^2}\right]\frac{\sin(2\omega t)}{x}\vec{r}. \tag{20}$$

Observe that the quantity involved in this current is the scalar field momentum density (3*b*). Although the flux of scalar radiation at large distances does not vanish, there is no net flux of energy, as can be seen by averaging the scalar current on a period of a scalar oscillation. We also see that the only transformation process is that of the scalar field energy density into the momentum density, and vice versa.

3.1. Φ^2 versus flat oscillaton

Let us now compare the flat oscillaton to the equilibrium configurations obtained from the EKG equations. First of all, the particular solution (16) resembles the ansatz (8), but the fundamental frequency in the flat oscillaton obeys $\Omega > 1$, contrary to $\Omega \leqslant 1$ in the Φ^2-oscillaton. The behaviour of both solutions for the energy density is quite similar, since they can be expanded as in (10) with only even Fourier modes.

It is clear now that when gravity is taken into account the final object is bounded, that is, it must be confined to a finite spatial region. The Φ^2-oscillaton does not decay as r^{-2} anymore, but gravity makes the energy density decay more rapidly, in fact exponentially, which ultimately leads to a finite total mass of the oscillaton (see section 2).

As in the flat case, we can perform a similar analysis of the conservation equation. After some easy algebra, the KG equation (6) reads

$$\frac{\partial \rho_\Phi}{\partial t} - \frac{1}{r^2}\frac{\sqrt{C}}{A}\frac{\partial}{\partial r}\left(\frac{r^2 \mathcal{P}_\Phi}{\sqrt{C}}\right) = -\frac{1}{2}\frac{\dot{A}}{A}(\rho_\Phi + p_r). \tag{21}$$

The quantity $\rho_\Phi + p_r = 2\rho_K$ is just the scalar field kinetic energy density

$$\rho_K = \frac{1}{2A}(C\dot{\Phi}^2 + \Phi'^2). \tag{22}$$

Equation (21) does not have an easy interpretation like its flat counterpart, but it still tells us how the energy locally transforms inside an oscillaton.

We have again a term that looks like a scalar current, which is not only related to the scalar momentum density, but also to the gravitational energy, in some sense represented by the metric functions A, C. In addition, we find a source term related to the scalar field kinetic energy and to the time-dependent part of the gravitational energy. This source term might be responsible for the existence of non-singular solutions [1, 2], since it is not present in the static case.

The evident interpretation of equation (21) is that there is always an interchange of energy among the scalar energy density, the momentum density and the gravitational field. The time-dependence of the scalar field energy density is a natural consequence of the KG equation. It is the total energy (kinetic, potential, momentum, gravitational) which is *locally* conserved [21], as can be directly seen from equations (19) and (21).

Finally, the flat oscillaton also indicates that spherical symmetry does not exclude the possibility of scalar (spin-0) radiation, a possibility related to the regular solutions associated with the condition $\Omega > 1$, and then to a dispersion relation in which the wave number $k^2 > 1$, i.e. a non-zero positive energy for scalar field waves. On the opposite side, the inclusion of gravity allows for regular and confined solutions for $\Omega \leqslant 1$, which is a condition obtained from the EKG equations (since the value of Ω can be seen as an output value). The latter could be interpreted as $k^2 < 1$ and then the scalar field waves does not have enough energy to escape from the gravitational well potential.

4. Newtonian oscillatons

Since the discovery of oscillatons [1] it was argued that there is no *post-Newtonian* procedure for them because the time derivatives of the field are of the same order as the field itself. But, the similarity between oscillatons and boson stars (see the so-called stationary limit procedure in [2]) suggests that we may find a Newtonian-like limit for oscillatons by linearizing the EKG equations adequately.

To see how this can be done, we will consider the case in which the scalar field Φ and the metric functions μ, ν in (1) are weak but the metric is not Minkowskian. Next, we make Fourier expansions for Φ, μ, ν as in (8). Before that, we first recall that the full EKG equations, when solved for weaker fields, suggest that we may only take one (two) term(s) in the Fourier expansion of Φ (μ, ν) [2]; see section 2. The other terms become more than two orders of magnitude smaller and then their contribution can be neglected compared to the contribution of the first terms. Therefore, we only take

$$\sqrt{\kappa_0}\Phi(x,t) = \sqrt{2}\phi(x)\cos(\omega t), \tag{23}$$

$$\mu(x,t) = \mu_0(x) + \mu_2(x)\cos(2\omega t), \tag{24}$$

$$\nu(x,t) = \nu_0(x) + \nu_2(x)\cos(2\omega t). \tag{25}$$

As usual, we take the EKG equations until the second order of the functions is involved but we should regard spatial and time derivatives as

$$\frac{\partial}{\partial r} \sim \frac{1}{r}, \qquad \frac{\partial}{\partial t} \sim \mathcal{O}(1). \tag{26}$$

Then, the resulting equations for weak fields are

$$\nu_0' = \frac{x}{2}\phi^2, \tag{27a}$$

$$\mu_0' = \frac{x}{2}\phi^2 - \frac{\nu_0 + \mu_0}{x}, \tag{27b}$$

$$(\nu_0 - \mu_0)\phi = \phi'' + \frac{2}{x}\phi', \tag{27c}$$

$$\mu_2' = \frac{x}{2}\phi^2 - \frac{\nu_2 + \mu_2}{x}, \tag{27d}$$

$$\nu_2 + \mu_2 = \frac{x}{2}\phi\phi'. \tag{27e}$$

Equations (27*a*)–(27*e*) can be written in a more adequate form if we define new functions

$$\alpha(x) \equiv \nu_0(x) + \mu_0(x), \tag{28a}$$

$$\beta(x) \equiv 2\mu_2(x), \tag{28b}$$

$$\gamma(x) \equiv \nu_0(x) - \mu_0(x), \tag{28c}$$

$$\delta(x) \equiv 2\nu_2(x). \tag{28d}$$

From the last equations, we find that the first Fourier coefficients in (8) are written now as $C_0(x) = 1 + \alpha(x) - \gamma(x)$, $C_2(x) = \beta(x)$, $A_0(x) = 1 + \alpha(x)$, $A_2 = (1/2)(\beta(x) + \delta(x))$. Finally, we find

$$(x\gamma)'' = x\phi^2, \tag{29a}$$

$$(x\phi)'' = x\gamma\phi, \tag{29b}$$

$$\alpha = x\gamma', \tag{29c}$$

$$\beta' = x\phi^2 - \phi\phi', \tag{29d}$$

$$\delta = x\phi\phi' - \beta. \tag{29e}$$

Equations (29*a*) and (29*b*) are well known because they appear in the Newtonian limit of boson stars [1, 11, 19] and also in the so-called SN equations [25]. Along this similarity, we shall call *Newtonian oscillatons* the solutions of equations (29*a*)–(29*e*). Therefore, we can take advantage of the earlier studies on Newtonian boson stars and the SN equations and give a series of analytical results for the Newtonian oscillatons for which we will follow closely the excellent works [19, 25]. However, the similarity is not exact and the differences should be studied; we will comment on this at the end of this section.

4.1. Scaling properties

A first good result is that in this limit equations (29*a*)–(29*d*) have acquired additional symmetries. In particular, they have a scale invariance of the form $\{\phi, \gamma, \alpha, \beta, \delta, x\} \rightarrow \{\lambda^2(\hat{\phi}, \hat{\gamma}, \hat{\alpha}, \hat{\beta}, \hat{\delta}), \lambda^{-1}\hat{x}\}$. This behaviour is not surprising because ϕ, μ_0, ν_0, μ_2, ν_2 are always of the same order of magnitude; see figure 1.

Taking advantage of such scale invariance, the space of possible solutions is reduced to the solution of the special case $\hat{\phi}(0) = 1$. Other solutions can be obtained by just an appropriate scaling. This also implies that the weak field limit $\phi \ll 1$ corresponds to $\lambda \ll 1$.

As a consequence of this, observe that equation (29*e*) transforms as

$$\hat{\delta} = -\hat{\beta} + \lambda^2\hat{x}\hat{\phi}\hat{\phi}', \tag{30}$$

which is just telling us that $\nu_2 = -\mu_2 + \mathcal{O}(\lambda^2)$, and that the oscillating component of the metric function g_{rr} is of order λ^2 with respect to the other quantities. Also, the number of variables is further reduced since we can plainly take $\hat{\delta} = -\hat{\beta}$.

The scalar field energy density (3*a*) of the Newtonian oscillaton also obeys a scaling law of the form $\rho_\Phi = \lambda^4 \hat{\rho}_\Phi \left(m_{\rm Pl}^2 m_\Phi^2 / 8\pi\right)$ where

$$\hat{\rho}_\Phi(\hat{x}) = \hat{\phi}^2(\hat{x})[1 - \lambda^2 \gamma_\infty \sin^2(\omega t)]. \tag{31}$$

The energy density is almost static, since its time-dependent part is λ^2 times smaller than the static one. On the other hand, the Schwarzschild mass of the system (11) obeys the scaling $M_\Phi = \lambda \hat{M} \left(m_{\rm Pl}^2 / m_\Phi\right)$ where

$$\hat{M} = \frac{1}{2} \lim_{\hat{x}\to\infty} \hat{x}^2 \hat{\gamma}' = \frac{1}{2} \int_0^\infty \hat{X}^2 \hat{\phi}^2 \, \mathrm{d}\hat{X}, \tag{32}$$

which could be seen as just the integrated mass from the energy density (31).

4.2. Boundary conditions

The boundary conditions can be derived from the relativistic case. Regularity at $\hat{x} = 0$ requires $\hat{\phi}'(0) = \hat{\alpha}(0) = 0$, and $\hat{\beta}(0) = -\hat{\delta}(0)$ which is satisfied straightforwardly.

Asymptotic flatness needs $\hat{\phi}(\infty) = \hat{\alpha}(\infty) = \hat{\beta}(\infty) = \hat{\delta}(\infty) = 0$, but $\hat{\gamma}(\infty) \equiv \gamma_\infty \neq 0$ because it gives the *output* value of the fundamental frequency through $\Omega^{-2} = 1 + \lambda^2 \gamma_\infty$. In fact, the same interpretation arises classically since in the case of the SN equations because the wavefunction has the usual stationary form $\psi \sim \mathrm{e}^{-\mathrm{i}mEt}$ where $\gamma_\infty = -2E$ [25]. Observe that $\Omega < 1$ in the weak field limit (although it is quite close to 1), suggesting then the presence of bounded solutions, unlike the flat oscillaton in which case $\Omega \geqslant 1$. Therefore, Newtonian oscillatons are very simple since they depend only on one set of initial eigenvalues $\hat{\gamma}(0) \equiv \gamma_0, \hat{\beta}(0) \equiv \beta_0$ once the number of nodes of the solution is specified. As in the relativistic case, we will consider only 0-node Newtonian oscillatons.

4.3. Analytical results

The following integral expressions are easily obtained from equations (29*a*)–(29*d*) (see [25]):

$$\hat{\phi}(\hat{x}) = 1 + \int_0^{\hat{x}} \hat{X}(1 - \hat{X}/\hat{x}) \hat{\phi}(\hat{X}) \hat{\gamma}(\hat{X}) \, \mathrm{d}\hat{X}, \tag{33a}$$

$$\hat{\gamma}(\hat{x}) = \gamma_0 + \int_0^{\hat{x}} \hat{X}(1 - \hat{X}/\hat{x}) \hat{\phi}^2(\hat{X}) \, \mathrm{d}\hat{X}, \tag{33b}$$

$$\hat{\alpha}(\hat{x}) = \frac{1}{\hat{x}} \int_0^{\hat{x}} \hat{X}^2 \hat{\phi}^2(\hat{X}) \, \mathrm{d}\hat{X}, \tag{33c}$$

$$\hat{\beta}(\hat{x}) = \beta_0 + \int_0^{\hat{x}} \hat{X} \hat{\phi}^2(\hat{X}) \, \mathrm{d}\hat{X}, \qquad \beta_0 \equiv -\int_0^\infty \hat{X} \hat{\phi}^2(\hat{X}) \, \mathrm{d}\hat{X}, \tag{33d}$$

in which we have taken into account the limit $\lambda \ll 1$. The last equation is just a consequence of asymptotic flatness.

From these expressions we can find the useful asymptotic behaviours near $\hat{x} = 0$,

$$\hat{\phi}(\hat{x}) = 1 + \tfrac{1}{6} \gamma_0 \hat{x}^2 + \mathcal{O}(\hat{x}^4), \tag{34a}$$

$$\hat{\gamma}(\hat{x}) = \gamma_0 + \tfrac{1}{6} \hat{x}^2 + \mathcal{O}(\hat{x}^4), \tag{34b}$$

$$\hat{\alpha}(\hat{x}) = \tfrac{1}{3} \hat{x}^2 + \mathcal{O}(\hat{x}^4), \tag{34c}$$

$$\hat{\beta}(\hat{x}) = \beta_0 + \tfrac{1}{2} \hat{x}^2, \tag{34d}$$

and for large $\hat{x}$

$$\hat{\phi}(\hat{x}) = \phi_\infty \exp(-\sqrt{\gamma_\infty}\hat{x})\hat{x}^k, \qquad k = -1 + \frac{\hat{M}}{\gamma_\infty}, \tag{35a}$$

$$\hat{\gamma}(\hat{x}) = \gamma_\infty - \frac{2\hat{M}}{\hat{x}} + \cdots, \qquad \gamma_\infty = \gamma_0 + \int_0^\infty \hat{X}\hat{\phi}^2\,\mathrm{d}\hat{X}, \tag{35b}$$

$$\hat{\alpha}(\hat{x}) = \frac{2\hat{M}}{x}. \tag{35c}$$

The Newtonian picture above shows that $\hat{\phi}$ vanishes exponentially[4], where ϕ_∞ is a value to be determined from the solution. This is just because the object is gravitationally bound.

Observe that Newtonian oscillatons are further simplified because the last set of equations imply $\beta_0 = \gamma_0 + \gamma_\infty$, and then the solution depends only on the initial eigenvalue γ_0.

4.4. Numerical results

The solution for the 0-node Newtonian case is given by the values $\phi_\infty = 3.3943, \gamma_0 = -0.918\,58, \gamma_\infty = 0.978\,96, \hat{M} = 1.734\,15$ [19, 25, 11]. For oscillatons, we also need the value $\beta_0 = -1.897\,54$. The numerical integration of equations (29*a*)–(29*e*) is shown in figure 4.

The values in the Newtonian limit coincide quite well with those given by the procedure in section 2 if $\sqrt{\kappa_0}\Phi(0,0) \leqslant 10^{-3}$. For instance, if $\sqrt{\kappa_0}\Phi(0,0) = 10^{-3} = \sqrt{2}\phi(0)$ then the scaling parameter is $\lambda^2 = 10^{-3}/\sqrt{2}$. Therefore, the mass of the oscillaton is $M_\Phi = \lambda\hat{M} = 0.046$, and the initial values of the first modes are $C_0(0) - 1 = \lambda^2\gamma_0 = 1.000\,65$, $C_2(0) = \lambda^2\beta_0 = -0.001\,34$, in good agreement, for all the integration range, with the full numerical relativistic result using the full expansions (8).

Both numerical results, relativistic and Newtonian, are compared in figure 4, where we also show the next-to-order Fourier coefficients A_2, C_4, which are not described by the weak field approximation. But note that these coefficients are at least λ^2 times smaller than the leading ones $A_0 - 1, C_0 - 1, C_2$. This is consistent with the approximations assumed for Newtonian oscillatons.

To complete this section, we make a last comment on the weak field limit of oscillatons. It is widely known that boson stars become exactly the SN equations in the post-Newtonian approximation [1, 10, 11, 19]. Moreover, the function $2\hat{\gamma}$ is identified with the usual Newtonian potential U_N.

The situation is different with Newtonian oscillatons. If we take the usual definition of U_N, we find

$$U_\mathrm{N} = \tfrac{1}{2}\lambda^2[\hat{\gamma} - \gamma_\infty - \hat{\beta}\cos(2\omega t)] \tag{36}$$

and then U_N would have to be interpreted as a time-dependent Newtonian potential. This fact would lead us to the question of whether Newtonian oscillatons could be related to a non-stationary solution of the SN equations in which the Newtonian potential could be explicitly time-dependent.

The answer is in the negative, because the extra time-dependent term in (36) is beyond the Newtonian theory. First of all, observe that equation (29*b*) is the Schrödinger equation, the non-relativistic limit of the KG equation, coupled to a Newtonian potential which obeys the Poisson equation (29*a*). Actually, the right-hand side, i.e. the source, of the latter is just the *static* energy density (31). This is the just the usual Newtonian picture.

[4] We have noted that the correct asymptotic expression for large $\hat{x}$ in the case of function ϕ is given in [19], which appears incomplete in [25].

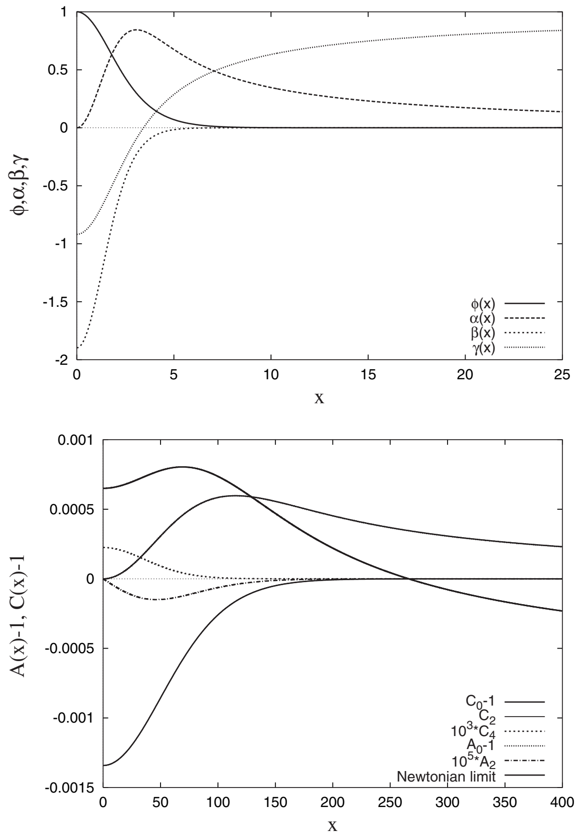

Figure 4. Numerical solutions for the Newtonian oscillaton. Top: scale-invariant functions $\hat{\phi}$, $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$. Bottom: comparison between numerical solutions using the full numerical equations (7*a*)–(7*d*) and the Newtonian oscillatons (29*a*)–(29*d*) with a scaling parameter $\lambda^2 = 10^{-3}/\sqrt{2}$; see text below. Higher modes are also shown, but they are, in general, at least λ^2 times smaller in amplitude, which is consistent with the weak field approximation of this section.

Going further and despite the form of equation (29*d*), we find that its right-hand side is not related to the energy density, but to the scalar radial pressure (3*c*). This is seen from equation (4*c*) by writing the scalar potential as $V(\Phi) = (1/2)(\rho_\Phi - p_\Phi)$. In the weak field limit, the scalar pressure obeys a scaling of the form $p_r = \lambda^4 \hat{p}_r \left(m_{\mathrm{Pl}}^2 m_\Phi^2 \big/ 8\pi\right)$ where

$$\hat{p}_r(\hat{x}) = -\hat{\phi}^2(\hat{x})[\cos(2\omega t) + \lambda^2 \gamma_\infty]. \tag{37}$$

This can be compared to the relativistic case (3*c*), in which p_r has, in general, a time-independent term.

The reason for the existence of equation (29*d*) is that we are not recovering just the Newtonian gravity from the Einstein theory: the scalar pressure of the system is of the same order of magnitude as the energy density[5]. Also observe that, contrary to equation (31), the static part in (37) is negligible with respect to the time-dependent one. This is the physical reason why the intrinsic property of oscillatons, their time-dependence, does not disappear

[5] Newtonian theory can be recovered from Einstein theory if three conditions are fulfilled: (1) low gravity ($g - 1 \ll 1$), (2) low velocities ($v/c \ll 1$) and (3) low pressures ($p \ll \rho$).

even in the weak field limit [1]. Just for comparison, the scalar radial pressure in the case of boson stars is completely negligible and one recovers exactly the Newtonian case.

5. The Schwarzschild oscillaton

In this section, we will give the solutions for an oscillaton with a Schwarzschild metric in the background. This issue is related to the numerical evolution of the gravitational collapse of oscillatons [1, 12, 14].

The final stage of this gravitational collapse is an oscillaton surrounded by a diffuse cloud of ejected scalar field. The dominant contribution to the gravitational field is due to the central oscillaton which matches the Schwarzschild solution far away from the centre. Thus, the ejected scalar field lives in that Schwarzschild background.

As in the case of the flat oscillaton, we will not solve the Einstein equations but only the KG equation (6) under the assumption that, in a very first approximation, the ejected scalar field does not modify the background metric.

Recalling that the Schwarzschild metric is

$$\mathrm{d}s^2 = -\left(1 - \frac{2MG}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2MG}{r}\right)^{-1} + r^2(\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\varphi^2), \qquad (38)$$

the KG equation to be solved is

$$\Phi'' + \Phi'\frac{2}{r}\left[1 + \frac{MG}{r - 2MG}\right] - \frac{m_\Phi^2}{1 - \frac{2MG}{r}}\Phi = \frac{\ddot{\Phi}}{\left(1 - \frac{2MG}{r}\right)^2}. \qquad (39)$$

This differential equation is separable. If we take $\Phi(t,r) = \phi(r)f(t)$ and $x = m_\Phi r$, then

$$\phi'' = -\left(1 - \frac{2Mm_\Phi}{m_{\mathrm{Pl}}^2 x}\right)^{-1}\left\{\left(1 - \frac{Mm_\Phi}{m_{\mathrm{Pl}}^2 x}\right)\frac{2\phi'}{x} + \left[1 - \frac{\alpha^2}{m_\Phi^2}\left(1 - \frac{2Mm_\Phi}{m_{\mathrm{Pl}}^2 x}\right)^{-1}\right]\phi\right\}, \qquad (40a)$$

$$\ddot{f} = -\alpha^2 f, \qquad (40b)$$

in which the evident solution for the temporal part may be $f(t) = \cos(\alpha t)$.

The solution to the radial part cannot be given exactly, and it depends on the ratios $\Theta \equiv \alpha/m_\Phi$ and $\Lambda \equiv 2Mm_\Phi/m_{\mathrm{Pl}}^2$. The latter becomes relevant if we recall that the parameter M is the total mass of the central oscillaton, which is calculated by formula (11). The latter would indicate that Λ is of order

$$\Lambda = \lim_{x\to\infty} x[1 - A^{-1}(t,x)] \leqslant 1.2. \qquad (41)$$

Since we will be analysing the solution outside the central oscillaton, where at least $x > 10$, $\Lambda/(2x) \ll 1$. Therefore, we will deal with the asymptotic form of equation (40*a*),

$$\phi'' \simeq -\frac{2\phi'}{x} + \left[\left(1 + \frac{\Lambda}{x}\right) - \Theta^2\left(1 + 2\frac{\Lambda}{x}\right)\right]\phi. \qquad (42)$$

5.1. Asymptotic solutions

Before going into the general solution to equation (42), we first analyse the special case that corresponds to $\Theta = 1$. In this case, the solution to equation (42) reads

$$\phi_1(x) = \frac{1}{\sqrt{x}}\left\{J_1(2\sqrt{\Lambda x}), Y_1(2\sqrt{\Lambda x})\right\}, \qquad (43)$$

where $J_1(z)$, $Y_1(z)$ are Bessel functions of the first kind and second kind, respectively.

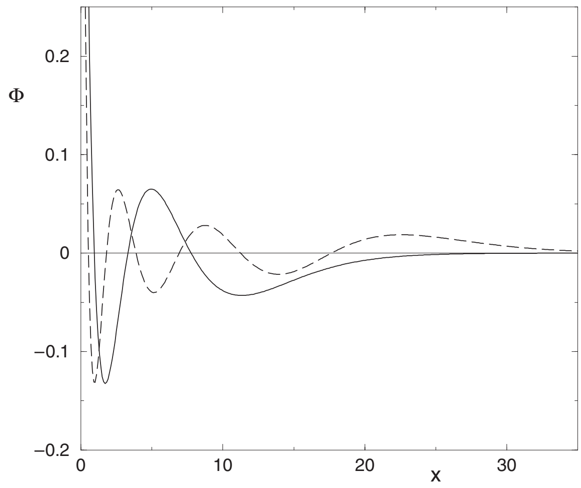

Figure 5. This figure shows $\phi(x) = M_{\mu 1/2}(x)/x$ for $\mu = 4, 7$ (solid and dashed curves respectively). The parameter μ is equal to the numbers of nodes +1.

These solutions radially oscillate and their amplitudes decay asymptotically as $x^{-3/4}$. Making a simple calculation, we find that the scalar energy density decays as $x^{-3/2}$, which directly implies that the integrated mass of the solution grows as $x^{1/2}$. Since this is the expected asymptotic behaviour for $x \gg 1$, we should conclude that this solution cannot represent a bound object, because it would imply an infinite integrated mass.

On the other hand, the general solution to equation (42) can be given in closed form in terms of the so-called Whittaker functions

$$\phi_2(x) = \frac{1}{x}\{M_{\mu 1/2}(z), W_{\mu 1/2}(z)\}, \qquad z = 2\sqrt{1-\Theta^2}x, \qquad \mu = \frac{1}{2}\frac{\Lambda(2\Theta^2-1)}{\sqrt{1-\Theta^2}}. \tag{44}$$

The Whittaker functions are defined as [26]

$$\{M_{\mu\nu}(z), W_{\mu\nu}(z)\} = \mathrm{e}^{-z/2}z^{\nu+1/2}\{{}_1F_1(\nu-\mu+1/2, 2\nu+1; z), U(\nu-\mu+1/2, 2\nu+1; z)\}, \tag{45}$$

where $[{}_1F_1, U](a, b; z)$ are the confluent hypergeometric and Kummer functions respectively.

Well-behaved and finite-mass solutions are obtained only if the argument of functions (44) is real and the parameter μ is zero or a positive integer, i.e. only if the confluent hypergeometric function becomes a polynomial of a real variable, which ultimately implies again that the frequency of the scalar field $\Theta^2 < 1$ can only have certain eigenvalues. In particular, the range allowed is $(1/2) \leqslant \Theta^2 < 1$.

Some instances of (44) are illustrated in figure 5 for $\mu = 4, 7$. It can be observed that these solutions are bound, since the field decays exponentially at large distances from the centre. Also, the solution can have several nodes, the number of which (n) is determined by the parameter μ through $n = \mu - 1$.

Going back to the original picture of a central oscillaton surrounded by a cloud of ejected scalar field, we may consider that the ejected scalar field would settle down into the bounded solution (which implies a finite integrated mass), and then it is more likely to behave as

$$\Phi(t, x) \simeq \phi_2(x)\cos(\alpha t). \tag{46}$$

As said above, we expect the ejected scalar field to settle down into (46), but for this it would be necessary to adjust its frequency in order to satisfy (44) with μ being a positive integer. If this scalar field was ejected from a central oscillaton, then its frequency will not, in general, coincide with the fundamental frequency of the central object. Actually, a frequency of order $\alpha \sim m_\Phi$ implies a high number of nodes, for which solution (43) ($\alpha = m_\Phi$) should be interpreted as the *infinite*-node limit of (44).

6. Conclusions

The study of oscillatons and all related features appears a non-trivial subject. In this paper, we have analysed the Φ^2-oscillaton in order to give complete solutions as possible, involving many terms in the Fourier expansions used and minimizing the nonlinearities present in the coupled EKG system. The analysis includes the calculation of the total mass and fundamental frequencies for a wide range of oscillatons using fast-convergence formulae.

In order to have a better understanding of oscillatons, we compared the Φ^2-case with the so-called flat oscillaton, which arises in the Minkowski spacetime. The latter has an exact solution which helped us to understand the internal energy transformation processes that occur inside an oscillaton. We showed that the interchange of energy, in the case of the flat oscillaton, occurs solely between the scalar field energy density and the momentum density. In the case of the Φ^2-oscillaton, the interchange of energy occurs between the scalar energy density, its momentum density and the gravitational field.

Going further in the same direction, we found the Newtonian-like limit for oscillatons and we called the corresponding solutions Newtonian oscillatons. The solutions presented possess a scaling symmetry and the differential equations are quite similar to those found in the Newtonian limit of boson stars, which is also related to the so-called SN equations. Unlike the latter, Newtonian oscillatons are still time-dependent (like their relativistic partners), but it was possible to extend the semi-analytical work of the SN equations to the new situation with oscillatons.

In addition, we studied the asymptotic behaviour of the Schwarzschild oscillaton which may be related to the final stage of gravitational collapse of oscillatons. The solutions presented would correspond, at first approximation, to a diffuse cloud of scalar field that surrounds a central oscillaton, and then it lies on a Schwarzschild background.

Many points remain for future investigation. First of all, it is necessary to make a similar complete study of self-interacting oscillatons which have been useful for modelling dark matter, with quartic or cosh self-interactions. This study would deal with a larger number of nonlinearities and also with the Newtonian-like limit including self-interactions (for work in this direction with boson stars see, for example, [10, 11]).

The latter becomes even more important if we recall that some proposals of galactic dark halos use only weak fields, in both the SN equations or the Newtonian boson stars [9–11]. Being Newtonian oscillaton time-dependent objects, they could provide new interesting features for scalar galactic halos. If realistic, this picture would be an appropriate hierarchical model of structure formation [12, 14].

Summarizing, oscillatons provide a wide range of new possibilities of gravitationally bounded objects that may be important within scalar field dark matter models.

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